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Asymptotic analysis of the Boltzmann equation for dark matter relics in the presence of a running dilaton and space-time defects

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The interplay of dilatonic effects in dilaton cosmology and stochastic quantum space-time defects within the framework of string/brane cosmologies is examined. The Boltzmann equation describes the physics of thermal dark-matter-relic abundances in the presence of rolling dilatons. These dilatons affect the coupling of stringy matter to D-particle defects, which are generic in string theory. This coupling leads to an additional source term in the Boltzmann equation. The techniques of asymptotic matching and boundary-layer theory, which were recently applied by two of the authors (Bender and Sarkar) to a Boltzmann equation, are used here to find the detailed asymptotic relic abundances for all ranges of the expectation value of the dilaton field. The phenomenological implications for the search for supersymmetric dark matter in current colliders, such as the LHC, are discussed.

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I. INTRODUCTION

To evaluate candidates for cold dark matter (DM) it is necessary to compute relic abundances including physics beyond the standard model. String-theory considerations provide a natural source of such physics.

Recently, a *global* asymptotic analysis was performed on the (Riccati-type) Boltzmann differential equation that describes the evolution of the thermal DM relic abundances in an expanding universe [1]. It was shown that boundary-layer theory, which makes use of asymptotic matching [2], can give a consistent approximate solution to this Riccati equation in two physically interesting cases: (i) standard Friedman-Robertson-Walker (FRW) cosmology [3], and (ii) dilatonic string cosmology [4,5]. In case (i) the freeze-out and post-freeze-out *regions* (we emphasize that these are regions and not isolated points) for the DM abundances were defined using this novel approach. In case (ii) the Boltzman equation of case (i) is modified by the addition of a rolling dilaton source term derivable from string theory and proportional to the dilaton cosmic rate $\frac{d\Phi}{dt}$. The effects of the rolling dilaton on cold DM abundances were calculated, and it was shown that there is a large-time power-law decay of the DM abundance (with calculable corrections). The latter results explain the findings of Ref. [6] on the dilution of DM relic abundances in the current epoch in supersymmetric theories with rolling dilatons. This dilution may significantly affect the available parameter space (after the appropriate cosmological constraints from WMAP [7] are taken into account) and, in turn, may affect the searches for supersymmetry at colliders such as the Large Hadron Collider (LHC) [8].

The analysis cited above does not include the effect of a cosmological background due to effectively pointlike defects (quantum space-time foam), which are generically found in models based on string theory [9]. Dilatons are coupled to the foam through the string coupling constant. This foam modifies the effect of the dilaton in the evolution of DM and can even *dominate* asymptotically in the absence of dilaton effects. In our model of space-time foam the universe is represented as a brane, with three large spatial longitudinal dimensions, embedded in a higher-dimensional bulk space. The "foamy" structures are provided by stringy membrane (D-brane) defects, compactified appropriately along extradimensional manifolds. From the point of view of a four-dimensional observer the defects appear to be pointlike (D particles).[1] As the D-brane world moves in the bulk, the D particles cross it and thus appear to the four-dimensional observer as stochastic space-time structures, flashing on and off. The stochasticity in target-space is attributed to quantum fluctuations of the D particles, viewed as stringy dynamical entities embedded in the bulk space. The dilaton Φ directly affects this process because its vacuum expectation value determines the string coupling g_s. This paper investigates the interplay between the dilaton and a background of space-time defects and their effects on the asymptotic behavior of relic abundances.

In Sec. II we review the main results of Ref. [1] concerning the DM relic density for asymptotically long times in standard and dilatonic cosmologies. This serves to

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[1] D particles are in the spectrum of some but not all string theories; that is, they exist in the spectrum of type IIA but not type IIB string theory. Even when D particles are not in the spectrum, compactified higher dimensional branes may seem like effective D particles for observers on the brane world.

introduce the powerful technique of asymptotic matching [2] used in Ref. [1]. In Sec. III we apply the analytical methods of Ref. [1] to the case of D-particle stochastic foam (D foam) in the presence of relaxing dilatons. The D foam is characterized by a source that differs from the source in dilatonic cosmology. Various classes of asymptotic behaviors are determined by the expectation value of Φ. In Sec. IV we discuss the phenomenology of these models. We address the combined effects of the running-dilaton and D-foam sources on the thermal DM relic abundances today and the associated constraints implied by the current LHC phenomenology. Finally, a technical discussion of the thermodynamic properties of the various types of universes in the presence of sources, which are examined in this article, is given in the Appendix. There, we explain in detail how it is possible to define an entropy function that is conserved in the presence of nontrivial source terms in the Boltzmann equation; this function allows a thermodynamic interpretation of the respective cosmological equations.

II. REVIEW OF ASYMPTOTIC SOLUTIONS TO THE BOLTZMANN EQUATIONS FOR RELIC ABUNDANCES IN STANDARD AND DILATONIC COSMOLOGIES

For a DM species X of mass m_X the evolution of $Y(x) \equiv \mathcal{N}/s$, the number density $\mathcal{N}$ per entropy density s, is governed by the Boltzmann equation [3]

$$Y'(x) = -\lambda x^{-n-2}[Y^2(x) - Y_{\rm eq}^2(x)], \tag{2.1}$$

where $x \equiv m_X/T$ is the dimensionless independent variable and T is the temperature. This Riccati equation does not include any dilatonic effects of string theory. We are primarily interested in epochs of the universe for which $m_X > T > T_0$, where T_0 is the current temperature of the universe. The integer $n = 0, 1, 2, \ldots,$ comes from a partial-wave analysis of the scattering of DM particles: $n = 0$ refers to s-wave scattering, $n = 1$ characterizes p-wave scattering, and so on. The parameter λ is a dimensionless measure of the scattering of DM particles and is regarded as a large number ($\lambda \gg 1$). If we parametrize the thermally averaged annihilation cross section $\langle \sigma v \rangle = \sigma_0 x^{-n}$ with $n = 0, 1, \ldots,$ for $(s, p, \ldots)$-wave DM annihilation, and the Hubble parameter as $H = H_m x^{-2}$, then $\lambda \equiv \sigma_0 m_X^3/H_m$ [3]. For bosonic remnants the function $Y_{\rm eq}(x)$ is the distribution [10]

$$Y_{\rm eq}(x) = A \int_0^\infty ds \frac{s^2}{e^{\sqrt{s^2+x^2}} - 1}, \tag{2.2}$$

where $A = 0.145 g/g_*$, g is the degeneracy factor for the DM species, and g_* counts the total number of massless degrees of freedom [3].

A closed-form analytical solution to the Riccati equation (2.1) is unavailable, so an approximate heuristic approach is customarily used to treat this equation: As the universe cools and x increases, the nature of the solution $Y(x)$ to (2.1) changes rapidly in the vicinity of a value $x = x_f$, the so-called *freeze-out* point, and as $x \to \infty$ the solution $Y(x)$ approaches the constant Y_∞, called the *relic abundance*. One approximation is made for $x < x_f$ and another is made for $x > x_f$. The solutions in the two regions are then patched at $x = x_f$. The value x_f is determined from equating the interaction rate of the DM particle and the expansion rate of the universe, a sensible physical criterion.

This approach gives a reasonably accurate determination of Y_∞ and, prior to the work of Ref. [1], it has been widely adopted [3]. However, this splitting into two regions is only a pragmatic convenience, and there is really no precise value x_f. Rather, there may be (in a sense to be specified) a freeze-out *region*. Because the differential equation (2.1) is first order, its solution is completely determined by *one* initial condition, namely $Y(0)$. The usual method of splitting (2.1) into two approximate first-order equations, which are valid in each of two regions, requires two conditions, an initial condition and a patching condition. The value of x_f becomes explicitly involved in the determination of Y_∞ even though the mathematical theory of differential equations does not require this. To avoid this unsatisfactory mathematical treatment (which is common in the literature), two of the current authors (Bender and Sarkar) presented in Ref. [1] a detailed analysis of the associated Riccati equations using applied mathematical methods commonly used in fluid mechanics. A key concept is that the freeze-out region can, at least in physically relevant cases, be considered as a boundary layer. The solutions in the two regions can then be matched asymptotically. Before reviewing the solution of (2.1) for large x we introduce the Boltzmann equation in the presence of a dilaton background.

In the case of rolling dilaton cosmologies [4] the thermal DM relic abundance is characterized by the presence of a linear *sink* term, which is proportional to the rate of the rolling dilaton field $\frac{d\Phi}{dt}$ [5]. The derivation of this sink term is reviewed in the Appendix. We also explain there that the "physical" frame, relevant for cosmological observations, is the so-called Einstein frame, obtained by an appropriate redefinition of the metric tensor and a time-coordinate transformation [cf. (A5)]. In that frame, the dilaton does not couple to the Einstein-Hilbert scalar curvature term in the gravitational part of the Lagrangian. We adopt the Einstein frame for our computations.

In theories with scale-factor duality [4], we have in this frame

$$\Phi(t) = \Phi_0 \log a(t), \tag{2.3}$$

where $a(t)$ is the scale factor of the expanding universe. In eras where the temperature T satisfies $m_X > T > T_0$,

$$Y'(x) = -\lambda x^{-n-2}[Y^2(x) - Y_{\rm eq}^2(x)] + \Phi_0 Y(x)/x. \tag{2.4}$$

Here, Φ_0 is a *negative* dimensionless constant of order 1 that appears in the general expression for the dilaton field as a function of cosmic time t. For $\Phi_0 = -\phi < 0$, the string coupling $g_s = e^{\Phi}$ becomes *perturbatively* small for large times and vanishes asymptotically as $t \to \infty$. Thus, the σ-model perturbative picture suffices to describe the features of cosmology at large times. As shown in Ref. [1], the presence of the particular dilaton source in (2.4) gives a solution for $Y(x)$ whose behavior is *qualitatively different* from the solution for $Y(x)$ in (2.1).

A. Boundary-layer theory

Since λ is large, the highest derivative in both Eqs. (2.1) and (2.4) is multiplied by a small parameter, which implies that these equations may be treated by using boundary-layer techniques [2] and leads to the concept of a freeze-out region as opposed to a freeze-out point [1]. We rewrite (2.4) as

$$\frac{1}{\lambda} Z'(x) = -x^{-n-2}[x^{-\phi} Z^2(x) - x^{\phi} Y_{\rm eq}^2(x)], \qquad (2.5)$$

where

$$Z(x) \equiv Y(x) x^{\phi}. \qquad (2.6)$$

The coefficient $1/\lambda$ of the highest-derivative term is very small. The number of terms on the right side has been reduced from three to two; this facilitates asymptotic matching. Outside a boundary layer (the outer region), $Z(x)$ varies slowly. Inside a boundary layer, $Z(x)$ varies rapidly.

