Chiral symmetry breaking and meson wave functions in soft-wall AdS/QCD

Thomas Gutsche,[1] Valery E. Lyubovitskij,[1,*] Ivan Schmidt,[2] and Alfredo Vega[3]

[1]*Institut für Theoretische Physik, Universität Tübingen, Kepler Center for Astro and Particle Physics,*
Auf der Morgenstelle 14, D-72076 Tübingen, Germany
[2]*Departamento de Física y Centro Científico Tecnológico de Valparaíso (CCTVal),*
Universidad Técnica Federico Santa María, Casilla 110-V Valparaíso, Chile
[3]*Departamento de Física y Astronomía, Universidad de Valparaíso, Avenida Gran Bretaña 1111, Valparaíso, Chile*

We consider mesons composed of light and heavy quarks and discuss the construction of the corresponding meson wave functions in soft-wall AdS/QCD. We specifically take care that constraints imposed by chiral symmetry breaking and by the heavy quark limit are fulfilled. The main results are (i) the wave functions of light mesons have a nontrivial dependence on the current quark mass, which gives rise to a mass spectrum consistent with the one including explicit breaking of chiral symmetry; (ii) the wave functions of heavy-light mesons generate their correct mass spectrum, the mass splittings of vector and pseudoscalar states, and the correct scaling of leptonic decay constants $f_{Q\bar{q}} \sim 1/\sqrt{m_Q}$; (iii) the wave functions of heavy quarkonia produce their correct mass spectrum and lead to a scaling behavior of the leptonic decay constants $f_{Q\bar{Q}} \sim \sqrt{m_Q}$ and $f_{c\bar{b}} \sim m_c/\sqrt{m_b}$ at $m_c \ll m_b$, consistent with potential models and QCD sum rules.

The last decade has been marked by significant progress in the development of AdS/QCD—a new class of approaches based on gauge/gravity duality [1], which try to model strong interactions in terms of fields propagating in extradimensional curved manifolds, e.g., in anti–de Sitter (AdS) space. Fields in AdS$_{d+1}$ space are classified by unitary, irreducible representations of the $SO(d, 2)$ group [2,3], which is isomorphic to the conformal group acting on the boundary of AdS space, where the dual conformal field theory (CFT) is living. With some assumptions the CFT can be truncated to QCD, and therefore the AdS fields could be holographically matched to QCD operators and bound states. There are two main types of AdS/QCD approaches: top-down (brane constructions in string theories leading to low-energy gauge theories with properties of QCD) and bottom-up models (phenomenological frameworks specifying the geometry of AdS space and bulk fields in order to incorporate the basic properties of QCD). In this paper we focus on one of the successful examples of bottom-up approaches—the soft-wall model (see e.g., Refs. [4–6]), which is based on a soft breaking of conformal invariance via the introduction of a dilaton field (holographic analogue of the gluon condensate [7]) in the exponential prefactor or the effective potential. It leads to a truncation of AdS space in the infrared and therefore provides confinement of bulk fields, which are expanded in a tower of massive Kaluza-Klein modes identified with radial excitations of hadrons. The dilaton field is also responsible for the mechanism of spontaneous breaking of chiral symmetry.

The profiles of bulk fields in extra dimension are matched to hadronic wave functions. It has been shown that the extradimensional coordinate can be identified with the transverse impact variable characterizing the separation of partons in a hadron in light-front QCD (light-front holography) [8]. It is essential to use the quadratic profile for the dilaton field to generate Regge trajectories for hadron masses and to reproduce the correct scaling of hadronic form factors at large values of Euclidean transverse momentum squared. It is also helpful to use this profile since most of the calculations can be performed analytically. Of course, not all features of QCD have been yet incorporated in the formalism of the soft-wall model. The work on providing an accurate matching to QCD is in progress. Therefore, some objections concerning the soft-wall model seem premature.

This work is addressed to the problem of constructing the hadronic wave functions using the AdS and light-front QCD correspondence. Originally the idea of such correspondence was proposed in Refs. [9]. It was shown that, from the matching of matrix elements for physical processes (e.g., from the electromagnetic or gravitational form factors of hadrons), one can relate the string mode— the bulk profile of the AdS field in a holographic dimension and the transverse part of the hadronic light-front wave function (LFWF) for the case of massless quarks. Later in Ref. [10], in the case of a two-parton state the LFWF was generalized by the explicit inclusion of the constituent quark masses in the LF kinetic energy $\sum_i (\mathbf{k}_{\perp i}^2 + m_i^2)/x_i$. In the LFWF this corresponds to the introduction of the longitudinal WF corresponding to the so-called Brodsky-Huang-Lepage (BHL) or Gaussian ansatz [11] (see also discussion in Ref. [12]). In Refs. [13,14] we studied the

problem of the longitudinal part of the LFWF. In particular, in Ref. [13] following the ideas of Ref. [10], we derived the longitudinal part of the LFWF using constraints of heavy quark effective theory. It was based on the BHL ansatz, where the dimensional parameter depends on the flavor of the constituent quarks. Note that the BHL ansatz deals with constituent quarks while the direct parameters of QCD are the current quarks. The main objective of this paper is to construct the longitudinal part of the LFWF in terms of current quark masses instead of constituent ones. In particular, we will show that this construction helps to introduce the mechanism of explicit breaking of chiral symmetry. This means that the explicit breaking of chiral symmetry is a property of the longitudinal part of the hadronic LFWF. The idea that explicit breaking of chiral symmetry is induced via the current quark mass dependence of the longitudinal LWFW was proposed in two-dimensional large N_c QCD [15]. This mechanism was later used in the context of the two-dimensional massive Schwinger model [16–18]. Recently this problem was reexamined in Refs. [19,20].

In our approach the breaking of the conformal and chiral symmetries is related to the presence of the dilaton field. We show that both symmetries are broken dynamically due to the interaction with a dilaton—background field living in the z direction of AdS space; in other words, it does not show up at the AdS boundary. The dilaton is a massless zero mode. As it was mentioned before we use the quadratic dilaton profile, which can be considered as the expectation value of the scalar bulk field with dimension 2, which is holographically dual to the dimension-2 gluon operator $\mathcal{O}_{A^2} = \langle A^2_{\mu,\min} \rangle$ [21]. In different types of AdS/QCD approaches a few scenarios of chiral symmetry breaking have been proposed due to the presence of specific background or scalar fields. These fields can be considered as duals to the dimension-4 gluon $\mathcal{O}_{G^2} = \mathrm{tr}(G^2_{\mu\nu})$ and the dimension-3 quark $O_{\bar{q}q} = \bar{q}q$ operators. Their couplings to the bulk fields integrated over z can be interpreted as holographic analogues of in-hadron condensates of the operators $\mathcal{O}_{A^2}$, $\mathcal{O}_{G^2}$, and $\mathcal{O}_{\bar{q}q}$. The reason for this assignment is the following: in AdS/QCD (especially in the soft-wall model) matrix elements are defined by integrals over profiles of bulk fields in the z direction, which are holographic images of the hadronic wave functions. The idea of in-hadron condensates was suggested in Ref. [22] and developed in Refs. [23,24]. Notice that in-hadron condensates can be related to vacuum condensates (e.g., in the pion case via current algebra), and therefore there is no contradiction or conflict with QCD and chiral perturbation theory [25]. In particular, the main idea of Refs. [22–24] is that the quark and gluon condensates, holographically interpreted as spatial effects of bulk fields in the z direction, can be interpreted according to the AdS/QCD dictionary as finite-volume effects in hadrons. As stressed in Ref. [23] the quark and gluon condensates have spatial support within hadrons.

