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Models with three Higgs doublets in the triplet representations of A_4 or S_4

R. González Felipe,[1,2,*] H. Serôdio,[3,†] and João P. Silva[1,2,‡]

[1]*Instituto Superior de Engenharia de Lisboa—ISEL, 1959-007 Lisboa, Portugal*

[2]*Centro de Física Teórica de Partículas (CFTP), Instituto Superior Técnico, Universidade Técnica de Lisboa, 1049-001 Lisboa, Portugal*

[3]*Departament de Física Teòrica and IFIC, Universitat de València-CSIC, E-46100 Burjassot, Spain*

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We consider the quark sector of theories containing three scalar $SU(2)_L$ doublets in the triplet representation of A_4 (or S_4) and three generations of quarks in arbitrary A_4 (or S_4) representations. We show that for all possible choices of quark field representations and for all possible alignments of the Higgs vacuum expectation values that can constitute global minima of the scalar potential, it is not possible to obtain simultaneously nonvanishing quark masses and a nonvanishing CP-violating phase in the Cabibbo-Kobayashi-Maskawa quark mixing matrix. As a result, in this minimal form, models with three scalar fields in the triplet representation of A_4 or S_4 cannot be extended to the quark sector in a way consistent with experiment.

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I. INTRODUCTION

There is a long history of articles considering discrete symmetries in the study of the leptonic sector (see for instance the recent reviews [1–4] and references therein), including many models predicting tribimaximal leptonic mixing [5], now disfavored by the measurement of a large mixing angle θ_{13} [6–9]. In the quark sector, models based on the A_4 symmetry as a possible family symmetry were first introduced in Refs. [10,11]. After the impact of the symmetry on the Yukawa matrices is known, some structure for the vacuum expectation values (VEV) has to be assumed before moving on to the mass matrices and respective phenomenological predictions. Occasionally, this has been performed without a full study of the scalar sector and without ensuring properly whether the assumed vacuum structure indeed corresponds to the global minimum. This may occur, in part, because finding local minima is easy (one just has to show that the gradient of the potential vanishes), while ensuring that there is no other, lower-lying minimum is often rather difficult. Recently, Degee *et al.* [12] have introduced a geometrical procedure to minimize highly symmetric scalar potentials and solved the problem for a three Higgs doublet model (3HDM) potential with an A_4 or an S_4 symmetry. Although it is not explicitly stated, Ref. [12] refers to a set of three Higgs fields in a triplet representation of the group.[1] This is a crucial point, since if one were to place each of the three Higgs fields in a singlet representation, one would end up with the most general 3HDM potential. It is found that the possible VEV alignments for the A_4 symmetric potential [14] that may correspond to a global minimum are [12]

$$\begin{aligned} &v(1,0,0), \qquad v(1,1,1), \\ &v(\pm 1, \eta, \eta^*) \quad \text{with} \quad \eta = e^{i\pi/3}, \\ &v(1, e^{i\alpha}, 0) \quad \text{with any } \alpha. \end{aligned} \tag{1}$$

Similarly, the possible VEV alignments corresponding to global minima in the S_4 symmetric potential are [12]

$$\begin{aligned} &v(1,0,0), \qquad v(1,1,1), \\ &v(\pm 1, \eta, \eta^*) \quad \text{with} \quad \eta = e^{i\pi/3}, \qquad v(1, i, 0). \end{aligned} \tag{2}$$

In each case, a VEV corresponding to some permutation of the fields is also a possible global minimum. Any other solution of the stationarity conditions may be a saddle point, a local maximum, or even a local minimum, but never the global minimum.

Besides a correct identification of global minima, one must also consider whether the specific discrete symmetry under study can be extended to the whole Lagrangian of the theory, in a way consistent with known data. In particular, in the quark sector there should be no massless quarks, no diagonal blocks in the Cabibbo-Kobayashi-Maskawa (CKM) matrix, and/or no vanishing CP-violating phase. As shown by Ferreira and Silva [15], these constraints place stringent limits on the type of mass matrices obtainable from Abelian symmetries in the 2HDM.

In this article, we consider models with three Higgs doublets Φ_i in a triplet representation of A_4 (Sec. II), or in a triplet representation of S_4 (Sec. III). This ensures that the only possible global VEV structures are those in Eqs. (1) and (2), respectively. The models contain only three generations of left-handed quark doublets Q_L, right-handed up-type quark singlets u_R, and right-handed down-type quark singlets d_R. Our conclusions are briefly summarized in Sec. IV.