We have two outer regions where $Z(x) = Z^{(1)}(x)$ and $Z(x) = Z^{(2)}(x)$, respectively. In the left outer region $Z^{(1)}(x) \approx Z_{\rm eq}(x) \equiv x^{\phi} Y_{\rm eq}(x)$. To be precise, we write

$$Z^{(1)}(x) \sim \sum_{k=0}^{\infty} \lambda^{-k} Z_k^{(1)}(x). \qquad (2.7)$$

On substituting $Z^{(1)}(x)$ into (2.5), we find that

$$Z_0^{(1)}(x) = A e^{-x} x^{\varphi+3/2}, \qquad Z_1^{(1)}(x) = x^{\varphi+n+2}/2, \qquad (2.8)$$

and so on. The entity x_f is defined to be the value of x for which

$$Z_0^{(1)}(x) = Z_1^{(1)}(x) \qquad (2.9)$$

and is a measure of where the equilibrium region ends. Equation (2.9) implies that

$$x_f \sim \log(2A\lambda) - (n + 1/2)\log(x_f). \qquad (2.10)$$

This analysis is somewhat simplified (see Ref. [1]). The higher order terms in (2.7) are not negligible, but they lead to a series with alternating signs that is Borel summable. The Borel sum of the series leads to a multiplicative renormalization of A by a factor close to 1. To keep the notation simple we have not distinguished A from the renormalized A. Solving the equation obtained by replacing in (2.10) the symbol $\sim$ by the equality sign gives x_f:

$$x_f = (n + 1/2) W\left[\frac{(2\lambda A)^{n+1/2}}{n + 1/2}\right], \qquad (2.11)$$

where $W(z)$ is a Lambert function [11]. Hence the asymptotic behavior is fully determined in terms of constants occurring in the Boltzmann equation.

The value x_f lies in the transition region from equilibrium to freeze-out, which is interpreted as a boundary layer. This interpretation can be justified by the method of asymptotic matching. We define an inner variable X as follows:

$$x = x_f + \kappa X. \qquad (2.12)$$

Then $|X|$ can be large compared to 1 but small compared to λ. Now, for $\mathcal{Z}(X)$ we have

$$\frac{1}{\kappa} \mathcal{Z}'(X) = -\lambda x_f^{-n-2-\phi}[\mathcal{Z}^2(X) - A^2 x_f^{3+2\phi} e^{-2x_f}] \approx -\lambda x_f^{-n-2-\phi} \mathcal{Z}^2(X). \qquad (2.13)$$

The exponential term is negligible because $x_f \approx 25$ for typical values $\lambda \approx 10^{14}$ and $A \approx 0.00145$.

From the principle of dominant balance [2] we have

$$\kappa = x_f^{n+2+\phi}/\lambda. \qquad (2.14)$$

The solution to (2.13) is

$$\mathcal{Z}(X) = 1/(X + D), \qquad (2.15)$$

where D is a constant of integration. This is the solution in the boundary-layer (or freeze-out) region.

To the right of this boundary layer there is a second outer region. For large x in this region

$$Z'(x) \sim -\lambda x^{-n-2-\phi} Z^2(x) \quad (x \gg 1), \qquad (2.16)$$

whose solution is

$$Z^{\text{post-freeze-out}}(x) \sim \frac{1}{1/C - \lambda x^{-n-1-\phi}/(n+1+\phi)}, \qquad (2.17)$$

where C is an integration constant.

The behaviors in the equilibrium outer region, the boundary-layer region, and the post-freeze-out outer region must be asymptotically matched. This matching determines the constants of integration C and D. We first match the solution in the equilibrium region to the boundary-layer solution:

$$Z^{\text{thermal-equilibrium}}(x) \sim 2A x^{3/2+\phi} e^{-x} \sim 2A(x_f + \kappa X)^{3/2+\phi} e^{-x_f} e^{-\kappa X}.$$

The factor of 2 is included because two lowest-order terms of the expansion in (2.7) are considered. Noting that κ and X/x_f are small, we get

$$Z^{\text{thermal-equilibrium}}(x) \sim \frac{x_f^{n+2+\phi}}{\lambda(1+\kappa X)} \sim \frac{1}{X + \lambda x_f^{-n-2-\phi}} \quad (2.18)$$

on using (2.14). Hence, comparing with (2.15), we deduce that

$$D = \lambda x_f^{-n-2-\phi}. \quad (2.19)$$

Similarly, (2.17) leads to

$$Z^{\text{post-freeze-out}}(x) \sim \frac{1}{\frac{1}{C} - \frac{\lambda}{n+1+\phi}(x_f + \kappa X)^{-n-1-\phi}},$$

from which we deduce that

$$Z^{\text{post-freeze-out}}(x) \sim \frac{1}{\frac{1}{C} - \frac{\lambda}{(n+1+\phi)x_f^{n+1+\phi}} + X}. \quad (2.20)$$

Comparing with (2.15), we get

$$D = \frac{1}{C} - \frac{\lambda}{(n+1+\phi)x_f^{n+1+\phi}}. \quad (2.21)$$

Finally, from (2.19) we deduce that

$$C = \frac{(n+1+\phi)x_f^{n+2+\phi}}{\lambda(n+1+\phi+x_f)}. \quad (2.22)$$

The leading behavior for large x in the post-freeze-out region is

$$Y(x) \sim \frac{(n+1+\phi)x_f^{n+2+\phi}}{\lambda(n+1+\phi+x_f)} x^{-\phi}. \quad (2.23)$$

We denote the solution to (2.1) as $Y_{ns}(x)$, where ns stands for *no source*. Its asymptotic value for large x is obtained from (2.23) by setting $\phi = 0$. The specific solution for x_f in (2.11) is denoted by $x_{f,ns}$.

The above calculation forms the basis of the following analysis that will be given for various parameter ranges and sources in the Boltzmann equation.

III. DM RELIC ABUNDANCES: THE CASE OF A STOCHASTIC STRINGY SPACE-TIME FOAM

The background of stochastic D-particle foam leads [9] to the inclusion of a positive source Γ (as opposed to the sink in dilaton cosmology) in the standard Boltzmann equation for the thermal relic abundance of the DM species X of mass m_X. In terms of the number density $\mathcal{N}$ it was shown in Ref. [9] that the Boltzmann equation reads

$$\frac{d\mathcal{N}}{dt} + 3H\mathcal{N} = \Gamma(t)\mathcal{N} + C[f], \quad (3.1)$$

where $C[f]$ denotes the Boltzmann interaction terms and

$$\Gamma(t) = 2Hm_X a^4(t)\frac{g_s^2}{M_s^2}T(9 + 2m_X/T)\langle\langle\Delta^2\rangle\rangle, \quad (3.2)$$

where M_s is the string mass scale. (The mass of a D-particle defect in the foam is M_s/g_s [9].) The quantity $\langle\langle\Delta^2\rangle\rangle$ is a dimensionless variable, which expresses the variance in the recoil velocities of the D-particle defects in the foam, during their collisions with the DM particles [9].

The symbol $\langle\langle\cdots\rangle\rangle$ denotes the average over the population of D particles on the three-dimensional-space brane world in a given epoch of the universe. The no-force (dustlike) behavior of the D particles implies the following scaling of $\langle\langle\Delta^2\rangle\rangle$ with the scale factor $a(t)$ of the four-dimensional (brane) universe:

$$\langle\langle\Delta^2\rangle\rangle = \langle\Delta^2\rangle_0 a^{-3}(t) = \langle\Delta^2\rangle_0 C_0^{-3}T^3 = \langle\Delta^2\rangle_0 m_X^3 C_0^{-3}x^{-3}. \quad (3.3)$$

Here $C_0 = a(t_0)T_0$ is a dimensionful constant that appears in the cooling law of the universe; that is,

$$a(t) = C_0/T = a(t_0)/(1+z), \quad (3.4)$$

where z is the redshift parameter. The values $z = 0$, $t = t_0$, and $T = T_0$ correspond to the current era. This source is positive (in contrast to the sink of dilaton cosmology) and is discussed in a more general framework in the Appendix.

We now discuss the collision term $\langle\sigma v\rangle[(\mathcal{N}^{(0)})^2 - \mathcal{N}^2]$ in (3.1), where $\mathcal{N}^{(0)}$ is the equilibrium value of the DM number density. Equation (3.1) now becomes

$$Y'(x) = -\lambda x^{-n-2}[Y^2(x) - Y_{\text{eq}}^2(x)] + \frac{2C_0^4 g_s^2}{m_X^2 M_s^2}\langle\langle\Delta^2\rangle\rangle x^2(9+2x)Y(x). \quad (3.5)$$

Hence, the Boltzmann equation (3.5) becomes

$$Y'(x) = -\lambda x^{-n-2}[Y^2(x) - Y_{\text{eq}}^2(x)] + g_s^2\frac{2C_0 m_X}{M_s^2}\langle\Delta^2\rangle_0(9+2x)Y(x)/x. \quad (3.6)$$

There is an implicit dilaton dependence in (3.6) that needs to be made explicit. The string coupling g_s is the exponential of the dilaton, $g_s = g_0\exp(\langle\Phi\rangle)$, and so

$$g_s \sim g_0 a^{-\phi} = g_0(C_0/m_X)^{-\phi}x^{-\phi}. \quad (3.7)$$

Hence, for consistency we must incorporate *both* the dilaton sink and the source induced by D-particle foam in the Boltzmann equation in a combined source. The resulting Boltzmann equation is

$$Y'(x) = -\lambda x^{-n-2}[Y^2(x) - Y_{\text{eq}}^2(x)] - \mathcal{S}(x,\phi)Y(x)/x, \quad (3.8)$$

where

$$S(x,\phi) = \phi - \zeta_\phi(9+2x)x^{-2\phi} \quad (3.9)$$

and

$$\zeta_\phi \equiv 2g_0^2 C_0^{1-2\phi}m_X^{1+2\phi}\langle\Delta^2\rangle_0/M_s^2. \quad (3.10)$$