We will show that our approach is consistent with model-independent relations and constraints valid in the regime of explicitly and spontaneously broken chiral symmetry. The pion is massless only when the chiral symmetry is spontaneously broken (for a vanishing current quark mass $\hat{m}$) while it becomes massive via the mechanism of explicit breaking of chiral symmetry encoded in the longitudinal part of its LFWF. In addition, we generate a finite splitting of the axial-vector and vector meson masses and reproduce the Weinberg relations between the masses of $\rho(770)$, $a_1(1270)$, and $f_0(600)$. When the current quark masses are included in the formalism through the longitudinal LFWF, the pion and other chiral pseudoscalar fields (kaon and η meson) acquire a mass according to the Gell-Mann-Okubo-Renner relations.

We would also like to mention that the classification of the bulk fields propagating in AdS_5 is similar to the classification of hadrons according to the chiral group [26,27]. As we stressed before, fields in AdS_5 are classified by unitary, irreducible representations of the $SO(4, 2)$ group [2,3]. The $SO(4, 2)$ group is decomposed with respect to its maximal subgroup $SO(4) \otimes SU(2)$, where $SO(4)$ is isomorphic to the group $SU(2) \otimes SU(2)$. Thus, the bulk fields are characterized by three quantum numbers—minimal energy E_0 and two spins J_1 and J_2—and belong to the representations denoted by $D(E_0, J_1, J_2)$, which have specific chiral properties, because $SO(4)$ is also isomorphic to the chiral group $SU_L(2) \otimes SU_R(2)$. Because of the gauge/gravity duality the energy of the bulk field E_0 is identified with Δ—the dimension of the corresponding CFT operator. Masses of bulk fields are expressed in terms of their energy. In particular, the scalar fields S belong to the representation $D(E_0, 0, 0)$ with mass $\mu^2 R^2 = E_0(E_0 - 4)$, which also belongs to the chiral representation $(0, 0)$ (here R is the AdS radius). There are two independent possibilities for a description of vector fields: in terms of vectors V_μ transforming according to the "vectorial" representation $D(E_0, \frac{1}{2}, \frac{1}{2})$ with mass $\mu^2 R^2 = (E_0 - 1)(E_0 - 3)$, which belong to the chiral representation $(1, 0) \oplus (0, 1)$, and in terms of antisymmetric tensors $W_{\mu\nu}$ transforming according to the "tensorial" representation $D(E_0, 1, 0) \oplus D(E_0, 0, 1)$ with mass $\mu^2 R^2 = (E_0 - 2)^2$, which belong to the chiral representation $(\frac{1}{2}, \frac{1}{2})$. Finally, the fermions ψ with spin $1/2$ belong to the representation $D(E_0, 0, \frac{1}{2}) \oplus D(E_0, \frac{1}{2}, 0)$ with mass $\mu R = E_0 - 2$. Fermions belong to the chiral representation $(0, \frac{1}{2}) \oplus (\frac{1}{2}, 0)$. The classification of chiral properties of bulk fields in AdS is identical to the classification according to a chiral basis [26,27], including two possible representations for vector fields, which correspond to vector and pseudotensor operators in QCD. In this sense, the case of exact chiral symmetry corresponds to gauge invariance for massless bulk fields in AdS. The chiral limit for vector mesons corresponds to conservation of the interpolating vector currents on the AdS boundary of AdS space. This means that their holographic analogues in

AdS—vector bulk fields—should be massless or have energy $E_0 = 2 + J = 3$ (with $J = 1$) for both representations $D(E_0, \frac{1}{2}, \frac{1}{2})$ and $D(E_0, 1, 0) \oplus D(E_0, 0, 1)$. Therefore in the case of exact chiral symmetry the twist dimension of interpolating operators of vector mesons must be $\Delta = 2 + J = 3$ (with $J = 1$). However, this is only true for a conformal field theory. In order to guarantee the correct scaling of hadronic form factors in QCD the assignment $E_0 \equiv \Delta$ must be connected with the twist τ of the corresponding interpolating operator expressed in terms of $L = \max |L_z|$—the maximal value of the z component of the quark orbital angular momentum in the LF wave function [5,28]. In the case of two-parton states (mesons) $\Delta_M = \tau_M = 2 + L$, while in the case of three-parton states (baryons) $\Delta_B = \tau_B + 1/2 = 7/2 + L$. This means that the scaling of operators in QCD depending on L corresponds to the picture of both broken conformal and chiral symmetries, and hadrons can be classified according to the nonrelativistic scheme $^{2S+1}L_J$. Conformal invariance is broken to the symmetry of the Poincaré group due to the dilaton field present (living) in the extra z dimension, which is dual to the gluon condensate [7]. The chiral group $SU(2)_L \otimes SU(2)_R$ is broken to the vector isospin group $SU(2)_V$ due to the coupling of bulk fields with the dilaton.