*ricardo.felipe@ist.utl.pt
†hugo.serodio@ific.uv.es
‡jpsilva@cftp.ist.utl.pt

[1]To be precise, the three scalar fields must be in a faithful representation of the group [13].

II. THE A_4 CASE

A_4 is the group of even permutations of four objects, and it has 12 elements divided into four irreducible representations, namely, three singlets $\mathbf{1}$, $\mathbf{1}'$, $\mathbf{1}''$ and one triplet $\mathbf{3}$. The multiplication rules are

$$\begin{aligned} &\mathbf{1}\otimes \text{any} = \text{any}, \quad \mathbf{1}'\otimes\mathbf{1}' = \mathbf{1}'', \quad \mathbf{1}'\otimes\mathbf{1}'' = \mathbf{1}, \\ &\mathbf{1}'\otimes\mathbf{3} = \mathbf{3}, \quad \mathbf{1}''\otimes\mathbf{1}'' = \mathbf{1}', \quad \mathbf{1}''\otimes\mathbf{3} = \mathbf{3}, \\ &\mathbf{3}\otimes\mathbf{3} = \mathbf{1}\oplus\mathbf{1}'\oplus\mathbf{1}''\oplus\mathbf{3_s}\oplus\mathbf{3_a}. \end{aligned} \tag{3}$$

We recall that for the corresponding entry of the Yukawa coupling matrix to be nonvanishing, the Yukawa Lagrangian must be in the invariant singlet representation $\mathbf{1}$. Since the three Higgs doublets are in the representation $\mathbf{3}$, we see from Eq. (3), that the product of left-handed and right-handed fermions must also be in a triplet representation. This means that at least one of the fermion fields in each charge sector must be in a triplet representation. The possibilities for the representations of the left-handed quark fields and for the up and down right-handed quarks are listed in Table I.

Since permutations of the three fields in each sector do not lead to new structures for the Yukawa matrices, the notation "three singlets" stands for the following independent possibilities for the fields in each of the three generations:

$$\begin{aligned} &(\mathbf{1},\mathbf{1},\mathbf{1}), \quad (\mathbf{1},\mathbf{1}',\mathbf{1}''), \quad (\mathbf{1},\mathbf{1},\mathbf{1}'), \quad (\mathbf{1}',\mathbf{1}',\mathbf{1}'), \\ &(\mathbf{1},\mathbf{1}',\mathbf{1}'), \quad (\mathbf{1}',\mathbf{1}',\mathbf{1}''), \quad (\mathbf{1},\mathbf{1},\mathbf{1}''), \\ &(\mathbf{1}',\mathbf{1}'',\mathbf{1}''), \quad (\mathbf{1},\mathbf{1}'',\mathbf{1}''), \quad (\mathbf{1}'',\mathbf{1}'',\mathbf{1}''). \end{aligned} \tag{4}$$

In order to use the VEVs given in Eq. (1), one must be sure to use a representation of the group that is consistent with the basis in which those VEVs were obtained in Ref. [12]. Indeed, if one starts from Higgs fields with the VEVs of Eq. (1), and one changes the scalar fields by a unitary transformation U, i.e.,

TABLE I. Possible representations of the left-handed quark doublets (Q_L), the right-handed up quark singlets (u_R), and the right-handed down quark singlets (d_R), when the three Higgs doublets are in a triplet representation **3**.

Q_L	u_R	d_R
3	**3**	**3**
3	**3**	Three singlets
3	Three singlets	**3**
3	Three singlets	Three singlets
Three singlets	**3**	**3**

$$\begin{pmatrix}\Phi_1\\ \Phi_2\\ \Phi_3\end{pmatrix} \to U \begin{pmatrix}\Phi_1\\ \Phi_2\\ \Phi_3\end{pmatrix}, \tag{5}$$

then the VEVs also transform as

$$\begin{pmatrix}\langle\Phi_1\rangle\\ \langle\Phi_2\rangle\\ \langle\Phi_3\rangle\end{pmatrix} \to U \begin{pmatrix}\langle\Phi_1\rangle\\ \langle\Phi_2\rangle\\ \langle\Phi_3\rangle\end{pmatrix}, \tag{6}$$

and, in general, will no longer have the form in Eq. (1). A suitable basis for the triplet representation of A_4 is given by

$$S = \begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{pmatrix}, \qquad T = \begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 1 & 0 & 0\end{pmatrix}. \tag{7}$$

In the notation of Sec. 6.4 of Ref. [16], $a_1 = S$, $b = T$, and $a_2 = T^{-1}ST$ is redundant. These matrices satisfy $S^2 = T^3 = (ST)^3 = 1$, showing that they indeed generate the group A_4. Equation (7) also coincide with the basis used in Ref. [17].