We are especially interested in the regime of temperatures $m_X \gg T$, that is, as $x \to \infty$. However, the asymptotic matching requires a knowledge of the solution for higher T as well. From (3.9) it is clear that for the case $\phi = 1/2$ the x dependence of the source and sink coincide for large x. Just as in (2.5), it is convenient to rewrite (3.8) in the form of a differential equation with the nonlinear terms on the right side. We introduce the function

$$g(x, \phi) \equiv x^{\phi} \exp\left[\zeta_\phi\left(\frac{9}{2\phi} - \frac{2x}{1-2\phi}\right)x^{-2\phi} + \frac{2\zeta_\phi}{1-2\phi}\right], \tag{3.11}$$

which is smooth at $\phi = 1/2$, and the function

$$f(x, \phi) \equiv x^{-n-2}/g(x, \phi). \tag{3.12}$$

We then define

$$Z(x, \phi) \equiv g(x, \phi)Y(x),$$

which is the analog of (2.6), and

$$Z_{\rm eq}(x, \phi) \equiv g(x, \phi)Y_{\rm eq}(x).$$

The Riccati equation satisfied by $Z(x, \phi)$ is

$$\frac{dZ(x, \phi)}{dx} = -\lambda f(x, \phi)[Z^2(x, \phi) - Z^2_{\rm eq}(x, \phi)], \tag{3.13}$$

which is similar in structure to (2.5). The explicit form of $f(x, \phi)$ is

$$f(x, \phi) = x^{-n-2-\phi} \exp\left[-\zeta_\phi\left(\frac{9}{2\phi} - \frac{2x}{1-2\phi}\right)x^{-2\phi} - \frac{2\zeta_\phi}{1-2\phi}\right]. \tag{3.14}$$

The dominant asymptotic behavior of $f(x, \phi)$ as $x \to \infty$ changes according to the value of ϕ; $f(x, \phi)$ decays for $\phi > 1/2$ and $f(x, \phi)$ increases exponentially for large x for $\phi < 1/2$. Note that the phenomenologically relevant quantity is the Hubble-constant-free-relic abundance, $\Omega h^2 = m\mathcal{N}/\rho_0^c$, where ρ_0^c is the critical density today and $\mathcal{N}$ is the number density of the DM species. This is the quantity that is measured in experiments. For DM species X with mass m_X it is given by [3]

$$\Omega_X h^2 = m_X^4 Y(x)/x^3. \tag{3.15}$$

The modification of Ω_X can be compared to the standard (source-free) relic density by considering the phenomenologically interesting ratio

$$\mathcal{R} \equiv \lim_{x\to\infty} \frac{\Omega_X}{\Omega_X^{\text{source-free}}} \sim \lim_{x\to\infty} \frac{Y(x)}{Y_{ns}(x)}, \tag{3.16}$$

where $\Omega_X^{\text{source-free}}$ denotes the relic density of the DM species X in the standard cosmology case with constant dilaton and no space-time foam. We now systematically consider the behavior of the solution to the Boltzmann equation for various values of ϕ.

A. The case of ϕ near 1/2

To investigate the behavior near $\phi = 1/2$ we let $\phi = 1/2 - \delta$ and treat δ as small. We write $Z_\delta(x) \equiv Z(x, 1/2 - \delta)$. The Riccati equation satisfied by $Z_\delta(x)$ in (3.13) is

$$Z'_\delta(x) = -\lambda f_\delta(x)[Z^2_\delta(x) - Z^2_{{\rm eq},\delta}(x)], \tag{3.17}$$

where $Z_{{\rm eq},\delta}(x) \equiv Z_{\rm eq}(x, 1/2 - \delta)$ for small δ and

$$f_\delta(x) \approx x^{-n-5/2+\delta+2\eta_\delta} \exp(-9\eta_\delta/x) \tag{3.18}$$

with $\eta_\delta \equiv \zeta_{1/2-\delta}$. Moreover, we have

$$Z_{{\rm eq},\delta}(x) \sim Ax^{2-\delta-2\eta_\delta}e^{-x}. \tag{3.19}$$

As we did in Sec. II, we argue that for large x in the post-freeze-out outer region, $Z_\delta(x) \approx Z_\delta^{\text{post-freeze-out}}(x)$, where

$$\frac{d}{dx}Z_\delta^{\text{post-freeze-out}}(x) = -\lambda f_\delta(x)[Z_\delta^{\text{post-freeze-out}}(x)]^2. \tag{3.20}$$

The solution to (3.20) is

$$Z_\delta = \frac{1}{\mathcal{C}_\delta^{-1} - \lambda\frac{x^{-n-3/2+\delta+2\eta_\delta}}{n+3/2-\delta-2\eta_\delta}}, \tag{3.21}$$

and $\mathcal{C}_\delta$ is an integration constant to be determined.

In the equilibrium outer region, following (2.5), (2.7), and (2.9), we substitute

$$Z_\delta(x) \sim \sum_{k=0}^{\infty} \lambda^{-k} Z_{k,\delta}(x) \tag{3.22}$$

into (3.17). This leads to $Z_{0,\delta}(x) = Z_{{\rm eq},\delta}(x)$ and

$$Z_{1,\delta}(x) = -\frac{1}{2f_\delta(x)}\frac{d}{dx}\log Z_{{\rm eq},\delta}.$$

The value $x = x_f$, which characterizes the freeze-out region, is determined by $Z_{0,\delta}(x) = Z_{1,\delta}(x)$, and we again obtain (2.10).

In the inner (freeze-out) region we introduce X as in (2.12). The resulting equation for $\mathcal{Z}_\delta(X)$ is

$$\frac{1}{\kappa}\frac{d}{dX}\mathcal{Z}_\delta(X) = -\lambda x_f^{-n-5/2+\delta+2\eta_\delta}[\mathcal{Z}^2_\delta(X) - A^2x_f^{4-2\delta-4\eta_\delta}e^{-2x_f}]. \tag{3.23}$$

The criterion of dominant balance requires that

$$\frac{1}{\kappa} = \lambda x_f^{-n-\frac{5}{2}+\delta+2\eta_\delta}. \tag{3.24}$$

Following earlier arguments [see (2.13)], in the inner region we have

$$\frac{d}{dX}\mathcal{Z}_\delta(X) \approx -\mathcal{Z}^2_\delta(X).$$

The solution to this equation is

$$\mathcal{Z}_\delta(X) = 1/(X + \mathcal{D}_\delta), \tag{3.25}$$

where $\mathcal{D}_\delta$ is a constant of integration. Matching (3.21) with (3.25) gives

$$\mathcal{D}_\delta = \frac{1}{\mathcal{C}_\delta} - \frac{\lambda}{(n+3/2-\delta-2\eta_\delta)x_f^{n+3/2-\delta-2\eta_\delta}}. \tag{3.26}$$

As in (2.19), the matching of the solutions in the equilibrium and freeze-out regions determines that

$$\mathcal{D}_\delta = \lambda x_f^{-n-5/2+2\eta_\delta+\delta}. \tag{3.27}$$

The analog of (2.22) is

$$\frac{1}{\mathcal{C}_\delta} = \lambda \frac{n+3/2-2\eta_\delta-\delta+x_f}{n+3/2-2\eta_\delta-\delta} x_f^{-n-5/2+2\eta_\delta+\delta}. \tag{3.28}$$

Here, $x_f = x_{f,ns}$. Consequently, for $\phi = 1/2 - \delta$ and δ small, the large-x asymptotic behavior of $Y(x)$ is

$$Y(x) \sim C_\delta x^{-1/2+2\zeta_\delta+\delta}. \tag{3.29}$$

In such a case C_δ and the freeze-out region are determined from (3.28) and from (2.10), while the freeze-out point is given by (2.11). We note that the limit $\delta \to 0$ is smooth.

B. The case of general $\phi > 0$ with ϕ not near 1/2

By the arguments given in Sec. II for large x in the post-freeze-out region the approximate solution to (3.13) is $Z(x,\phi) \approx Z^{\text{post-freeze-out}}(x,\phi)$, where

$$\frac{d}{dx} Z^{\text{post-freeze-out}}(x,\phi) = -\lambda f(x,\phi)[Z^{\text{post-freeze-out}}(x,\phi)]^2. \tag{3.30}$$

The solution to this equation is

$$Z^{\text{post-freeze-out}}(x,\phi) = \frac{1}{C_\phi^{-1} + \lambda \int dx f(x,\phi)}, \tag{3.31}$$

where C_ϕ is a positive constant. Equation (3.31) is valid for general $\phi > 0$.

It is convenient to rewrite (3.11) and (3.14) using the function

$$h(x,\phi) \equiv (1 - x^{1-2\phi})/(1-2\phi). \tag{3.32}$$

We then have

$$g(x,\phi) = x^\phi \exp\left(\frac{9\zeta_\phi}{2\phi} x^{-2\phi}\right) \exp[2\zeta_\phi h(x,\phi)]$$

and

$$f(x,\phi) = x^{-n-2-\phi} \exp\left(-\frac{9\zeta_\phi}{2\phi} x^{-2\phi}\right) \exp[-2\zeta_\phi h(x,\phi)].$$

In the limit as $x \to 0$

$$h(x,\phi) \to \begin{cases} \frac{1}{1-2\phi}, & \text{for } \phi < 1/2, \\ -\frac{1}{2\phi-1}, & \text{for } \phi > 1/2, \end{cases}$$

and so $f(x,\phi) \to 0$. Furthermore as $x \to \infty$, $Z_{\text{eq}}(x,\phi)$ is negligible because in this limit

$$h(x,\phi) \to \begin{cases} -\infty & \text{for } \phi < 1/2, \\ -\frac{1}{2\phi-1} & \text{for } \phi > 1/2. \end{cases}$$

1. The case $0 < \zeta_\phi < \phi \ll 1/2$

Next, we consider the case for which $\zeta_\phi/\phi \sim O(1)$ and $\zeta_\phi x_f \ll 1$. This case illustrates the competition between space-time foam and dilaton sources in their effect on the relic abundance. In this case

$$g(x,\phi) \sim x^\phi \exp\left(\frac{9\zeta_\phi}{2\phi}\right), \quad f(x,\phi) \sim x^{-n-2-\phi} \exp\left(\frac{9\zeta_\phi}{2\phi}\right), \tag{3.33}$$

for x in the freeze-out region and $x \gg x_f$. The analog of (2.5) is similar except that λ is replaced by $\lambda \exp(-\frac{9\zeta_\phi}{2\phi})$.