In a series of papers two versions of soft-wall models based on the use of positive [5,28,29] and negative [13,30] dilaton profiles were developed. We showed that these models are equivalent in the case of the bound state problem due to a specific dilaton field-dependent redefinition of the bulk field. It could be shown that after such a redefinition the action for the bulk field $\Phi_J = \Phi_{M_1 \cdots M_J}(x, z)$ with spin J reads [5,13,28–30]

$$S_J = \int d^d x dz \sqrt{g} \, [\partial_M \Phi_J \partial^M \Phi^J - (\mu_J^2 + V_J(z))\Phi_J \Phi^J],$$
$$(1)$$

where the AdS metric is specified as $ds^2 = e^{2A(z)}(dx_\mu dx^\mu - dz^2)$, $g = e^{5A(z)}$, $A(z) = \log(R/z)$, and R is the AdS radius. Here Φ_J is the symmetric, traceless tensor classified by the representation $D(E_0, J/2, J/2)$ with energy $E_0 = \Delta = 2 + L$, which is related to the bulk mass μ_J as $\mu_J^2 R^2 = (E_0 - J)(E_0 - 4 + J) = L^2 - (2 - J)^2$. This action is most convenient in order to study the bound state problem. Versions of the soft-wall model with so-called positive or negative dilaton profiles are just different manifestations of the action (1) after an appropriate dilaton-dependent redefinition of the bulk field $\Phi_J \rightarrow \Phi_J e^{\pm \varphi(z)/2}$. $V_J(z)$ is the effective dilaton potential, which has an analytical expression in terms of the field $\varphi(z)$ and the "metric" field $A(z)$ without referring to any specific form of their z profiles:

$$V_J(z) = e^{-2A(z)}(\varphi''(z) + (d - 1 - 2J)\varphi'(z)A'(z)). \quad (2)$$

The potential $V_J(z)$ breaks both conformal and chiral invariance spontaneously. It was originally suggested in Ref. [4] to use a quadratic profile for the dilaton field (more precisely its z profile) $\varphi(z) = \kappa^2 z^2$, where κ is a scale parameter, and the conformal metric $A(z) = \log(R/z)$ in order to obtain Regge behavior for the hadronic mass spectra. Note that such a form of the profile arises immediately if one considers the free action for the dilaton in the following form:

$$S_\chi = \int d^d x dz \sqrt{g} [\partial_M \chi(x, z) \partial^M \chi(x, z) - \mu_\chi^2 \chi^2(x, z)], \quad (3)$$

where $\mu_\chi^2 = \Delta(\Delta - 4) = -4$ is the bulk mass of the dilaton with $\Delta = 2$ (therefore, the dilaton field is the pure scalar field). We propose that the Kaluza-Klein expansion for the dilaton field is trivial:

$$\chi(x, z) = \varphi(z), \quad (4)$$

where $\varphi(z)$ is the dilaton z profile, and therefore the dilaton is only living in the z direction. The dilaton should be massless (it does not appear in the observable hadronic mass spectra). Solving the equation of motion for $\varphi(z)$ with $\mu_\chi^2 R^2 = -4$ and $A(z) = \log(R/z)$ and assuming the power behavior of $\varphi(z)$, we find $\varphi(z) \sim z^2$ which confirms the conjecture of Ref. [4].

As we stressed before, the quadratic dilaton can be considered as the expectation value of the scalar bulk field with dimension 2, which is holographically dual to the dimension-2 gluon operator $\langle A_{\mu,\min}^2 \rangle$ [21]—a nonlocal gauge-invariant gluon condensate coinciding with the gauge-noninvariant gluon condensate of the same dimension $\langle A_\mu^2 \rangle$ in the Landau gauge. The dimension-2 gluon condensate has been discussed in the literature (see e.g., Refs. [6,21,31]). Note, the interpretation of the dilaton as the quantity dual to the condensate of the dimension-2 operator has been given in the framework of the soft-wall model [6] where the dilaton was introduced in the warping factor, breaking the conformal-invariant background metric. Inclusion of a more complicated form of the dilaton potential (e.g., taking into account self-interaction terms) can be viewed as a further extension of the soft-wall models based on the quadratic dilaton profile.

Notice that a quadratic form of the z profile of the dilaton is not unique. Fore example, in the Liu-Tseytlin model (a type of top-down AdS/QCD approach) [32] the conformal invariance was violated by the dilaton taken in the form $e^{\varphi(z)} = 1 + qz^4$, where the parameter q, according to the AdS/QCD dictionary [33], was related to the matrix element of a QCD operator: the scalar $\langle \text{tr}(G_{\mu\nu}^2) \rangle$ and pseudoscalar $\langle \text{tr}(G_{\mu\nu}\tilde{G}_{\mu\nu}) \rangle$ gluon condensates. On the other hand, the scalar dimension-4 gluon condensate is connected to the quark vacuum condensate via the decoupling relation [34]

$$\langle 0|\bar{q}q|0 \rangle \simeq -\frac{1}{12m} \langle 0|\frac{\alpha_s}{\pi} \text{tr}(G_{\mu\nu}^2)|0 \rangle \quad (5)$$

derived in the leading order of a $1/m$ expansion, where m is the mass of a heavy quark or the constituent mass in the case

of light quarks. Therefore, nonzero dimension-4 gluon condensates signal the existence of nonzero dimension-6 quark condensates, which are manifestations of the spontaneous breaking of chiral symmetry. Therefore we see that the breaking of conformal symmetry is connected with the breaking of chiral symmetry. Actually, this idea is not new (see e.g., discussion in Ref. [35]).

Next we explain why the dilaton is also responsible for the spontaneous breaking of chiral symmetry. With the quadratic z profile of the dilaton field $\varphi(z) = \kappa^2 z^2$ the meson spectrum is given by the master formula [13]

$$M_{nJ}^2 = 4\kappa^2 \left(n + \frac{L + J}{2} \right), \qquad (6)$$

where κ is the dilaton parameter, which signals the spontaneous breaking of conformal and chiral symmetry and leads to a discrete mass spectrum of hadrons. An additional mechanism of spontaneous breaking of chiral symmetry is encoded in the bulk mass μ_J^2, which explicitly depends on L and forbids parity doubling (e.g., between vector and axial mesons). The corresponding breaking term in the mass formula is $\delta M^2 = 2\kappa^2 (L - J)$. It means that we can interpret Eq. (6) as

$$M^2 = \bar{M}^2 + \delta M^2, \qquad (7)$$

where $\bar{M}^2 = 4\kappa^2 (n + J)$ is the term corresponding to the parity doubling limit.

The hadronic wave functions are identified with the profiles of AdS modes $\Phi_n(z)$ in the z direction,

$$\Phi_{nL}(z) = \sqrt{\frac{2\Gamma(n + 1)}{\Gamma(n + L + 1)}} \kappa^{L+1} z^{2+L} e^{-\kappa^2 z^2/2} L_n^L(\kappa^2 z^2),$$

$$(8)$$

which have the correct behavior in both the ultraviolet and infrared limits,

$$\begin{aligned} \Phi_{nL}(z) &\to z^{2+L} \quad \text{at small } z, \\ \Phi_{nL}(z) &\to 0 \quad \text{at large } z, \end{aligned} \qquad (9)$$

and are normalized according to the condition

$$\int_0^\infty \frac{dz}{z^3} \Phi_{nL}^2(z) = \int_0^\infty dz\, \phi_{nL}^2(z) = 1, \qquad (10)$$

where $\Phi_{nL}(z) = z^{3/2} \phi_{nL}(z)$. At $z \to 0$ the scaling of the bulk profile is identified with the scaling of the corresponding mesonic interpolating operator $\tau = 2 + L$. As we mentioned in the Introduction, τ depends on L (instead of J as in CFT) because we model QCD and should reproduce the scaling of hadronic form factors. As we stressed before, dependence on L means spontaneous breaking of chiral invariance, which is expected because after the introduction of the dilaton field we broke the conformal or gauge invariance acting in AdS space.