One way to confirm that we are indeed using a basis consistent with Ref. [12] is to check that imposing S and T on the 3HDM potential, we recover

$$\begin{aligned} V = &-\frac{M_0}{\sqrt{3}}(|\Phi_1|^2+|\Phi_2|^2+|\Phi_3|^2) + \frac{\Lambda_0}{3}(|\Phi_1|^2+|\Phi_2|^2+|\Phi_3|^2)^2 + \frac{\Lambda_3}{3}[|\Phi_1|^4+|\Phi_2|^4+|\Phi_3|^4-|\Phi_1|^2|\Phi_2|^2 \\ &-|\Phi_2|^2|\Phi_3|^2-|\Phi_3|^2|\Phi_1|^2] + \Lambda_1[(\mathrm{Re}\Phi_1^\dagger\Phi_2)^2+(\mathrm{Re}\Phi_2^\dagger\Phi_3)^2+(\mathrm{Re}\Phi_3^\dagger\Phi_1)^2] + \Lambda_2[(\mathrm{Im}\Phi_1^\dagger\Phi_2)^2+(\mathrm{Im}\Phi_2^\dagger\Phi_3)^2 \\ &+(\mathrm{Im}\Phi_3^\dagger\Phi_1)^2] + \Lambda_4[(\mathrm{Re}\Phi_1^\dagger\Phi_2)(\mathrm{Im}\Phi_1^\dagger\Phi_2)+(\mathrm{Re}\Phi_2^\dagger\Phi_3)(\mathrm{Im}\Phi_2^\dagger\Phi_3)+(\mathrm{Re}\Phi_3^\dagger\Phi_1)(\mathrm{Im}\Phi_3^\dagger\Phi_1)], \end{aligned} \tag{8}$$

as in Eq. (9) of Ref. [12].[2]

In A_4, with the basis of Eq. (7), the product of two triplets, $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$, gives [1,17]

$$\begin{aligned} &(a\otimes b)_{\mathbf{1}} = a_1b_1 + a_2b_2 + a_3b_3, \quad (a\otimes b)_{\mathbf{1}'} = a_1b_1 + \omega^2 a_2b_2 + \omega a_3b_3, \quad (a\otimes b)_{\mathbf{1}''} = a_1b_1 + \omega a_2b_2 + \omega^2 a_3b_3, \\ &(a\otimes b)_{\mathbf{3}_s} = (a_2b_3 + a_3b_2, a_3b_1 + a_1b_3, a_1b_2 + a_2b_1), \quad (a\otimes b)_{\mathbf{3}_a} = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1), \end{aligned} \tag{9}$$

[2]Equation (9) of Ref. [12] coincides with the sum of Eqs. (38) and (39) of Ref. [16], with the substitutions $\Lambda_0 = 3\lambda + \lambda'$, $\Lambda_1 = \lambda'' + 2\,\mathrm{Re}(\tilde{\lambda})$, $\Lambda_2 = \lambda'' - 2\,\mathrm{Re}(\tilde{\lambda})$, $\Lambda_3 = -\lambda'$, $\Lambda_4 = -4\,\mathrm{Im}(\tilde{\lambda})$.

where $\omega = e^{2i\pi/3}$, and s, a stand for the symmetric and antisymmetric triplet components, respectively.

We will also need the product of three triplets, a, b, and $c = (c_1, c_2, c_3)$,

$$\begin{aligned} (a \otimes b \otimes c)_s &= a_1(b_2c_3 + b_3c_2) + a_2(b_3c_1 + b_1c_3) \\ &\quad + a_3(b_1c_2 + b_2c_1), \\ (a \otimes b \otimes c)_a &= a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) \\ &\quad + a_3(b_1c_2 - b_2c_1). \end{aligned} \tag{10}$$

We are now ready to construct the Yukawa matrices for the various cases. We have built a program to test all possibilities automatically. As a first example, let us consider the case $\Phi \sim \mathbf{3}$, $(\bar{Q}_{L1}, \bar{Q}_{L2}, \bar{Q}_{L3}) \sim (\mathbf{1}, \mathbf{1}, \mathbf{1}')$, $d_R \sim \mathbf{3}$, and $u_R \sim \mathbf{3}$. We start with the down sector. Since $\bar{Q}_{L1}$ is in the $\mathbf{1}$ representation, it must couple to the $(\Phi \otimes d_R)_{\mathbf{1}}$ combination obtained from Eq. (9). The same is true for $\bar{Q}_{L2}$, with an independent coefficient. This leads to the Yukawa terms,

$$\begin{aligned} &\alpha_1 \bar{Q}_{L1}[\Phi_1 d_{R1} + \Phi_2 d_{R2} + \Phi_3 d_{R3}] \\ &\quad + \alpha_2 \bar{Q}_{L2}[\Phi_1 d_{R1} + \Phi_2 d_{R2} + \Phi_3 d_{R3}]. \end{aligned} \tag{11}$$