The analog of (2.10) is

$$\begin{aligned} x_f &\sim \log\left[2A\lambda \exp\left(-\frac{9\zeta_\phi}{2\phi}\right)\right] - (n+1/2)\log(x_f) \\ &= \log(2A\lambda) - (n+1/2)\log(x_f) - \frac{9\zeta_\phi}{2\phi}. \end{aligned} \tag{3.34}$$

The previous analysis then implies that for large x we have the following asymptotic behavior for $Y(x)$:

$$Y(x) \sim \frac{(n+1+\phi)x_f^{n+2+\phi}}{\lambda(n+1+\phi+x_f)} x^{-\phi}, \tag{3.35}$$

where

$$x_f = (n+1/2)W\left(\left[2\lambda A \exp\left(-\frac{9\zeta_\phi}{2\phi}\right)\right]^{n+1/2} \Big/ (n+1/2)\right). \tag{3.36}$$

We denote this value of x_f by $x_{f,1}$. The scaling (3.35) is formally similar to the pure time-dependent dilaton case in (2.23), but the effects of the D-foam are incorporated only in the shifted value of the freeze-out point x_f in (3.36).

2. The case $\phi \gg 1/2$

For $\phi \gg 1/2$ and $x \gg x_f$ we have

$$\frac{1}{Z} = -\lambda \exp\left(\frac{2\zeta_\phi}{2\phi-1}\right) \frac{x^{-n-1-\phi}}{n+1+\phi} + \frac{1}{\mathcal{C}}, \tag{3.37}$$

where $\mathcal{C}$ is a constant. To leading order the analog of (2.9) for this case is independent of ϕ and ζ_ϕ, so x_f is determined by (2.10). In the inner (boundary-layer) region we again write $x = x_f + \kappa X$ and $Z(X) = Z(X_F + \kappa X)$. Hence,

$$\frac{1}{\kappa}\frac{dZ}{dX} \simeq -\lambda(x_f + \kappa X)^{-n-2-\phi} \times \exp\left(\frac{2\zeta_\phi}{2\phi-1}\right)[Z^2(X) - Z_{\text{eq}}^2(x_f)],$$

where $Z_{\rm eq}^2(x_f) = \frac{1}{4\lambda^2} x_f^{4+2\phi+2n} \exp(-\frac{4\zeta_\phi}{2\phi-1})$. The principle of dominant balance then implies that

$$\frac{1}{\kappa} = \lambda x_f^{-n-2-\phi} \exp\left(\frac{2\zeta_\phi}{2\phi-1}\right).$$

Hence, $\frac{d\mathcal{Z}}{dX} = -\mathcal{Z}^2$ with the solution $\mathcal{Z}(X) = 1/(x+\mathcal{D})$, where D is a constant.

Matching the equilibrium region to the boundary layer gives

$$2Z_{\rm eq}(x_f + \kappa X) \approx 1/(x+\mathcal{D}).$$

This implies that

$$\mathcal{D} = \lambda x_f^{-n-2-\phi} \exp\left(\frac{2\zeta_\phi}{2\phi-1}\right). \tag{3.38}$$

Matching the freeze-out-region solution to the post-freeze-out-region solution (3.37), we find that

$$\frac{1}{\mathcal{C}} = \lambda \exp\left(\frac{2\zeta_\phi}{2\phi-1}\right) x_f^{-n-2-\phi}\left(1 + \frac{x_f}{n+1+\phi}\right). \tag{3.39}$$

Finally, we find that as $x \to \infty$,

$$Y(x) \sim \frac{1}{\lambda}\exp\left(-\frac{2\zeta_\phi}{2\phi-1}\right) \times \frac{x^{-\phi}}{x_f^{-n-2-\phi} + (x_f^{-n-1-\phi} - x^{-n-1-\phi})/(n+1+\phi)}, \tag{3.40}$$

and $x_f = x_{f,ns}$.

3. The approach to ϕ=0

The integrating factor in (3.11) is singular as $\phi \to 0^+$. However, the function g is only determined up to an x-independent factor. To study the limit $\phi \to 0$ we consider a modified $g(x,\phi)$ and an associated $f(x,\phi)$, which we denote $\tilde{g}(x,\phi)$ and $\tilde{f}(x,\phi)$, respectively. These functions have the following form:

$$\tilde{g}(x,\phi) = x^\phi \exp[2\zeta_\phi h_1(x,\phi)]\exp[9\zeta_\phi h_2(x,\phi)], \tag{3.41}$$

where

$$h_1(x,\phi) \equiv \frac{1-x^{1-2\phi}}{1-2\phi}, \quad h_2(x,\phi) \equiv \frac{x^{-2\phi}-1}{2\phi} - \frac{2}{9}, \tag{3.42}$$

and, as before, we have the relation

$$\tilde{f}(x,\phi) \equiv x^{-n-2}/\tilde{g}(x,\phi).$$

The limits $x \to \infty$ and $\phi \to 0$ do *not* commute (a feature that is common to other limits involving x). Parallel to the discussion of Ref. [9], we take the limit $\phi \to 0$ first. It is straightforward to show that for large x but ζx still small one obtains

$$1/\mathcal{C} = \mathcal{D} + \lambda x_f^{-n-1+9\zeta}/(n+1-9\zeta). \tag{3.43}$$

By matching the equilibrium region to the freeze-out region we obtain

$$\mathcal{D} = \lambda x_f^{-n-2+9\zeta}. \tag{3.44}$$

These formulas are similar to the case of dilaton cosmology in the absence of space-time foam with the crucial difference that ϕ is now replaced by -9ζ. Finally, we obtain

$$Y(x) \sim \frac{x^{9\zeta}}{\mathcal{C}^{-1} - \lambda x^{-n-1+9\zeta}/(n+1-9\zeta)}, \tag{3.45}$$

which indicates the role of D foam as a source of particle production in this case, in the sense that Y increases as x increases. Also, in this case $x_f = x_{f,ns}$. Notice that the behavior (3.44), which indicates an increase of the DM thermal relic abundance with decreasing temperature, is compatible with our earlier numerical investigations in Ref. [9]. For $x \gg x_f$ (as in the current universe) the abundance (3.45) can be approximated by

$$Y(x) \sim \lambda^{-1} x_f^{n+2}(x/x_f)^{9\zeta} \quad (x \gg x_f), \tag{3.46}$$

which we use in Sec. IV to discuss the phenomenology of these models.

IV. PHENOMENOLOGICAL IMPLICATIONS

As mentioned earlier, the phenomenologically relevant quantity that can be compared directly with experiments is the Hubble-constant-free relic abundance, $\Omega h^2 = m\mathcal{N}/\rho_0^c$, where ρ_0^c is the current critical density and $\mathcal{N}$ is the number density of the DM species. For DM species X with mass m_X this quantity is given by (3.15) [3]. The behavior of Ω_X is then readily obtained for all cases studied in this work.

The analysis in the previous sections indicates that time-dependent sources in our cosmological models lead to modified relic abundances for DM species, as compared to those computed within the standard cosmology. This modification can be quantified by considering the ratio (3.16) in which the numerator and denominator may involve different freeze-out temperatures. Since both expressions are known theoretically, the ratio (3.16) is computable explicitly for all cases studied above.

Before proceeding with the phenomenology of the various sources discussed in this article, we make some generic remarks. If the sources are such that there is *dilution* of DM relic abundance *relative* to the prediction of standard cosmology, this can have important phenomenological implications for new physics, such as supersymmetry (SUSY) at colliders [5,6,8]. In such a case there is a *larger* portion of the available parameter space of the SUSY model, which is compatible with the WMAP and other cosmological/astrophysical data [7].

More room for supersymmetry implies heavier partners, which in turn may have interesting signatures at colliders, such as the LHC. If the relic density of the neutralino $\tilde{\chi}_1^0$, which is the dominant DM in SUGRA-like models, is diluted by a factor of about $1/10$ in the presence of sources, then the final states expected at the LHC consist of Z bosons, Higgs bosons, and τ leptons. Such states are produced when one looks at the decay chains of the dominant SUSY production mechanism of squark $\tilde{q}$ and gluino $\tilde{g}$ pairs at the LHC:

$$\tilde{q}\to q\tilde{\chi}_2^0\to q\tau\tilde{\tau}_1\to q\tau\tau\tilde{\chi}_1^0, \quad \tilde{\chi}_2^0\to h^0\tilde{\chi}_1^0, \quad \tilde{\chi}_2^0\to Z\tilde{\chi}_1^0,$$

where $\tilde{\chi}_2^0$ is the next-to-lightest neutralino, and h^0 is the Higgs particle. In Ref. [8] a detailed analysis in the standard parameter space $m_{1/2}$, m_0 (where $m_{1/2}$ and m_0 are the gaugino and scalar masses) of mSUGRA models has been performed. In this analysis the parametric regions for the dominant decay patterns at the LHC:

(1) Higgs + jets + missing transverse energy,
(2) Z + jets + missing transverse energy,
(3) 2τ + jets + missing transverse energy,

have been predicted. Dilution factors of about $1/100$ or more are compatible with the analysis in this paper for reasonable values of the parameters. Such dilutions may even push the parameter spaces of minimal supersymmetric models beyond the reach of the LHC (assuming standard-model-like Higgs particle masses of about 125 GeV). For instance, in the constrained minimal supersymmetric standard model (CMSSM) with Higgs-mass range 123–128 GeV and $\tan\beta$ of about 50, Lahanas and Spanos discussed the dilaton-induced dilution factor [6]. On including the effect of the dilution factor, they showed that the constraints placed on the parameter space of CMSSM, from the current ATLAS and CMS SUSY searches for DM, were not sufficient to exclude the model.

We proceed to discuss the phenomenology of the cases discussed above by giving the corresponding values of the ratio (3.16) today. We assume that the freeze-out points $x_{f,ns}$ in the absence of sources are about 30, as expected in typical phenomenological models in which the DM is identified as a supersymmetric partner, such as a neutralino.