As we noted before, the chiral group is isomorphic to the subgroup of $SO(4, 2)$.

Let us discuss the consistency of the mass formula (6). First of all, we reproduce the massless pion $M_\pi^2 = 0$ for $n = L = J = 0$, which is consistent with the picture of spontaneous breaking of chiral symmetry. Also, at $z = 0$ all bulk profiles vanish, which corresponds to pointlike hadrons (zero scale limit) and we restore conformal, chiral, and gauge invariance associated with the AdS group $SO(4, 2)$. In this sense we do not have any contradictions with chiral invariance as mentioned in Ref. [27], because for $z = 0$ it is restored for harmonics with any L and for finite z we live in the phase of spontaneously broken chiral invariance (see further details in Ref. [36]). For finite z the profile for the ground state ρ meson is defined by their leading twist corresponding to $L = 0$, which is consistent with the statement of Ref. [27] that the ground state ρ meson is almost a pure S wave in the infrared. On the other hand, the bulk profile dual to the ground state ρ meson is defined in the extra dimension. Therefore it has no correspondence to the chiral-invariant superposition of Fock states with $L = 0$ (S wave) and $L = 2$ (D wave) discussed in Ref. [27]. Note that both the electromagnetic field and $J = L = 1$ vector mesons have the same interpolating twist-3 operator. They also have the same dual field in AdS space—a massless vector bulk field and the bulk-to-boundary propagator of the electromagnetic field in the timelike region [28,37], which has the poles corresponding to the excited states with $J = L = 1$:

$$p^2 = M^2 = 4\kappa^2 (n + 1) \qquad (11)$$

as expected in the chiral-invariant limit.

Let us mention other interesting results deduced from the meson mass formula (6). There are two relations between the masses of the ground states of vector $\rho(770)$, axial-vector $a_1(1270)$, and scalar $f_0(600)$ mesons,

$$M_{a_1} = M_\rho \sqrt{2} = 2\kappa, \qquad M_{f_0} = M_\rho = \sqrt{2}\kappa, \quad (12)$$

which are consistent with the predictions done in Ref. [38] in the same limit when chiral symmetry is spontaneously broken. Also, the vector and axial-vector multiplets are not degenerate in our approach (even for higher values of J), because of the finite mass splitting of axial-vector and vector mesons states:

$$\Delta_{VA} = M_A^2 - M_V^2 = 2\kappa^2. \qquad (13)$$

This means that we do not have parity doubling. An amazing fact is that the same prediction relating the masses of ρ and a_1 mesons was also obtained [39] on the basis of spectral function sum rules at $M_\pi = 0$ and using the Kawarabayashi-Suzuki-Riazuddin-Fayyazuddin formula [40] for the $\rho\pi\pi$ coupling.

For the value $\kappa = 500$ MeV used in this paper the masses of the ρ and a_1 mesons are well reproduced in comparison to data

$$M_\rho = 721 \text{ MeV}, \quad (\text{data: } 775.49 \pm 0.34 \text{ MeV}),$$
$$M_{a_1} = 1010 \text{ MeV}, \quad (\text{data: } 1230 \pm 40 \text{ MeV}), \quad (14)$$

while our result for the f_0 mass should be considered as a prediction. We get $M_{f_0} = 721$ MeV, which perfectly agrees with a model-independent result based on analyticity and unitarity of the S matrix [41]: $M_{f_0} = 735.0 \pm 6.1$ MeV. But also note that a recent compilation [42] indicates a mass of $M_{f_0} = 446 \pm 6$ MeV as deduced from the pole position in the process amplitude.

Next we consider the inclusion of the longitudinal part of the LFWF. We will show that this extension is important to include in our formalism the dependence on the current quark masses and therefore explicit breaking of chiral symmetry. As we stressed before, the first step in this direction was done in Refs. [10]. The authors proposed to write down the mesonic two-parton wave function in a factorized form, as a product of transverse $\phi_{nL}(\zeta)$, longitudinal $f(x, m_1, m_2)$, and angular $e^{im\phi}$ modes. In Ref. [13] we further proposed to do such separation in a more convenient form, factorizing in addition $\sqrt{x(1-x)}$—the Jacobian of the $\zeta \to |\mathbf{b}_\perp|$ coordinate transformation:

$$\psi_{q_1\bar{q}_2}(x, \zeta, m_1, m_2) = \frac{\phi_{nL}(\zeta)}{\sqrt{2\pi\zeta}} f(x, m_1, m_2) e^{im\phi} \sqrt{x(1-x)}. \quad (15)$$

In Ref. [13] we used a Gaussian ansatz for the longitudinal part of the LFWF and treated the quark masses m_1 and m_2 as constituent quark masses. Here we follow Refs. [15,16] and consider current quark masses, and by an appropriate choice of the longitudinal wave function $f(x, m_1, m_2)$ we get consistency with QCD in both sectors of light and heavy quarks. We generate the masses of light pseudoscalar mesons in agreement with the picture resulting from explicit breaking of chiral symmetry—in the leading order of the chiral expansion the masses of pseudoscalar mesons are linear in the current quark mass. In this vein we also guarantee that the pseudoscalar meson masses satisfy the Gell-Mann-Oakes-Renner (GMOR) relation for the pion mass

$$M_\pi^2 = 2\hat{m}B \quad (16)$$

and the Gell-Mann-Okubo (GMO) relation between the masses of pion, kaon, and η meson:

$$4M_K^2 = M_\pi^2 + 3M_\eta^2, \quad (17)$$

In previous equations $\hat{m} = (m_u + m_d)/2$ is the average mass of u and d quarks (we work in the isospin limit $m_u = m_d$), $B = |\langle 0|\bar{u}u|0\rangle|/F_\pi^2$ is the quark condensate parameter, and F_π is the leptonic decay constant. Note, the condensate parameter B is related to the coupling constant of the pseudoscalar density to the pion G_π [25] (or in-hadron condensate of pion [23,24]):

$$B = \frac{G_\pi}{2F_\pi}, \qquad \langle 0|\bar{q}i\gamma_5\tau^i q|\pi^k\rangle = \delta^{ik}G_\pi. \quad (18)$$