Once the fields Φ_i are substituted by their VEVs v_i, these terms give the first and second row of the down-type quark mass matrix, M_d, respectively. Since $\bar{Q}_{L3}$ is in the $\mathbf{1}'$ representation, we can only obtain a singlet with the $\mathbf{1}''$ combination $(\Phi \otimes d_R)_{\mathbf{1}''}$ in Eq. (9). This leads to a term

$$\alpha_3 \bar{Q}_{L3}[\Phi_1 d_{R1} + \omega \Phi_2 d_{R2} + \omega^2 \Phi_3 d_{R3}], \tag{12}$$

which will fill the third row of M_d. Thus, the down-type quark mass matrix reads

$$M_d = \begin{pmatrix} \alpha_1 v_1 & \alpha_1 v_2 & \alpha_1 v_3 \\ \alpha_2 v_1 & \alpha_2 v_2 & \alpha_2 v_3 \\ \alpha_3 v_1 & \omega \alpha_3 v_2 & \omega^2 \alpha_3 v_3 \end{pmatrix}, \tag{13}$$

with arbitrary complex constants α_i.

Recalling that the up-quark Yukawa terms involve the combinations $\bar{Q}_L \tilde{\Phi} u_R$, a similar analysis of the up-type quark sector yields

$$M_u = \begin{pmatrix} \beta_1 v_1^* & \beta_1 v_2^* & \beta_1 v_3^* \\ \beta_2 v_1^* & \beta_2 v_2^* & \beta_2 v_3^* \\ \beta_3 v_1^* & \omega \beta_3 v_2^* & \omega^2 \beta_3 v_3^* \end{pmatrix}, \tag{14}$$

where β_i are arbitrary complex constants.

In order to find the most relevant features of the quark sector, we define the Hermitian matrices

$$H_d = M_d M_d^\dagger, \qquad H_u = M_u M_u^\dagger, \tag{15}$$

whose eigenvalues coincide with the squared masses in each quark sector. Moreover, the CKM CP-violating phase is proportional to the determinant [18]

$$J = \mathrm{Det}(H_d H_u - H_u H_d). \tag{16}$$

We must now substitute (v_1, v_2, v_3) by each of the possible VEV alignments in Eq. (1), including all possible permutations, and study the properties of H_d, H_u, and J. As an example, consider the possibility that $(v_1, v_2, v_3) = v(1, e^{i\alpha}, 0)$, for any phase α. Then

$$M_d = v \begin{pmatrix} \alpha_1 & \alpha_1 e^{i\alpha} & 0 \\ \alpha_2 & \alpha_2 e^{i\alpha} & 0 \\ \alpha_3 & \omega \alpha_3 e^{i\alpha} & 0 \end{pmatrix}, \tag{17}$$

$$M_u = v \begin{pmatrix} \beta_1 & \beta_1 e^{-i\alpha} & 0 \\ \beta_2 & \beta_2 e^{-i\alpha} & 0 \\ \beta_3 & \omega \beta_3 e^{-i\alpha} & 0 \end{pmatrix}. \tag{18}$$

As a result, we predict one massless quark with charge $-1/3$ and one massless quark with charge $2/3$, contrary to experimental evidence. It is interesting to note that, in this case, H_d and H_u do not depend on α but, nevertheless, $J \neq 0$. This means that the model predicts one massless quark in each charge sector but displays explicit CP violation in the CKM matrix.[3]

As a second example, let us consider the case $\Phi \sim \mathbf{3}$, $(\bar{Q}_{L1}, \bar{Q}_{L2}, \bar{Q}_{L3}) \sim (\mathbf{1}, \mathbf{1}', \mathbf{1}'')$, $d_R \sim \mathbf{3}$, and $u_R \sim \mathbf{3}$. We find

$$M_d = \begin{pmatrix} \alpha_1 v_1 & \alpha_1 v_2 & \alpha_1 v_3 \\ \alpha_2 v_1 & \omega \alpha_2 v_2 & \omega^2 \alpha_2 v_3 \\ \alpha_3 v_1 & \omega^2 \alpha_3 v_2 & \omega \alpha_3 v_3 \end{pmatrix}, \tag{19}$$

$$M_u = \begin{pmatrix} \beta_1 v_1^* & \beta_1 v_2^* & \beta_1 v_3^* \\ \beta_2 v_1^* & \omega \beta_2 v_2^* & \omega^2 \beta_2 v_3^* \\ \beta_3 v_1^* & \omega^2 \beta_3 v_2^* & \omega \beta_3 v_3^* \end{pmatrix}. \tag{20}$$