The temperature of the universe, which is used in the definition of x today x_0, is that of the cosmic microwave background (CMB) temperature $T_{\rm CMB}\approx 2.35\times 10^{-13}$ GeV. Thus, for DM masses in the range $m_X\approx$ 300 GeV–1 TeV, we have

$$x_0\equiv m_X/T_{\rm CMB}\approx 10^{15}\text{–}10^{16}. \tag{4.1}$$

Moreover, we assume that the source-free relic abundance $Y_{ns}(x)$ currently, which approaches a constant as $x\to\infty$ [3], as the boundary-layer analysis of Ref. [1] confirms, is given by

$$\lim_{x\to\infty} Y_{ns}\approx \frac{(n+1)x_{f,ns}^{n+2}}{\lambda(n+1+x_{f,ns})}. \tag{4.2}$$

Recall that the freeze-out point in the source-free case $x_{f,ns}$ indicates a range of values of x in the vicinity of (2.10) with $A\approx 0.000145$ [1]. For all but the case $0<\zeta_\phi<\phi\ll 1/2$ the freeze-out point $x_f=x_{f,ns}$. However, as is evident from (3.34), even in the case $0<\zeta_\phi<\phi\ll 1/2$, the freeze-out point is shifted by an amount less than $9/2$: $x_f^{\phi\ll 1/2}\sim x_{f,ns}-\frac{9\zeta_\phi}{2\phi}$. In the models we consider here $x_{f,ns}\approx 30$, so such a shift is not significant. Thus, from now on we treat $x_f\approx x_{f,ns}$ in all cases. This simplifies the arguments and allows an easy estimate of the ratio $\mathcal{R}$ in (3.16).

As a starting point, we take the case of a time-dependent dilaton source of the form (2.3) in the absence of D foam; that is, $\zeta_\phi=0$. This case was discussed in Refs. [5,6] and was revisited in Ref. [1] using asymptotic matching techniques. From (2.23) and (4.2) the ratio (3.16) becomes (upon setting $x_f\sim x_{f,ns}$)

$$\mathcal{R}^{\rm dilaton}(x=x_0)\sim\frac{n+1+\phi}{n+1}\frac{n+1+x_{f,sn}}{n+1+\phi+x_{f,ns}}(x_{f,ns}/x_0)^{\phi} \tag{4.3}$$

with x_0 given in (4.1).

From (3.35) we then notice that (4.3) also applies to the case of nontrivial D foam but with $0<\zeta_\phi<\phi\ll 1/2$. For $x_{f,sn}$ about 30 and for s-wave scattering ($n=0$) the approximate thermal DM relic dilution factor (4.3) is determined by $(x_{f,ns}/x_0)^\phi\approx 10^{-16\phi}$ for DM masses m_x in the range 0.3–1 TeV. Thus, to obtain a dilution factor of order $1/10$, which is relevant for LHC phenomenology, we need values of ϕ near $1/16$, which is small compared with $1/2$ and which is consistent. However, the case of phenomenologically significant dilution requires that $\zeta_\phi x_f\ll 1$ and thus $\zeta_\phi\ll\phi$. For the pure dilation case, in the absence of D foam, one may have larger values of ϕ that lead to acceptable phenomenology; for instance, a dilution of about 10^{-2} can be obtained with $\phi\approx 1/8$.

On the other hand, in the case where the space-time defect (D-foam defect) dominates the time-dependent dilaton effect, that is, when the strength of the foam fluctuations is such that $\zeta\gg\phi\to 0$, we have an *enhancement* of the DM relic abundances rather than a dilution as the temperature decreases. This becomes clear from (3.45) and (3.46). In such cases there is *less room* for supersymmetry available in the relevant parameter space as compared with the source-free case after cosmological (WMAP) constraints [7] are taken into account.

The enhancement factor scales like

$$\mathcal{R}\,(x=x_0)\sim\frac{n+1+x_f}{n+1}(x_0/x_f)^{9\zeta}. \tag{4.4}$$

For s-wave scattering and with x_0 given by (4.1) this implies that $\mathcal{R}\sim(10)^{(136-154)\zeta}$. Such models lead to more severe constraints on the available supersymmetry parameter space if the enhancement is observable.

Therefore, for these models to be phenomenologically viable today, this requires $\mathcal{R}$ to be $\mathcal{O}(1)$ within experimental error, so that the increase compared to the source-free (standard) case is not appreciable. This requires that $\zeta < 10^{-3}$, so that the error in calculating abundances would match the per mil level of the current errors in experimental astrophysical measurements [7]. Because of (3.10), this implies that

$$\zeta = 2x_0(g_0 m_X/M_s)^2\langle\Delta^2\rangle_0 < 10^{-3}. \tag{4.5}$$

For the range of x_0 in (4.1) this is satisfied for heavy D particles of masses $M_s/g_0 \geq (10^{11}–10^{13})/\langle\Delta^2\rangle_0$ GeV, and m_X in the range 0.3–1 TeV and $\langle\Delta^2\rangle_0 \leq 1$.

Next, we discuss the case of a time-dependent dilaton with $\phi \gg 1/2$ in (3.40). Since we always assume weak foam fluctuations, $\zeta_\phi < 1$, this case is also dilaton dominated, and hence we expect a significant dilution of the relic abundance. Indeed, because of (3.40) and (4.2), in this case we obtain the following expression in the limit $x \to x_0 \gg x_f = x_{f,ns}$ for the ratio $\mathcal{R}$ in Eq. (3.16):

$$\mathcal{R} \sim \exp\left(-\frac{2\zeta_\phi}{2\phi-1}\right)\left(\frac{n+1+\phi}{n+1}\right)\left(\frac{n+1+x_{f,ns}}{x_{f,ns}}\right) \times (x_f/x_{f,ns})^{n+1}(x_f/x)^{\phi}, \tag{4.6}$$

where we assume that the freeze-out points $x_f \sim x_{f,ns}$ are about 30 or more. Thus, we have significant dilution of the DM relic densities at late epochs of the universe. For instance, in the present era and for DM masses in the range $m_X \approx 300$ GeV, the ratio is $\mathcal{R} \approx \exp(-\frac{2\zeta_\phi}{2\phi-1}) \times (\approx 200) \times 10^{-15\phi}$ for s- or p-wave scattering ($n = 1, 2$). Thus, we see that for $\zeta \ll 1$, which is natural in the case of D foam with D particles whose masses are higher than a tera-electron volt [9], the main factor that drives the dilution is the value of the dilaton parameter ϕ.

In the case $\phi \approx 5$, for instance, the dilution factor is already enormous (it is of order 10^{-75}), so in such models practically all DM today will have disappeared. This may rule these models out phenomenologically, although the situation with DM and its nature is currently unclear, as there is no concrete evidence for it apart from the galactic motion. For this reason alternative theories with no dark matter but modified gravity at galactic scales have been considered extensively in the literature. We do not consider them here, since in our opinion the evidence against them, especially from galactic lensing measurements, is significant. Thus, all we can say is that this type of supersymmetric DM (satisfying $x_f \approx 30$) would be diluted in this model, and it would be practically absent today. Other types of DM that would not couple to the dilaton might survive.

Next we discuss the cases for which ϕ is near 1/2. Now, using (3.28) and (3.29), we obtain for $x \to x_0 \gg x_f \sim x_{f,ns}$:

$$\mathcal{R} \sim \frac{n+3/2-2\zeta_{1/2-\delta}-\delta}{n+3/2-2\zeta_{1/2-\delta}+x_{f,ns}}\,\frac{n+1+x_{f,ns}}{n+1} \times (x_{f,ns}/x)^{1/2-2\zeta_{1/2-\delta}-\delta}. \tag{4.7}$$

We observe that for $\delta \to 0$ and $\zeta_{1/2} < \phi = 1/2$, the main dilution comes from the dilaton effects and scales with x as $(x_{f,ns}/x)^{1/2-\zeta_{1/2}}$. Thus the dilution due to the dilaton is compensated by foam fluctuation effects, so for $\zeta_{1/2} = \phi/2 = 1/4 < \phi = 1/2$ there is no appreciable dilaton-driven dilution, and the ratio (4.7) tends to one ($R_{\zeta_{1/2}=1/4} \approx 1$) for any $n > 0$.

We observe from (3.10) that $\zeta_{1/2}$ is independent of x_0:

$$\zeta_{1/2} = 2g_0^2\langle\Delta^2\rangle_0 m_X^2/M_s^2 \tag{4.8}$$

and the condition $\zeta = 1/4$ implies that $\langle\Delta^2\rangle_0 = M_s^2/(8g_0^2 m_X^2) < 1$, where the inequality on the right side ensures naturalness in the fluctuations of a weak foam, which we have assumed throughout. The latter condition necessitates $m_X \approx M_s/g_s$. We stress that in this case the result for the relic abundance today turns out to be equal to the standard source-free case independent of the actual freeze-out point, and hence in principle m_X is only constrained to be of the same order of magnitude as the D-particle mass M_s/g_0.

Finally, we mention that one may consider a $\delta \geq 1/8$ to produce dilution of order $\mathcal{R} \leq 10^{-2}$ in the relic abundance (4.7), thereby opening the possibility of pushing this class of supersymmetric models out of the reach of the LHC, according to the analysis in Ref. [6]. However, in this case, (3.10) implies that the condition $1/4 = \zeta_\delta \sim 2g_0^2 x_0^{-\delta}\langle\Delta^2\rangle_0 m_X^2/M_s^2$ for x_0 in the region (4.1) can be satisfied for $g_0^2 m_X^2/M_s^2 \sim (1/8)10^{15\delta}\langle\Delta^2\rangle_0$. To ensure that $m_X \leq M_s/g_0$ this would imply naturally small fluctuations in the foam $\langle\Delta^2\rangle_0 \sim 10^{-15\delta}$ with $\delta \geq 1/8$.

The above predictions are quite generic and hence they are largely independent of the details of the underlying microscopic model. Nevertheless, the cosmology of the models, in particular the precise dependence of the dilaton on the cosmic time at various eras of the universe, is an open issue. The lack of detailed microscopic models that would determine the form of the dilaton potential and provide rigorous information on the region of validity of the dilaton cosmological solution (2.3) complicates matters. Nevertheless, one may perform phenomenological searches on the compatibility of such solutions at various epochs of the universe. For the DM searches mentioned above, all one needs is the dominance of the time-dependent dilaton at early epochs of the universe before the big-bang nucleosynthesis. Nevertheless, a dilaton of the form (2.3) can be compatible (notably at the same level as the ΛCDM model) with cosmological data even at low redshifts of order $z = O(1)$, where large scale structure in the universe (galaxies and clusters of them) is formed, as demonstrated recently in Ref. [12]. On the other hand, D-foam dominance at late eras (such as the end of radiation or matter-dominated era [13]) has been argued to play a role in galactic growth itself. Thus, considering models with combined dilaton and D-foam sources, as in the current article, may be desirable from the point of view of constructing realistic cosmologies in such frameworks.