In the sector of heavy quarks we get agreement with heavy quark effective theory and potential models for heavy quarkonia. In the heavy quark mass limit $m_Q \to \infty$ we obtain the correct scaling of the leptonic decay constants for both heavy-light mesons $f_{Q\bar{q}} \sim 1/\sqrt{m_Q}$ and heavy quarkonia $f_{Q\bar{Q}} \sim \sqrt{m_Q}$ and $f_{c\bar{b}} \sim m_c/\sqrt{m_b}$ at $m_c \ll m_b$. In this limit we also generate the correct expansion of heavy meson masses

$$M_{Q\bar{q}} = m_Q + \bar{\Lambda} + \mathcal{O}(1/m_Q),$$
$$M_{Q\bar{Q}} = 2m_Q + E + \mathcal{O}(1/m_Q), \quad (19)$$

where $\bar{\Lambda}$ is the approximate difference between the masses of the heavy-light meson and the heavy quark, E is the binding energy in heavy quarkonia, and their splittings, e.g., between vector and pseudoscalar states of heavy-light mesons, become

$$M_{Q\bar{q}}^V - M_{Q\bar{q}}^P \sim \frac{1}{m_Q}. \quad (20)$$

We choose the longitudinal wave function in the form

$$f(x, m_1, m_2) = Nx^{\alpha_1}(1-x)^{\alpha_2}, \quad (21)$$

where N is the normalization constant fixed from

$$1 = \int_0^1 dx f^2(x, m_1, m_2), \quad (22)$$

and α_1, α_2 are parameters that will be fixed in order to get consistency with QCD.

In the present paper the physical quantities of interest are the mass spectrum M_{nJ}^2 and lepton decay constants f_M of mesons, which are given by the expressions [13]

$$M_{nJ}^2 = 4\kappa^2\left(n + \frac{L+J}{2}\right)$$
$$+ \int_0^1 dx\left(\frac{m_1^2}{x} + \frac{m_2^2}{1-x}\right)f^2(x, m_1, m_2), \quad (23)$$

$$f_M = \kappa\frac{\sqrt{6}}{\pi}\int_0^1 dx\sqrt{x(1-x)}f(x, m_1, m_2). \quad (24)$$

Using our ansatz for the longitudinal wave function (21) we calculate the leptonic decay constant and the correction to the mass spectrum analytically [13]:

$$M_{nJ}^2 = 4\kappa^2\left(n + \frac{L+J}{2}\right) + (1 + 2\alpha_1 + 2\alpha_2)\left(\frac{m_1^2}{2\alpha_1} + \frac{m_2^2}{2\alpha_2}\right),$$

$$f_M = \kappa\frac{\sqrt{6}}{\pi}\frac{\Gamma(\frac{3}{2}+\alpha_1)\Gamma(\frac{3}{2}+\alpha_2)}{\Gamma(3+\alpha_1+\alpha_2)}\sqrt{\frac{\Gamma(2+2\alpha_1+2\alpha_2)}{\Gamma(1+2\alpha_1)\Gamma(1+2\alpha_2)}}. \quad (25)$$

Next we analyze different types of mesons. We start with the light pseudoscalar mesons. Here we want to incorporate

a mechanism for explicit breaking of chiral symmetry in order to reproduce the GMOR and GMO relations in the leading order of the chiral expansion. For this purpose we set $\alpha_i = m_i/(2B)$, which is consistent with the ideas and results of Refs. [15,16] and which means that α_i vanishes in the chiral limit $m_i \to 0$. The leptonic decay constants of both the pion and kaon are finite and degenerate in the chiral limit [9,13]:

$$f_P = \frac{\kappa\sqrt{6}}{8}. \qquad (26)$$

Concerning the longitudinal wave functions of other light mesons, the choice of the α_i parameters does not necessarily coincide with the one used for the pseudoscalar mesons. But for simplicity we will use the same universal longitudinal wave function for all light mesons. In this case the leptonic decay constants of π, K, and ρ^0 mesons are degenerate and the corresponding constants for ω and ϕ mesons are related via the $SU(3)$ flavor conditions:

$$f_\pi = f_K = f_\rho = 3f_\omega = \frac{3f_\phi}{\sqrt{2}} = \kappa\frac{\sqrt{6}}{8}. \qquad (27)$$

Next we consider the heavy-light mesons. Here we should reproduce the heavy quark mass expansion of the heavy-light meson masses, the mass splitting of vector and pseudoscalar states, and the $1/\sqrt{m_Q}$ scaling of the leptonic decay constants. All these constraints are fulfilled if the α_i parameters are fixed as $\alpha_Q = \alpha$, a flavor-independent constant, while the parameter α_q must be

$$\alpha_q = \frac{2\alpha_Q}{m_Q}\left(1 + \frac{\bar{\Lambda}}{2m_Q}\right) - \frac{1}{2}. \qquad (28)$$

For this choice the results for the mass spectrum and leptonic decay constants, in the leading order of the chiral expansion, and the leading and next-to-leading order of the heavy quark mass expansion, are

$$M_{Q\bar{q}}^2 = 4\kappa^2\left(n + \frac{L+J}{2}\right) + (m_Q + \bar{\Lambda})^2, \qquad (29)$$

$$f_{Q\bar{q}} = \frac{\kappa\sqrt{6}}{\pi}\frac{2\sqrt{\alpha}}{\alpha + \frac{3}{2}}\sqrt{\frac{\bar{\Lambda}}{m_Q}} \sim \sqrt{\frac{1}{m_Q}}. \qquad (30)$$

Finally, for heavy quarkonia we fix the parameters α_{Q_i} as

$$\alpha_{Q_i} = \frac{m_{Q_i}}{4E}\left(1 - \frac{E}{2(m_{Q_1} + m_{Q_2})}\right) + \mathcal{O}\left(\frac{1}{m_{Q_i}}\right), \qquad (31)$$

and then we get the following results for the spectrum:

$$M_{Q_1\bar{Q}_2}^2 = 4\kappa^2\left(n + \frac{L+J}{2}\right) + (m_{Q_1} + m_{Q_2} + E)^2. \qquad (32)$$