For the VEV alignments $v(1, 1, 1)$ and $v(\pm 1, \eta, \eta^*)$ of Eq. (1), this leads to

$$H_d = 3v^2 \begin{pmatrix} |\alpha_1|^2 & 0 & 0 \\ 0 & |\alpha_2|^2 & 0 \\ 0 & 0 & |\alpha_3|^2 \end{pmatrix}, \tag{21}$$

$$H_u = 3v^2 \begin{pmatrix} |\beta_1|^2 & 0 & 0 \\ 0 & |\beta_2|^2 & 0 \\ 0 & 0 & |\beta_3|^2 \end{pmatrix}, \tag{22}$$

[3] One could envisage a more complicated setup where the light quark masses appear radiatively.

meaning that, in these cases, all quark masses are nonvanishing and nondegenerate. However, we find a diagonal CKM matrix and no *CP* violation, in blatant contradiction with experiment.

The particular case where $\bar{Q}_L$, u_R, and d_R (in addition to Φ) are all in a triplet representation of A_4 has been considered in Refs. [10,11] for the first three VEVs given in Eq. (1). Reference [10] solves the problem by adding a fourth scalar as a singlet of A_4; Ref. [11] considers symmetry breaking in stages.

Having gone through all cases in Table I and all possible VEV alignments in Eq. (1) (including permutations), we find that in all situations one obtains either massless quarks or a vanishing CKM phase.

In Table II we present, for each choice of representations and for each VEV alignment given in Eq. (1), the different quark mass spectra and the number of CKM mixing angles not predicted by the discrete symmetry, i.e., the number of parameter-dependent mixing angles (PDMA).

By itself, requiring nonvanishing quarks restricts the representations of $\{Q_L; u_R; d_R\}$ to the five possibilities $\{\mathbf{s};\mathbf{3};\mathbf{3}\}$, $\{\mathbf{3};\mathbf{s};\mathbf{s}\}$, $\{\mathbf{3};\mathbf{s};\mathbf{3}\}$, $\{\mathbf{3};\mathbf{3};\mathbf{s}\}$, and $\{\mathbf{3};\mathbf{3};\mathbf{3}\}$, where $\mathbf{s}$ stands for $(\mathbf{1}, \mathbf{1}', \mathbf{1}'')$, with the VEVs restricted to $v(1, 1, 1)$ or $v(\pm 1, \eta, \eta^*)$. In all these special cases, the CKM matrix equals the unit matrix. Thus, it is not possible to extend the A_4 symmetry to the quark sector, with only three generations of quarks and the three scalar fields in a triplet of A_4.

It is conceivable that this problem can be evaded by adding quark generations. More commonly, one considers other representations for the three scalar fields and/or one adds extra scalars to the theory in other representations of A_4. But, in such cases one *must* prove that the local minimum does indeed correspond to a global minimum. One can see from the treatment of A_4 that this endeavor is far from trivial [12].

TABLE II. Quark mass spectra and number of arbitrary CKM parameter-dependent mixing angles (PDMA) in the A_4 case. The symbol $\times$ stands for 0 or $m_i \neq 0$; $\mathbf{s}$ stands for $\mathbf{1}$, $\mathbf{1}'$ or $\mathbf{1}''$.