However, the rate of galactic growth is in principle capable of discriminating the various models (2.3) corresponding to different values of ϕ when more data become available in the near future. In all such theories, of course, an important requirement is that the big-bang-nucleosynthesis conditions at MeV temperatures are not disturbed.

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APPENDIX: THERMODYNAMIC PROPERTIES OF A UNIVERSE IN THE PRESENCE OF SOURCES

The purpose of this Appendix is to demonstrate that it is possible to define an appropriate entropy density (scaling with temperature as T^3), even in the presence of nontrivial backgrounds, such as a time-dependent dilaton and/or space-time foam. This allows the entropy density to be used in this paper as a fiducial quantity in the definition of the thermal relic abundance $Y(x)$.

In the presence of such nontrivial backgrounds the continuity equations of cosmic fluids corresponding to matter and radiation are modified relative to standard FRW cosmology. These modifications could affect the thermodynamic properties of the universe, such as the relation between the scale factor and the temperature T (the cooling law). It is the relativistic degrees of freedom that dominate the entropy and the cooling law. In the case of a FRW universe the continuity equation is the conservation of the stress-energy tensor $\nabla^\mu T_{\mu\nu} = 0$, which is compatible with Einstein's equations of general relativity and admits a thermodynamic interpretation. This equation can be manipulated to appear as the first law of thermodynamics for the total internal energy ρV in a co-moving volume $V \sim a^3$ and pressure p:

$$d(\rho V) + p dV = 0, \qquad V \sim a^3. \tag{A1}$$

This interpretation in terms of the first law is consistent with an *adiabatic* expansion at temperature T. A constant entropy function $S(T, V)$, which is analytic in T and V, can be constructed:

$$S = V \frac{\rho + p}{T}. \tag{A2}$$

The construction involves the application of the thermodynamic Maxwell relations to cast the right side of (A1) into the form TdS; that is,

$$0 = TdS = d(\rho V) + p dV. \tag{A3}$$

[Strictly speaking, (A3) should be modified by a term involving the chemical potential μ. Hence, TdS should be replaced by $TdS + \mu dN$. However, as is standard in cosmology, μ is ignored because μ/T is much smaller than one [3], which is consistent with the dominance of the relativistic degrees of freedom in the entropy.] The dominance of relativistic degrees of freedom in the entropy S is therefore consistent with the constancy of S and the cooling law $a \sim 1/T$. (Recall that $\rho = 3p \sim T^4$ for radiation.) From (A2) and (A3) it is then straightforward to see that the entropy density $s \equiv S/V$ scales with temperature T as T^3. For cosmologies with nontrivial time-dependent dilaton and/or space-time D-particle foam backgrounds we also construct entropy functions that are constant during the evolution of the universe.

(i). Dilaton cosmology

In the context of a FRW cosmology we review here the modification of the Boltzmann equation for dark matter relic abundances [5] in the presence of a rolling dilaton. In dilaton cosmologies inspired by string theory, the starting point is the existence of two kinds of target-space metric, related by an appropriate field redefinition and some coordinate transformations. The first is the so-called string-frame metric, $g^\sigma_{\mu\nu}$, which is the target-space metric that appears in the stringy world-sheet conformal field theory (σ model) describing the propagation of a string in a gravitational background. The conditions for world-sheet conformal invariance are equivalent to the equations of motion derived from a four-dimensional target-space effective action (after appropriate compactifications); the gravitational part of the effective action is a power series in space-time derivatives. The lowest-nontrivial order is quadratic (in derivatives) and is given by a Brans-Dicke scalar curvature term

$$S^\sigma = \frac{1}{16\pi G_N} \int d^4x \sqrt{-g^\sigma} e^{-2\Phi} R(g^\sigma) + \cdots, \tag{A4}$$

where Φ is the dilaton field, the superscript σ denotes quantities evaluated in the σ-model frame, G_N is the four-dimensional gravitational (Newton) constant, and $\cdots$ indicates the presence of kinetic terms for the dilaton and other terms.

We can avoid the noncanonical normalization of the Einstein-Hilbert term in (A4) by redefining the metric tensor in order to pass to the so-called Einstein frame so that the action (A4) acquires the canonical Einstein-Hilbert form. In this frame the coefficient in front of the scalar curvature tensor is independent of the dilaton-scale factor and is chosen to be $1/(16\pi G_N)$. The passage from the string frame to the Einstein frame also involves a transformation of the time coordinate [14]. To be explicit, we have

$$g_{\mu\nu} = e^{-2\Phi} g^\sigma_{\mu\nu}, \qquad \frac{\partial t}{\partial t^\sigma} = e^{-\Phi}. \tag{A5}$$

It is the Einstein frame that is relevant for a cosmological observer. In this frame G_N is truly constant in space time, in accord with current observations. (On the other hand, in the string frame, the effective gravitational "constant" $e^{2\Phi}G_N$ would depend on the dilaton field, which in general could be space-time dependent.)

Consider the phase space density of a DM species X, which is assumed to be coupled to the metric and rolling dilaton background terms:

$$f(|\vec{p}|, t, \Phi(t), g_{\mu\nu}(t)). \tag{A6}$$

It is important to stress that the phase-space distribution function (A6) is evaluated in the string frame, although its arguments are expressed in terms of quantities in the "physical" Einstein frame [5]. This is because in the context of the underlying string theory model, the string frame is the frame where matter excitations, represented as strings, are defined. The Einstein metric is assumed to be of FRW type, with $g_{00} = -1$, $g_{ij} = a(t)^2\delta_{ij}$, where $a(t)$ is the universe scale factor.

For a generic DM particle of mass m in the comoving frame we have $(p^\sigma)^\mu = m dx^\mu/d\tau = m(dx^\mu/dt)\times(dt/dt_\sigma) = m p^\mu e^{-\Phi}$. It can then be readily seen that

$$|\vec{p}^\sigma| \equiv (p^{i,\sigma}p^{j,\sigma}g^\sigma_{ij})^{1/2} = (p^i p^j g_{ij})^{1/2} \equiv |\vec{p}| = a(t)\left(\sum_{i=1}^{3} p^i p^i\right)^{1/2}. \tag{A7}$$

We make use of the above relations in what follows.

Below, we are interested in studying the action of the relativistic Liouville operator $\hat{L}$ on the phase-space density (A6). We commence our analysis by recalling the relativistic form of the Liouville operator in conventional general relativity:

$$\hat{L}[f] = \left(p^\alpha \frac{\partial}{\partial x^\alpha} + \Gamma^\alpha_{\beta\sigma}p^\beta p^\sigma \frac{\partial}{\partial p^\alpha}\right)f. \tag{A8}$$

The second term on the right side of the equation denotes the relativistic force, which follows from the geodesic equation. For a FRW universe, only the time-energy part survives from the first term; that is,

$$p^\alpha \frac{\partial}{\partial x^\alpha} = E\frac{\partial}{\partial t}, \tag{A9}$$

where E is the energy of the DM species in the Einstein frame. Moreover, the connection parts receive nontrivial contributions only from the terms $\sum_i \Gamma^0_{ii} p^i p^i \frac{\partial}{\partial E}$, $i = 1, 2, 3$, a spatial index (assuming for concreteness an already compactified string theory, or a theory on a three-brane world), with $\Gamma^0_{ii} = -a\dot{a}$, where the overdot denotes derivative with respect to the cosmic FRW time t, identified in our string theory with the Einstein-frame time (A5).

The dependence of (A6) on the time-dependent dilaton source implies that there will be corrections to the conventional Liouville operator associated with the action of the time derivatives on the phase-space density. These are due to the implicit time dependence of the dilaton background source. Schematically, we can denote the action of the full operator on f (A6) as $(\hat{L}_{\rm conv} + \hat{L}_{\rm dil})f$. Here, $\hat{L}_{\rm conv}$ is the conventional (dilaton-independent) Liouville operator, which in a FRW universe reads [3]

$$\hat{L}_{\rm conv} = E\frac{\partial}{\partial t} - a\dot{a}\sum_{i=1}^{3} p^i p^i \frac{\partial}{\partial E} = E\frac{\partial}{\partial t} - \frac{\dot{a}}{a}\sum_{i=1}^{3}|\vec{p}|^2\frac{\partial}{\partial E}. \tag{A10}$$

Also, $\hat{L}_{\rm dil}$ denotes the dilaton-source-induced corrections:

$$\hat{L}_{\rm dil} = E\dot{\Phi}\frac{\partial}{\partial\Phi}.$$

Therefore, the Boltzmann equation in the presence of a rolling dilaton background reads [5]

$$(\hat{L}_{\rm conv} + \hat{L}_{\rm dil})f = C[f] \Rightarrow \frac{\partial f}{\partial t} = \frac{\dot{a}}{a}\frac{|\vec{p}|^2}{E}\frac{\partial f}{\partial E} - \dot{\Phi}\frac{\partial f}{\partial \Phi} + \frac{1}{E}C[f], \tag{A11}$$

where $C[f]$ is the Boltzman collision term. This equation takes into account the implicit dependence of $f(|\vec{p}|, t, \Phi)$ on g_{ii} through $|\vec{p}| = (\sum_{i=1}^3 p^i p^i g_{ii})^{1/2}$, which implies that $\partial f/\partial g_{ii} = (\partial|\vec{p}|/\partial g_{ii})\partial f/\partial|\vec{p}| = \frac{p_i p_i}{2|\vec{p}|}\frac{\partial f}{\partial|\vec{p}|}$.

Upon considering the action of the above operator on the number density of a given DM species X, $n \equiv \int d^3p f$, we arrive, after some straightforward momentum integration by parts, at the *modified* Boltzmann equation for a four-dimensional effective field theory (after string compactification or restriction on three-brane worlds) in the presence of time-dependent dilaton source terms:

$$\frac{dn}{dt} = \frac{\dot{a}}{a}\int d^3p \frac{|\vec{p}|^2}{E}\frac{\partial f}{\partial E} - \dot{\Phi}\int d^3p\frac{\partial f}{\partial\Phi} + \int d^3p\frac{C[f]}{E}$$
$$\Rightarrow \frac{dn}{dt} + 3\frac{\dot{a}}{a}n = -\dot{\Phi}\int d^3p\frac{\partial f}{\partial\Phi} + \int\frac{d^3p}{E}C[f]. \tag{A12}$$

There are two types of dependence of f on Φ: (i) *Explicit* dependence of the form $e^{-4\Phi}$, which arises because in our approach the phase space density is constructed as a quantity in the string frame, which is then expressed in terms of quantities in the Einstein frame. As such, it is by definition (as a density) *inversely* proportional to the proper string-frame volume

$$V^\sigma = \int d^4x\sqrt{-g^\sigma} \propto e^{4\Phi},$$

because of (A5). (ii) *Implicit* dependence corresponding to a dependence on Φ through the Einstein-frame metric g_{ii} (A5).