It was shown in Ref. [43] that the trajectories of bottomia states deviate from linearity as we also discussed in Ref. [13]. This effect can be related to the one-gluon

exchange term, which results in an additional Coulomb-like interaction between quarks $V(r) = -4\alpha_s/3r$, where α_s is the strong coupling constant. Its contribution to the mass spectrum M^2 is negative and proportional to the quark mass squared [43]. For light and heavy-light mesons this term can be safely neglected, while this is not the case for heavy quarkonia (especially for bottomia states). Extending the results of Refs. [43] to the general case of a meson containing constituent quarks with masses m_{Q_1} and m_{Q_2}, we get the following expression for the shift of M^2 due to the color Coulomb potential:

$$\Delta M_{Q_1Q_2}^2 = -\frac{64\alpha_s^2 m_{Q_1} m_{Q_2}}{9(n+L+1)^2}, \qquad (33)$$

where α_s is the strong coupling considered as a free parameter. Therefore, the final expression for the heavy quarkonia spectra is given by the master formula:

$$M_{Q_1\bar{Q}_2}^2 = 4\kappa^2\left(n + \frac{L+J}{2}\right) + (m_{Q_1} + m_{Q_2} + E)^2 - \frac{64\alpha_s^2 m_{Q_1} m_{Q_2}}{9(n+L+1)^2}. \qquad (34)$$

For the leptonic decay constants of unflavored quarkonia we get the following result in leading order of the $1/m_Q$ expansion:

$$f_{Q\bar{Q}} = \frac{\kappa\sqrt{6}}{(2\pi)^{3/4}}\left(\frac{E}{m_Q}\right)^{1/4}, \qquad (35)$$

and for the decay constant of B_c meson:

$$f_{c\bar{b}} = \frac{2\kappa\sqrt{6}}{\pi^{3/4}}\frac{(m_c/E)^{3/4}}{(m_b/E)}. \qquad (36)$$

In the case of the B_c meson we additionally apply the condition $m_c \ll m_b$. Equations (35) and (36) can be combined into a general formula for the leptonic decay of heavy quarkonia:

$$f_{Q_1\bar{Q}_2} = \frac{2\kappa\sqrt{6}}{\pi^{3/4}}\frac{(\mu_{Q_1Q_2}/E)^{3/4}}{(M_0/E)}, \qquad (37)$$

where $\mu_{Q_1Q_2} = m_{Q_1}m_{Q_2}/(m_{Q_1} + m_{Q_2})$ is the reduced mass of heavy quarkonia and $M_0 = m_{Q_1} + m_{Q_2}$. In order to get the correct scaling [44] of the leptonic decay constants of heavy quarkonia we propose the following: in the case of heavy quarkonia the dilaton parameter κ should be flavor dependent and scale as $\mu_{Q_1Q_2}^{1/4} \cdot M_0^{1/2}$. Note that different dilaton parameters for light mesons and heavy quarkonia were also considered before (see e.g., Ref. [45]). Here we use the following ansatz for κ:

$$\kappa = \beta\left(\frac{\mu_{Q_1Q_2}}{E}\right)^{1/4}\left(\frac{M_0}{E}\right)^{1/2}, \qquad (38)$$

where $\beta = \mathcal{O}(m_Q^0)$. For unflavored quarkonia the final result for the leptonic decay constant reads:

TABLE I. Masses of light mesons.

Meson	n	L	S	Mass [MeV]			
π	0, 1, 2, 3	0	0	$M_{\pi(140)} = 140$	$M_{\pi(1300)} = 1010$	$M_{\pi(1800)} = 1421$	$M_{\pi(4s)} = 1738$
K	0	0, 1, 2, 3	0	$M_K = 495$	$M_{K_1(1270)} = 1116$	$M_{K_2(1770)} = 1498$	$M_{K_3} = 1801$
η	0, 1, 2, 3	0	0	$M_{\eta(1s)} = 566$	$M_{\eta(2s)} = 1149$	$M_{\eta(3s)} = 1523$	$M_{\eta(4s)} = 1822$
$f_0[\bar{n}n]$	0, 1, 2, 3	1	1	$M_{f_0(1p)} = 721$	$M_{f_0(2p)} = 1233$	$M_{f_0(3p)} = 1587$	$M_{f_0(4p)} = 1876$
$f_0[\bar{s}s]$	0, 1, 2, 3	1	1	$M_{f_0(1p)} = 985$	$M_{f_0(2p)} = 1404$	$M_{f_0(3p)} = 1723$	$M_{f_0(4p)} = 1993$
$\rho(770)$	0, 1, 2, 3	0	1	$M_{\rho(770)} = 721$	$M_{\rho(1450)} = 1233$	$M_{\rho(1700)} = 1587$	$M_{\rho(4s)} = 1876$
$\omega(782)$	0, 1, 2, 3	0	1	$M_{\omega(782)} = 721$	$M_{\omega(1420)} = 1233$	$M_{\omega(1650)} = 1587$	$M_{\omega(4s)} = 1876$
$\phi(1020)$	0, 1, 2, 3	0	1	$M_{\phi(1s)} = 985$	$M_{\phi(2s)} = 1404$	$M_{\phi(3s)} = 1723$	$M_{\phi(4s)} = 1993$
$a_1(1260)$	0, 1, 2, 3	1	1	$M_{a_1(1p)} = 1010$	$M_{a_1(2p)} = 1421$	$M_{a_1(3p)} = 1738$	$M_{a_1(4p)} = 2005$

$$f_{Q\bar{Q}} = \beta \frac{\sqrt{3}}{\pi^{3/4}} \sqrt{\frac{m_Q}{E}}. \tag{39}$$

For the B_c meson we get

$$f_{c\bar{b}} = 2\beta \frac{\sqrt{6}}{\pi^{3/4}} \frac{m_c/E}{\sqrt{m_b/E}}. \tag{40}$$

Finally we present the numerical results. The parameters are fixed to the following values. For the current quark masses we use

$$m_u = m_d = \hat{m} = 7 \text{ MeV}, \qquad m_s = 24\hat{m} = 168 \text{ MeV},$$

$$m_c = 1.275 \text{ GeV}, \qquad m_b = 4.18 \text{ GeV}. \tag{41}$$

For the strong coupling constants α_s we use the following set of parameters:

$$\alpha_s(c\bar{c}) = 0.45, \qquad \alpha_s(c\bar{b}) = 0.383, \qquad \alpha_s(b\bar{b}) = 0.27. \tag{42}$$

Note that for the light and heavy-light mesons we use the universal parameter $\kappa = 500$ MeV. The parameters $\bar{\Lambda}$ for heavy-light mesons are fixed from the mass difference of the experimental values of ground state of D, D_s, B, B_s mesons and corresponding heavy quark mass as

$$\bar{\Lambda}_{cq} = 0.595 \text{ GeV}, \qquad \bar{\Lambda}_{cs} = 0.695 \text{ GeV},$$

$$\bar{\Lambda}_{bq} = 1.1 \text{ GeV}, \qquad \bar{\Lambda}_{bs} = 1.19 \text{ GeV}. \tag{43}$$

The binding energies for heavy quarkonia are fixed as

$$E_{cc} = 0.795 \text{ GeV}, \qquad E_{cb} = 1.25 \text{ GeV}, \qquad E_{bb} = 1.45 \text{ GeV}. \tag{44}$$

The set of β couplings, defining the heavy flavor dependence of the dilaton parameter κ, is fixed in the case of heavy quarkonia as

$$\beta(c\bar{c}) = 0.36 \text{ GeV}, \qquad \beta(c\bar{b}) = 0.32 \text{ GeV},$$

$$\beta(b\bar{b}) = 0.41 \text{ GeV}. \tag{45}$$

These parameters are nearly the same for all quarkonia states, which is consistent with proposed scaling $\beta = \mathcal{O}(m_Q^0)$. In Tables I, II, III, IV, V, and VI we present the numerical results both for the mass spectrum and

TABLE II. Masses of heavy-light mesons.