VEV	Q_L	u_R	d_R	Number of PDMA	Mass spectrum
(1, 0, 0)	**3**	**3**	**3**	0	$(0, m_{u,d}, m'_{u,d})$
	3	**3**	s	0	$(0, m_u, m'_u)$ $(0, 0, m_d)$
	3	s	**3**	0	$(0, 0, m_u)$ $(0, m_d, m'_d)$
	3	s	s	0	$(0, 0, m_{u,d})$
	s	**3**	**3**	2	$(0, 0, m_{u,d})$
$(1, 1, 1)$, $(\pm 1, \eta, \eta^*)$	**3**	**3**	**3**	0	$(m_{u,d}, m'_{u,d}, m''_{u,d})$
	3	**3**	s	0	(m_u, m'_u, m''_u) $(\times, \times, m_d)$
	3	s	**3**	0	$(\times, \times, m_u)$ (m_d, m'_d, m''_d)
	3	s	s	0	$(\times, \times, m_{u,d})$
	s	**3**	**3**	0 1 2	$(m_{u,d}, m'_{u,d}, m''_{u,d})$ $(0, m_{u,d}, m'_{u,d})$ $(0, 0, m_{u,d})$
$(1, e^{i\alpha}, 0)$	**3**	**3**	**3**	1	$(0, m_{u,d}, m_{u,d})$
	3	**3**	s	1	$(0, m_u, m_u)$ $(0, \times, m_d)$
	3	s	**3**	1	$(0, \times, m_u)$ $(0, m_d, m_d)$
	3	s	s	1	$(0, \times, m_{u,d})$
	s	**3**	**3**	3 2	$(0, m_{u,d}, m'_{u,d})$ $(0, 0, m_{u,d})$

III. THE S_4 CASE

S_4 is the group of all permutations of four objects. It has 24 elements divided into five irreducible representations: two singlets $\mathbf{1}_1$, $\mathbf{1}_2$, one doublet $\mathbf{2}$, and two triplets $\mathbf{3}_1$, $\mathbf{3}_2$. The multiplication rules are

$$\begin{gathered} \mathbf{1_1} \otimes \text{any} = \text{any}, \quad \mathbf{1_2} \otimes \mathbf{1_2} = \mathbf{1_1}, \quad \mathbf{1_2} \otimes \mathbf{2} = \mathbf{2}, \\ \mathbf{1_2} \otimes \mathbf{3_1} = \mathbf{3_2}, \quad \mathbf{1_2} \otimes \mathbf{3_2} = \mathbf{3_1}, \quad \mathbf{2} \otimes \mathbf{2} = \mathbf{1_1} \oplus \mathbf{1_2} \oplus \mathbf{2}, \\ \mathbf{2} \otimes \mathbf{3_1} = \mathbf{3_1} \oplus \mathbf{3_2}, \quad \mathbf{2} \otimes \mathbf{3_2} = \mathbf{3_1} \oplus \mathbf{3_2}, \\ \mathbf{3_1} \otimes \mathbf{3_1} = \mathbf{1_1} \oplus \mathbf{2} \oplus \mathbf{3_1} \oplus \mathbf{3_2}, \quad \mathbf{3_1} \otimes \mathbf{3_2} = \mathbf{1_2} \oplus \mathbf{2} \oplus \mathbf{3_1} \oplus \mathbf{3_2}, \\ \mathbf{3_2} \otimes \mathbf{3_2} = \mathbf{1_1} \oplus \mathbf{2} \oplus \mathbf{3_1} \oplus \mathbf{3_2}. \end{gathered} \tag{23}$$

Since A_4 is a subgroup of S_4, this case will have at least the same unphysical restrictions. Yet, for model building, it is useful to go through the analysis in detail, uncovering the specific constraints that should be corrected when enlarging the model.

Let us start by assuming that the three Higgs doublets are in the representation $\mathbf{3_1}$. By looking at Eq. (23), we see that the product of left-handed and right-handed fermions

TABLE III. Possible representations of u_R and d_R when the three Higgs doublets are in a $\mathbf{3_1}$ representation and all Q_L are in a triplet representation $\mathbf{3_1}$ or $\mathbf{3_2}$.