Hence, the general structure of f has the form

$$f(\Phi, \vec{p}, \vec{x}, g^\sigma_{\mu\nu} = e^{2\Phi}g_{\mu\nu}; t) \propto e^{-4\Phi}\mathcal{F}(|\vec{p}|, \vec{x}, t). \tag{A13}$$

This implies that

$$\int d^3p \frac{\partial f}{\partial \Phi} = -4\int d^3p f + \sum_{i=1}^{3}\int d^3p \frac{\partial g_{ii}}{\partial \Phi}\frac{\partial f}{\partial g_{ii}}$$
$$= -4n - 2\int d^3p \sum_{i=1}^{3} g_{ii}\frac{\partial|\vec{p}|}{\partial g_{ii}}\frac{\partial f}{\partial|\vec{p}|}$$
$$= -4n - \int d^3p|\vec{p}|\frac{\partial f}{\partial|\vec{p}|}$$
$$= -4n + 3\int d^3p f(|\vec{p}|, t) = -n, \qquad \text{(A14)}$$

where in the last step we have performed appropriate partial (momentum-space) integrations.

The final form of the Liouville operation (A12) is then [5]

$$\frac{dn}{dt} + 3\left(\frac{\dot{a}}{a}\right)n - \dot{\Phi}n = \int \frac{d^3p}{E} C[f]. \qquad \text{(A15)}$$

We next remark that for a FRW cosmology in four space-time dimensions the presence of a rolling dilaton leads to a modification of the continuity equation for the total energy density ρ and pressure p [4–6,15]:

$$\dot{\rho} + 3H(\rho + p) - \dot{\Phi}(\rho - 3p) = 0, \qquad \text{(A16)}$$

where the dot denotes derivative with respect to the cosmic time t. In fact, this equation follows from the Einstein equations of motion for the graviton field in a cosmology with a rolling dilaton in the Einstein frame:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G_N T^m_{\mu\nu} + T^{\Phi}_{\mu\nu}, \qquad \text{(A17)}$$

where the term $T^m_{\mu\nu} = \frac{2}{\sqrt{-g}}(\delta\mathcal{L}^{\rm matter}\sqrt{-g})/\delta g_{\mu\nu}$ is associated with matter/radiation degrees of freedom, and $T^{\Phi}_{\mu\nu} = \frac{2}{\sqrt{-g}}\delta(\mathcal{L}^{\Phi}\sqrt{-g})/\delta g_{\mu\nu}$ denotes the contributions from the dilaton dependent (kinetic and other possible potential/dark energy) terms of the string effective action.

The covariant conservation law in the Einstein term for the total stress tensor $T^{m\mu\nu}{}_{;\nu} + T^{\Phi\mu\nu}{}_{;\nu} = 0$, which follows from (A17) because of the properties of the Riemann tensor in the Einstein frame, implies an energy flow between the matter and dilaton parts. This leads to the modified continuity equation (A16). In fact, the latter is derived by covariant derivation and further manipulations of the component forms of Eq. (A17) [5,15]:

$$3H^2 - \tilde{\varrho}_m - \varrho_\Phi = 0, \quad 2\dot{H} + \tilde{\varrho}_m + \varrho_\Phi + \tilde{p}_m + p_\Phi = 0, \qquad \text{(A18)}$$

where $\tilde{\varrho}_m = \frac{8\pi G_N}{\omega}\rho_m$ (and $\tilde{p}_m = \frac{8\pi G_N}{\omega}p_m$) denotes the matter energy density (and pressure) including dark matter contributions and ω is a dimensional constant having units of inverse time, as specified below. ϱ_Φ (and p_Φ) are the corresponding quantities for the dilaton dark-energy fluid with [5,15] $\rho_\Phi = \dot{\Phi}^2 + \hat{V}_{\rm all}/2$, $p_\Phi = \dot{\Phi}^2 - \hat{V}_{\rm all}/2$. Here, we assume for the sake of generality a dilaton potential $\hat{V}_{\rm all}$, which in the case of string theory may come from quantum string loops, breaking the scale invariance of target-space theory. The overdots in these equations denote derivatives with respect to the Einstein time t, which is proportional to the Robertson-Walker cosmic time $t_{\rm RW}$, $t = \omega t_{\rm RW}$ so that the Einstein time t is dimensionless. Without loss of generality we take [5,15] $\omega = \sqrt{3}H_0$, where H_0 is the present day Hubble constant. With this choice for ω the densities appearing in (A18) are in units of the critical density.

In addition to the equations of motion for the graviton, one also has an equation for the dilaton field Φ obtained by variation of the effective action with respect to Φ:

$$\ddot{\Phi} + 3H\dot{\Phi} + \frac{1}{4}\frac{\partial\hat{V}_{\rm all}}{\partial\Phi} + \frac{1}{2}(\tilde{\varrho}_m - 3\tilde{p}_m) = 0. \qquad \text{(A19)}$$

We now note that

(i) From (A16) the dilaton source terms do not play a role for radiation. One obtains the standard scaling of $\rho \sim a^{-4}$ in the radiation dominated era of the universe.

(ii) For dust, $p = 0$. Also, for DM with mass m_X, $\rho_X = m_X n_X$, where n_X is the number density. The source-independent part of (A16) yields the collisionless Boltzmann equation for thermal relic abundance. The dilaton-dependent term is a classical source term $\dot{\Phi}n$.

This is consistent with the fact that the Boltzmann equation (A12) is compatible with the conservation equation (A16) as well as the (modified) Einstein equations (A18).

The nonlinear part of the Boltzmann equation comes from two-body annihilations of DM particles. On assuming the functional dependence $\rho = \rho(a)$ and a dilaton source of the form (2.3), we obtain from (A16)

$$d(\rho V) + pdV - |\Phi_0|(\rho - 3p)dV/3 = 0. \qquad \text{(A20)}$$

Here, we have assumed that $\dot{\Phi} = -|\Phi_0|H$, where $H = \dot{a}/a$ is the Hubble parameter, $V \sim a^3$ is the comoving volume, and ρa^3 is the total (internal) energy in that volume.

We thus observe that the presence of a rolling dilaton affects the standard thermodynamic properties of the FRW universe. Our aim is to ascertain whether the total entropy in the comoving volume V remains constant in time after the inclusion of the dilaton source (2.3). A naive application of the first law of thermodynamics would identify $d(\rho V) + pdV$ with TdS, where T is the temperature, and S is the total entropy in the volume V. It would seem that a dilaton source leads to the nonconservation of entropy. However, this is incorrect. To show this, we first replace the zero of the right side of (A20) by $Td\mathcal{S}$, where $\mathcal{S}$ is the quantity that represents the entropy; the entropy is assumed to depend on T and V, so $\mathcal{S} = \mathcal{S}(T, V)$. From (A20) we find that

$$d\mathcal{S} = \frac{1}{T}d(\rho V) + \frac{p}{T}dV - \frac{|\Phi_0|}{3T}(\rho - 3p)dV$$
$$= V\frac{d\rho}{dT}dT + \frac{1}{T}[(1 + |\Phi_0|)p + (1 - |\Phi_0|/3)\rho]dV. \qquad \text{(A21)}$$

As in the case of conventional cosmology, it has been assumed that the cooling law of (3.4) holds, and that $\rho = \rho(T)$ and $p = p(T)$, since both ρ and p depend upon the scale factor a, which is a function of temperature, $a = a(T)$. The function $\mathcal{S}(T, V)$ is assumed to be a differentiable function of T and V. This implies the condition

$$\frac{\partial^2 \mathcal{S}}{\partial T \partial V} = \frac{\partial^2 \mathcal{S}}{\partial V \partial T}. \tag{A22}$$

From (A21) and (A22) we then obtain

$$\frac{1}{T^2}[(1 + |\Phi_0|)p + (1 - |\Phi_0|)/3)\rho] = \frac{(1 + |\Phi_0|)}{T}\frac{dp}{dT} - \frac{|\Phi_0|}{3T}\frac{d\rho}{dT}. \tag{A23}$$

The expression (A21) for $d\mathcal{S}$ can be rewritten as

$$\begin{aligned} d\mathcal{S} &= \frac{1}{T}d[(\rho + p)V] - \frac{V}{T}\frac{dp}{dT}dT - \frac{|\Phi_0|}{3T}d[(\rho - 3p)V] + \frac{|\Phi_0|V}{3T}\frac{d\rho}{dT}dT - \frac{|\Phi_0|V}{T}\frac{dp}{dT}dT \\ &= d\left[\frac{1}{T}\left(\rho\left[1 - \frac{|\Phi_0|}{3}\right] + p[1 + |\Phi_0|]\right)V\right] + \frac{V}{T^2}\left[(1 + |\Phi_0|)p + \left(1 - \frac{|\Phi|}{3}\right)\rho\right]dT - \frac{(1 + |\Phi_0|)V}{T}\frac{dp}{dT}dT + \frac{|\Phi_0|V}{3T}\frac{d\rho}{dT}dT \\ &= d\left[\frac{1}{T}\left(\rho\left[1 - \frac{|\Phi_0|}{3}\right] + p[1 + |\Phi_0|]\right)V\right]. \end{aligned} \tag{A24}$$

In the last equality on the right side we have used (A23). From (A24) we conclude that the quantity

$$\mathcal{S}(T, V) \equiv [\rho(1 - |\Phi_0|/3) + p(1 + |\Phi_0|)]V/T \tag{A25}$$

is constant upon using the classical equations of motion [or equivalently, the continuity equation (A16) for the case of dilaton cosmology (2.3)]. $\mathcal{S}$ may be identified with the total entropy in the comoving volume V. The corresponding entropy density s is then

$$s = \frac{1}{T}[\rho(1 - |\Phi_0|/3) + p(1 + |\Phi_0|)], \tag{A26}$$

which, in view of (A25), scales with the size of the universe as $a^{-3} = (T/C_0)^3$, upon assuming (3.4). We stress that the energy density ρ and pressure p in the above formulas pertain to the *total* degrees of freedom of the fluid including the relativistic ones. It is the latter, for which the dilaton source effects are *irrelevant* [see (A16)], that provide the dominant contributions to the entropy; otherwise, the entropy would not remain constant. Indeed, in the case of DM dust, $p = 0$, the entropy density is $s = \rho(1 - |\Phi_0|/3)/T$, which does not leave the entropy function (A25) constant. This is satisfied only for relativistic degrees of freedom that have an energy density scaling like $\rho \sim T^4$ with the temperature T.