Meson	J^P	n	L	S	Mass [MeV]			
$D(1870)$	0^-	0	0, 1, 2, 3	0	1870	2000	2121	2235
$D^*(2010)$	1^-	0	0, 1, 2, 3	1	2000	2121	2235	2345
$D_s(1969)$	0^-	0	0, 1, 2, 3	0	1970	2093	2209	2320
$D_s^*(2107)$	1^-	0	0, 1, 2, 3	1	2093	2209	2320	2425
$B(5279)$	0^-	0	0, 1, 2, 3	0	5280	5327	5374	5420
$B^*(5325)$	1^-	0	0, 1, 2, 3	1	5336	5374	5420	5466
$B_s(5366)$	0^-	0	0, 1, 2, 3	0	5370	5416	5462	5508
$B_s^*(5413)$	1^-	0	0, 1, 2, 3	1	5416	5462	5508	5553

TABLE III. Masses of heavy quarkonia $c\bar{c}$, $b\bar{b}$, and $c\bar{b}$.

Meson	J^P	n	L	S	Mass [MeV]			
$\eta_c(2980)$	0^-	0, 1, 2, 3	0	0	2975	3477	3729	3938
$\psi(3097)$	1^-	0, 1, 2, 3	0	1	3097	3583	3828	4032
$\chi_{c0}(3415)$	0^+	0, 1, 2, 3	1	1	3369	3628	3843	4038
$\chi_{c1}(3510)$	1^+	0, 1, 2, 3	1	1	3477	3729	3938	4129
$\chi_{c2}(3555)$	2^+	0, 1, 2, 3	1	1	3583	3828	4032	4219
$\eta_b(9390)$	0^-	0, 1, 2, 3	0	0	9337	9931	10224	10471
$\Upsilon(9460)$	1^-	0, 1, 2, 3	0	1	9460	10048	10338	10581
$\chi_{b0}(9860)$	0^+	0, 1, 2, 3	1	1	9813	10110	10359	10591
$\chi_{b1}(9893)$	1^+	0, 1, 2, 3	1	1	9931	10224	10471	10700
$\chi_{b2}(9912)$	2^+	0, 1, 2, 3	1	1	10048	10338	10581	10808
$B_c(6277)$	0^-	0, 1, 2, 3	0	0	6277	6719	6892	7025

TABLE IV. Decay constants f_P of pseudoscalar mesons in MeV.

Meson	Data [42]	Our
π^-	$130.4 \pm 0.03 \pm 0.2$	153
K^-	$156.1 \pm 0.2 \pm 0.8$	153
D^+	206.7 ± 8.9	207
D_s^+	257.5 ± 6.1	224
B^-	193 ± 11	163
B_s^0	$253 \pm 8 \pm 7$	170
B_c	$489 \pm 5 \pm 3$ [46]	489

TABLE V. Decay constants f_V of vector mesons with open flavor in MeV.

Meson	Data	Our
ρ^+	210.5 ± 0.6 [42]	216
D^*	$245 \pm 20^{+3}_{-2}$ [47]	207
D^*_s	$272 \pm 16^{+3}_{-20}$ [48]	224
B^*	$196 \pm 24^{+39}_{-2}$ [47]	223
B^*_s	$229 \pm 20^{+41}_{-16}$ [47]	170

TABLE VI. Decay constants f_V of vector mesons with hidden flavor in MeV.

Meson	Data [42]	Our
ρ^0	154.7 ± 0.7	153
ω	45.8 ± 0.8	51
ϕ	76 ± 1.2	72
J/ψ	277.6 ± 4	223
$\Upsilon(1s)$	238.5 ± 5.5	170

the leptonic decay constants of light, heavy-light, and heavy quarkonia in leading order of the chiral and heavy-quark mass expansion. Results are for most of the part in reasonable agreement with data. We would like to stress that we significantly improved the description of mesonic properties in comparison to our previous efforts [13].

In conclusion, we demonstrated that in the soft-wall model, where conformal or $SO(4, 2)$ gauge invariance is broken, the same applies for chiral invariance which is also broken (spontaneously). This is manifested in the L dependence of twists of interpolating operators of hadrons and in their observables such as mass spectra and form factors. In the limit of chiral invariance one could expect a dependence on J. However, the J dependence is specific for conformal field theories and not for QCD. Namely, the L dependence is dictated by the scaling of hadronic form factors at higher Q^2. A restriction to a specific L (in our case to the minimal L for a specific hadron) is a manifestation of spontaneous breaking of chiral symmetry. Chiral symmetry is restored in the exact limit $z = 0$ (not for small z), which is trivial and corresponds to the restoration of chiral symmetry for all hadrons having components with an adjustable value of L. We demonstrated explicitly that the present approach is

consistent with model-independent predictions obtained in the case of spontaneous broken chiral symmetry: (i) a massless pion, (ii) the Weinberg sum rule relating masses of ρ, a_1, and f_0 mesons. In the chiral limit the hadron eigenstates are superpositions of components with different values of L. Finally, we demonstrated how to consistently construct the longitudinal LFWF and include the current quark mass dependence. The latter is consistent, in the light quark sector, with the mechanism of explicit breaking of chiral symmetry, and in the heavy quark sector with the heavy quark spin-flavor symmetry.