Q_L	u_R	d_R	Q_L	u_R	d_R
$\mathbf{3_1}$	$\mathbf{1_1}, \mathbf{1_1}, \mathbf{1_1}$	$\mathbf{1_1}, \mathbf{1_1}, \mathbf{1_1}$	$\mathbf{3_2}$	$\mathbf{1_2}, \mathbf{1_2}, \mathbf{1_2}$	$\mathbf{1_2}, \mathbf{1_2}, \mathbf{1_2}$
	$\mathbf{1_1}, \mathbf{1_1}, \mathbf{1_1}$	$\mathbf{2}, \mathbf{1_1}$		$\mathbf{1_2}, \mathbf{1_2}, \mathbf{1_2}$	$\mathbf{2}, \mathbf{1_2}$
	$\mathbf{1_1}, \mathbf{1_1}, \mathbf{1_1}$	$\mathbf{3_1}$		$\mathbf{1_2}, \mathbf{1_2}, \mathbf{1_2}$	$\mathbf{3_1}$
	$\mathbf{1_1}, \mathbf{1_1}, \mathbf{1_1}$	$\mathbf{3_2}$		$\mathbf{1_2}, \mathbf{1_2}, \mathbf{1_2}$	$\mathbf{3_2}$
	$\mathbf{2}, \mathbf{1_1}$	$\mathbf{1_1}, \mathbf{1_1}, \mathbf{1_1}$		$\mathbf{2}, \mathbf{1_2}$	$\mathbf{1_2}, \mathbf{1_2}, \mathbf{1_2}$
	$\mathbf{2}, \mathbf{1_1}$	$\mathbf{2}, \mathbf{1_1}$		$\mathbf{2}, \mathbf{1_2}$	$\mathbf{2}, \mathbf{1_2}$
	$\mathbf{2}, \mathbf{1_1}$	$\mathbf{3_1}$		$\mathbf{2}, \mathbf{1_2}$	$\mathbf{3_1}$
	$\mathbf{2}, \mathbf{1_1}$	$\mathbf{3_2}$		$\mathbf{2}, \mathbf{1_2}$	$\mathbf{3_2}$
	$\mathbf{3_1}$	$\mathbf{1_1}, \mathbf{1_1}, \mathbf{1_1}$		$\mathbf{3_1}$	$\mathbf{1_2}, \mathbf{1_2}, \mathbf{1_2}$
	$\mathbf{3_1}$	$\mathbf{2}, \mathbf{1_1}$		$\mathbf{3_1}$	$\mathbf{2}, \mathbf{1_2}$
	$\mathbf{3_1}$	$\mathbf{3_1}$		$\mathbf{3_1}$	$\mathbf{3_1}$
	$\mathbf{3_1}$	$\mathbf{3_2}$		$\mathbf{3_1}$	$\mathbf{3_2}$
	$\mathbf{3_2}$	$\mathbf{1_1}, \mathbf{1_1}, \mathbf{1_1}$		$\mathbf{3_2}$	$\mathbf{1_2}, \mathbf{1_2}, \mathbf{1_2}$
	$\mathbf{3_2}$	$\mathbf{2}, \mathbf{1_1}$		$\mathbf{3_2}$	$\mathbf{2}, \mathbf{1_2}$
	$\mathbf{3_2}$	$\mathbf{3_1}$		$\mathbf{3_2}$	$\mathbf{3_1}$
	$\mathbf{3_2}$	$\mathbf{3_2}$		$\mathbf{3_2}$	$\mathbf{3_2}$

TABLE IV. Possible representations of u_R and d_R when the three Higgs doublets are in a $\mathbf{3_1}$ representation and two of the Q_L are in the doublet representation **2**.

Q_L	u_R	d_R
$\mathbf{2}$, $\mathbf{1_1}$	$\mathbf{3_1}$	$\mathbf{3_1}$
$\mathbf{2}$, $\mathbf{1_2}$	$\mathbf{3_2}$	$\mathbf{3_2}$

must also be in a $\mathbf{3_1}$ representation (or else, the Yukawa Lagrangian would not be in the invariant $\mathbf{1_1}$ representation). The possibilities for the representations of the up and down right-handed quarks are listed in Table III, when Q_L is in a triplet representation.

When two of the Q_L are in the doublet **2** representation, the possibilities are $(Q_L, u_R, d_R) \sim (\mathbf{2}, \mathbf{3_1}, \mathbf{3_1})$, $(\mathbf{2}, \mathbf{3_1}, \mathbf{3_2})$, $(\mathbf{2}, \mathbf{3_2}, \mathbf{3_1})$, or $(\mathbf{2}, \mathbf{3_2}, \mathbf{3_2})$. Similarly, when one of the Q_L is in a singlet representation, there are only two possibilities: either $(Q_L, u_R, d_R) \sim (\mathbf{1_1}, \mathbf{3_1}, \mathbf{3_1})$, or $(Q_L, u_R, d_R) \sim (\mathbf{1_2}, \mathbf{3_2}, \mathbf{3_2})$. But, in this case, the third Q_L field must be in a singlet representation that yields a Yukawa Lagrangian in the singlet representation. Otherwise, the mass matrix would have a row of zeros, and there would be a massless quark. As a result, when two of the Q_L are in the doublet **2** representation, the only viable possibilities for u_R and d_R are the ones listed in Table IV.

Finally, requiring that there are no massless quarks, when all the Q_L are in a singlet representation, the possibilities for u_R and d_R are listed in Table V.