We have the following relation between Y and the number density n_X of the DM species X:

$$Y = n_X T^{-3} \longrightarrow n_X = m_X^3 Y x^{-3}, \tag{A27}$$

as in the standard cosmology case. The energy density ρ_X of the DM relic satisfies $\rho_X = m_X n_X$. The current relic abundance,

$$\Omega_X h^2 \sim \frac{m_X^4}{\rho_0^c}\frac{Y_0}{x_0^3}, \tag{A28}$$

occurs for $x = x_0 = m_X/T_0$, with T_0 the current (CMB) temperature of the universe and ρ_0^c the current critical density. Since the latter is proportional to h^2, the above expression is independent of the value of the Hubble constant.

In practice, $x_0 \gg 1$. Hence, the asymptotic regime $Y(x)$ with $x \longrightarrow \infty$ is relevant. In the current literature one usually replaces Y_0 by Y_∞; that is,

$$\Omega_X h^2 \sim \frac{m_X^4}{\rho_0^c}\frac{Y_\infty}{x_0^3}. \tag{A29}$$

For standard cosmology in (3.39) $\lim_{x\to\infty} Y(x) = \text{const}$. This constant value of the freeze-out is identified with the current relic abundance of the weakly interacting massive particle $\Omega_X h^2 \sim 1/\langle\sigma v\rangle$.

We remark that the scaled Hubble-constant-independent relic abundance of the DM species X behaves as

$$\Omega_X h^2 \sim m_X^4 x^{-3} Y_\infty \qquad (x \longrightarrow \infty).$$

Also, for the case of standard cosmology $Y_\infty = \text{const}$ [see (3.39)],

$$\Omega_X h^2 \sim x^{-3} \longrightarrow 0 \qquad (x \longrightarrow \infty). \tag{A30}$$

For dilaton cosmology [see (2.3)] one has a modified law

$$\Omega_X h^2 \sim x^{-3-|\Phi_0|} \longrightarrow 0 \qquad (x \longrightarrow \infty). \tag{A31}$$

Finally we remark that for the dilaton case $\Phi_0 > 0$, the string coupling would increase for large times, and the theory would become strongly coupled and thus intractable. Nonperturbative string corrections would need to be incorporated. Nevertheless, the formal solution for $Y(x)$ behaves asymptotically as $Y(x) \sim x^{|\Phi_0|}$. This would still imply an asymptotically vanishing relic abundance provided that $\Phi_0 < 3$ because we have $\Omega_X h^2 \sim x^{-3+|\Phi_0|} \longrightarrow 0$ as $x \longrightarrow \infty$.

(ii). Stochastic D-particle-foam cosmology

It is known that in the background of D-particle spacetime foam (for constant dilatons), the Boltzmann equation for the number density n_X of the DM species X assumes the form [9]

$$\frac{d}{dt}n_X + 3Hn_X = \Gamma_{D\text{-foam}}(t)n_X + C[n], \tag{A32}$$

where $\Gamma_{D\text{-foam}}(t) = 2Ha^4 m_X\langle\langle\Delta^2\rangle\rangle\frac{g_s^2}{M_s^2}T(9 + 2m_X/T)$. The notation and conventions here are those of Ref. [9], where $C[n] = \langle\sigma v\rangle[(n_X^{\rm eq})^2 - (n_X)^2]$ is the standard nonlinear interaction term and $n_X^{\rm eq}$ is the thermal equilibrium number density of X. As discussed in Ref. [9] and reviewed in the text [see (3.3)], the recoil fluctuations of the D-foam $\langle\langle\Delta^2\rangle\rangle$ averaged over populations of D-particle defects have the scaling $\langle\langle\Delta^2\rangle\rangle \sim \langle\Delta^2\rangle_0 a^{-3}$.

To this end, we use the cooling law (3.4) and ignore the nonlinear interaction term $C[n]$. From (A32) and (3.3), for the regime of temperatures $m_X \gg T$, the energy density $\rho_X = m_X n_X$ then satisfies the continuity equation

$$\frac{d}{dt}\rho_X + 3H\rho_X = \tilde{\Gamma}_{D\text{-foam}}(t)H\rho_X, \tag{A33}$$

where $\tilde{\Gamma}_{D\text{-foam}}(t) = 4C_0\frac{g_s^2}{M_s^2}\frac{m_X^2}{T}\langle\Delta^2\rangle_0$. For weak foam effects we have $\tilde{\Gamma}_{D\text{-foam}} < 1$ in the range of temperatures we are interested in; that is, from the early universe until today ($T \geq C_0$). Equivalently, for an expanding universe where $\dot{a} > 0$ we have

$$d(\rho_X V) - \rho\tilde{\Gamma}_{D\text{-foam}}dV/3 = 0. \tag{A34}$$

Equation (A34) implies that the thermodynamic interpretation of heavy DM dust in the foam background is that of a gas with an adiabatic expansion of its volume. During the expansion entropy is constant, and the effective pressure $p_{\text{eff}-X}$ of the gas is *negative* (indicating cosmological instabilities):

$$p_{\text{eff}-X} = -\rho\tilde{\Gamma}_{D\text{-foam}}/3. \tag{5.35}$$

Note that $p_{\text{eff}-X}$ has a nontrivial dependence on the temperature.

From the cooling law (3.4), we may write [see (A33)] $\tilde{\Gamma} \equiv \tilde{\gamma}a$, where $\tilde{\gamma}$ is a constant much less than one. Hence, the scaling of the dust energy density, due to its interaction with the D foam, is easily obtained from (A34) to be (in units of a_0)

$$\rho_X \sim a^{-3}e^{\tilde{\gamma}\int_1^a da} \sim T^3 e^{\tilde{\gamma}(C_0/T-1)}. \tag{A36}$$

To find the entropy function that remains constant it is essential to consider the total energy density ρ, including relativistic degrees of freedom, and not only ρ_X. In a similar spirit to the dilaton case the relativistic degrees of freedom are insensitive to the heavy D-foam source effects. In this sense they satisfy an equation of the form (A1) by themselves with equation of state $p = \rho/3$, which can be added to (A34) to give the equation

$$d((\rho^{\rm rad} + \rho_X)V) + (p^{\rm rad} + p_{\text{eff}-X})dV \equiv d(\rho V) + p_{\rm eff}dV = 0. \tag{A37}$$

Equation (A37) is the analog of the continuity equation in the case of D foam. We stress that in (A37) ρ and $p_{\rm eff}$ refer to the *total* energy density and pressure, including relativistic degrees of freedom and D-foam background effects.

Taking into account that ρ is a function of T, we can formally replace the right side of (A37) by $Td\mathcal{S}$ to determine the (constant) entropy function $\mathcal{S}$ (ignoring chemical potential terms, a valid assumption for weak D foam); only at the very end of the computation will we set $d\mathcal{S}$ to zero. We then have

$$d\mathcal{S} = \frac{V}{T}\frac{d\rho}{dT}dT + \frac{\rho + p_{\rm eff}}{T}dV. \tag{A38}$$

$\mathcal{S}$ is considered to be a smooth function of T and V, which are treated as independent variables. From the requirement (A22) we deduce the condition

$$-\frac{\rho + p_{\rm eff}}{T^2} + \frac{1}{T}\frac{dp_{\rm eff}}{dT} = 0. \tag{A39}$$

We then see immediately from (A38) and (A39) that

$$d\mathcal{S} = d\left[\frac{\rho + p_{\rm eff}}{T}V\right] - \frac{V}{T}\frac{dp_{\rm eff}}{dT}dT + \frac{\rho + p_{\rm eff}}{T^2}VdT = d\left[\frac{\rho + p_{\rm eff}}{T}V\right],$$

which upon setting $d\mathcal{S} = 0$ implies the constancy of the effective entropy function in the comoving volume V:

$$\mathcal{S} = S_{\rm eff} = \frac{\rho + p_{\rm eff}}{T}V = \text{const}. \tag{A40}$$

Note that we have used (A36) and the cooling law (3.4); ρ and $p_{\rm eff}$ refer to the total energy density and pressure including relativistic components, which is essential for consistency. As in the previous cases, the relativistic degrees of freedom dominate the entropy. The entropy density s associated with $\mathcal{S}$ is given by an expression similar in form to that in standard cosmology,

$$s_{\text{eff }D\text{-foam}} = \rho + p_{\rm eff}/T, \tag{A41}$$

and scales with the temperature as T^3. Hence, s can be treated as a fiducial quantity to define $Y(x)$ just as in the dilaton cosmology case (i) above.

Notice also that for the case of dust in dilaton cosmology, the effective entropy function (A26) is reproduced upon replacing the source $\tilde{\Gamma}_{D\text{-foam}}$ by the corresponding source of the running dilaton (2.3) cosmology $\tilde{\Gamma}_{\text{running dil}} = -|\Phi_0|$. (With our definitions we have $\Gamma_{\text{running dil}} = \dot{\Phi} = -|\Phi_0|H \equiv \tilde{\Gamma}_{\text{running dil}}H$.)

In the paper we considered the combined source case, where the foam appears together with a nontrivial running dilaton of the form (2.3). The string coupling $g_s = e^{\Phi}$ exhibits a nontrivial scaling with the scale factor and also with temperature. The combined source is taken to be the *algebraic sum* of the respective two source terms; that is,

$$\Gamma_{\rm total} \equiv \tilde{\Gamma}_{\rm total} H = \left[-|\Phi_0| + 4 \frac{g_{s0}^2 m_X^2}{M_s^2} \langle \Delta^2 \rangle_0 \left(\frac{C_0}{T} \right)^{1-2|\Phi_0|} \right] H \qquad \text{(A42)}$$

in the asymptotic region $m_X \gg T$ of interest, where we have assumed the cooling law (3.4).

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