[1] J. M. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998); Int. J. Theor. Phys. **38**, 1113 (1999); S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, Phys. Lett. B **428**, 105 (1998); E. Witten, Adv. Theor. Math. Phys. **2**, 253 (1998).
[2] M. Flato and C. Fronsdal, J. Math. Phys. (N.Y.) **22**, 1100 (1981); E. Angelopoulos, M. Flato, C. Fronsdal, and D. Sternheimer, Phys. Rev. D **23**, 1278 (1981).
[3] S. Ferrara and C. Fronsdal, Phys. Lett. B **433**, 19 (1998); S. Ferrara, C. Fronsdal, and A. Zaffaroni, Nucl. Phys. **B532**, 153 (1998).
[4] A. Karch, E. Katz, D. T. Son, and M. A. Stephanov, Phys. Rev. D **74**, 015005 (2006).
[5] S. J. Brodsky and G. F. de Teramond, Phys. Rev. Lett. **96**, 201601 (2006).
[6] O. Andreev, Phys. Rev. D **73**, 107901 (2006).
[7] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri, and Y. Oz, Phys. Rep. **323**, 183 (2000).
[8] G. F. de Teramond and S. J. Brodsky, Phys. Rev. Lett. **94**, 201601 (2005); **102**, 081601 (2009).
[9] S. J. Brodsky and G. F. de Teramond, Phys. Rev. D **77**, 056007 (2008); **78**, 025032 (2008).
[10] S. J. Brodsky and G. F. de Teramond, arXiv:0802.0514.
[11] S. J. Brodsky, T. Huang, and G. P. Lepage, Springer Tracts Mod. Phys. **100**, 81 (1982).
[12] T. Huang, B.-Q. Ma, and Q.-X. Shen, Phys. Rev. D **49**, 1490 (1994).
[13] T. Branz, T. Gutsche, V. E. Lyubovitskij, I. Schmidt, and A. Vega, Phys. Rev. D **82**, 074022 (2010).
[14] A. Vega, I. Schmidt, T. Branz, T. Gutsche, and V. E. Lyubovitskij, Phys. Rev. D **80**, 055014 (2009).
[15] G. 't Hooft, Nucl. Phys. **B75**, 461 (1974).
[16] H. Bergknoff, Nucl. Phys. **B122**, 215 (1977).
[17] Y. Ma and J. R. Hiller, J. Comput. Phys. **82**, 229 (1989).

[18] Y.-Z. Mo and R. J. Perry, J. Comput. Phys. **108**, 159 (1993).

[19] S. S. Chabysheva and J. R. Hiller, arXiv:1207.7128.

[20] J. R. Forshaw and R. Sandapen, Phys. Rev. Lett. **109**, 081601 (2012).

[21] F. V. Gubarev, L. Stodolsky, and V. I. Zakharov, Phys. Rev. Lett. **86**, 2220 (2001); F. V. Gubarev and V. I. Zakharov, Phys. Lett. B **501**, 28 (2001).

[22] A. Casher and L. Susskind, Phys. Rev. D **9**, 436 (1974).

[23] S. J. Brodsky and R. Shrock, Proc. Natl. Acad. Sci. U.S.A. **108**, 45 (2011); arXiv:0803.2541; arXiv:0803.2554.

[24] S. J. Brodsky, C. D. Roberts, R. Shrock, and P. C. Tandy, Phys. Rev. C **82**, 022201 (2010).

[25] S. Weinberg, Physica (Amsterdam) **96A**, 327 (1979); J. Gasser and H. Leutwyler, Ann. Phys. (N.Y.) **158**, 142 (1984).

[26] L. Y. Glozman, Phys. Lett. B **587**, 69 (2004).

[27] L. Y. Glozman and A. V. Nefediev, Phys. Rev. D **80**, 057901 (2009).

[28] S. J. Brodsky and G. F. de Teramond, Phys. Rev. D **77**, 056007 (2008); G. F. de Teramond and S. J. Brodsky, Nucl. Phys. B, Proc. Suppl. **199**, 89 (2010).

[29] S. J. Brodsky and G. F. de Teramond, arXiv:0802.0514; G. F. de Teramond and S. J. Brodsky, AIP Conf. Proc. **1257**, 59 (2010).

[30] T. Gutsche, V. E. Lyubovitskij, I. Schmidt, and A. Vega, Phys. Rev. D **85**, 076003 (2012); **86**, 036007 (2012); **87**, 016017 (2013); A. Vega, I. Schmidt, T. Gutsche, and V. E. Lyubovitskij, Phys. Rev. D **83**, 036001 (2011).

[31] L. S. Celenza and C. M. Shakin, Phys. Rev. D **34**, 1591 (1986); K. G. Chetyrkin, S. Narison, and V. I. Zakharov, Nucl. Phys. **B550**, 353 (1999); arXiv:hep-ph/9811275; A. E. Dorokhov and W. Broniowski, Eur. Phys. J. C **32**, 79 (2003); E. Ruiz Arriola, P. O. Bowman, and W. Broniowski, Phys. Rev. D **70**, 097505 (2004); E. Ruiz Arriola and W. Broniowski, Phys. Rev. D **73**, 097502 (2006).

[32] H. Liu and A. A. Tseytlin, Nucl. Phys. **B553**, 231 (1999).

[33] I. R. Klebanov and E. Witten, Nucl. Phys. **B556**, 89 (1999).

[34] M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, Nucl. Phys. **B147**, 385 (1979).

[35] R. J. Crewther, Phys. Rev. D **3**, 3152 (1971); **4**, 3814(E) (1971).

[36] G. F. de Teramond and S. J. Brodsky, arXiv:0903.4922.

[37] H. R. Grigoryan and A. V. Radyushkin, Phys. Rev. D **76**, 095007 (2007).

[38] S. Weinberg, Phys. Rev. Lett. **65**, 1177 (1990).

[39] S. Weinberg, Phys. Rev. Lett. **18**, 507 (1967).

[40] K. Kawarabayashi and M. Suzuki, Phys. Rev. Lett. **16**, 255 (1966); Riazuddin and Fayyazuddin, Phys. Rev. **147**, 1071 (1966).

[41] Y. S. Surovtsev, P. Bydzovsky, and V. E. Lyubovitskij, Phys. Rev. D **85**, 036002 (2012).

[42] J. Beringer et al. (Particle Data Group Collaboration), Phys. Rev. D **86**, 010001 (2012).

[43] M. N. Sergeenko, Yad. Fiz. **56**, 140 (1993) [Phys. At. Nucl. **56**, 365 (1993)]; M. N. Sergeenko, Z. Phys. C **64**, 315 (1994); S. S. Gershtein, A. K. Likhoded, and A. V. Luchinsky, Phys. Rev. D **74**, 016002 (2006).

[44] V. V. Kiselev, Nucl. Phys. **B406**, 340 (1993).

[45] M. Fujita, T. Kikuchi, K. Fukushima, T. Misumi, and M. Murata, Phys. Rev. D **81**, 065024 (2010).

[46] T. W. Chiu and T. H. Hsieh (TWQCD Collaboration), Proc. Sci. LAT2006 (2007) 180.

[47] D. Becirevic, P. Boucaud, J. P. Leroy, V. Lubicz, G. Martinelli, F. Mescia, and F. Rapuano, Phys. Rev. D **60**, 074501 (1999).

[48] C. Aubin et al., Phys. Rev. Lett. **95**, 122002 (2005).