A suitable basis for the $\mathbf{3_1}$ representation of S_4, consistent with the notation of Ref. [12], can be found in Ref. [19],

$$F_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \qquad G_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \tag{24}$$

Notice that G_3 coincides with T in Eq. (7). Imposing F_3 and G_3 on the 3HDM potential, we recover Eq. (8), with $\Lambda_4 = 0$. The $\mathbf{3_2}$ representation of S_4 can be identified with the matrices $-F_3$ and G_3. These matrices satisfy $F_3^2 = G_3^3 = (F_3 G_3)^4 = 1$, showing that they indeed generate the group S_4. As for the explicit form of the tensor products, we will use the Appendix of Ref. [19]. For example, the product of two $\mathbf{3_1}$ triplets, $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$, gives

TABLE V. Possible representations of u_R and d_R when the three Higgs doublets are in a $\mathbf{3_1}$ representation and all Q_L are in a singlet representation $\mathbf{1_1}$ or $\mathbf{1_2}$.

Q_L	u_R	d_R
$\mathbf{1_1}$, $\mathbf{1_1}$, $\mathbf{1_1}$	$\mathbf{3_1}$	$\mathbf{3_1}$
$\mathbf{1_2}$, $\mathbf{1_2}$, $\mathbf{1_2}$	$\mathbf{3_2}$	$\mathbf{3_2}$

$$\begin{aligned} (a\otimes b)_{\mathbf{1_1}} &= a_1 b_1 + a_2 b_2 + a_3 b_3, \\ (a\otimes b)_{\mathbf{2}} &= (a_1 b_1 + \omega a_2 b_2 + \omega^2 a_3 b_3, a_1 b_1 \\ &\quad + \omega^2 a_2 b_2 + \omega a_3 b_3), \\ (a\otimes b)_{\mathbf{3_1}} &= (a_2 b_3 + a_3 b_2, a_3 b_1 + a_1 b_3, a_1 b_2 + a_2 b_1), \\ (a\otimes b)_{\mathbf{3_2}} &= (a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1). \end{aligned} \tag{25}$$

For illustration, let us consider the case $\Phi \sim \mathbf{3_1}$, $(\bar{Q}_{L1}, \bar{Q}_{L2}) \sim \mathbf{2}$, $\bar{Q}_{L3} \sim \mathbf{1_1}$, $d_R \sim \mathbf{3_1}$, and $u_R \sim \mathbf{3_1}$. We start with the down sector. The fact that $(\bar{Q}_{L1}, \bar{Q}_{L2})$ is in the doublet representation **2** means that we must pick up the doublet combination $(\Phi \otimes d_R)_{\mathbf{2}}$ obtained from Eq. (25), leading to

$$\begin{aligned} &\alpha_1 \bar{Q}_{L1}[\Phi_1 d_{R1} + \omega \Phi_2 d_{R2} + \omega^2 \Phi_3 d_{R3}] \\ &\quad + \alpha_1 \bar{Q}_{L2}[\Phi_1 d_{R1} + \omega^2 \Phi_2 d_{R2} + \omega \Phi_3 d_{R3}]. \end{aligned} \tag{26}$$

On the other hand, $\bar{Q}_{L3} \sim \mathbf{1_1}$ couples to $(\Phi \otimes d_R)_{\mathbf{1_1}}$ in Eq. (25), yielding

$$\alpha_2 \bar{Q}_{L3}[\Phi_1 d_{R1} + d_{R2} + \Phi_3 d_{R3}]. \tag{27}$$

Hence,

$$M_d = \begin{pmatrix} \alpha_1 v_1 & \omega \alpha_1 v_2 & \omega^2 \alpha_1 v_3 \\ \alpha_1 v_1 & \omega^2 \alpha_1 v_2 & \omega \alpha_1 v_3 \\ \alpha_2 v_1 & \alpha_2 v_2 & \alpha_2 v_3 \end{pmatrix}. \tag{28}$$

Similarly,

TABLE VI. Quark mass spectra and number of arbitrary CKM parameter-dependent mixing angles (PDMA) in the S_4 case, for the VEV $v(1, 0, 0)$. In all cases, $\Phi \sim \mathbf{3_1}$.

VEV	Q_L	u_R	d_R	Number of PDMA	Mass spectrum
	$\mathbf{3_1}$	$\mathbf{1_1}$	$\mathbf{1_1}$	0	$(0, 0, m_{u,d})$
		$\mathbf{1_1}$	$\mathbf{2}$, $\mathbf{1_1}$	0	$(0, 0, m_{u,d})$
		$\mathbf{1_1}$	$\mathbf{3_i}$	0	$(0, 0, m_u)$
					$(0, m_d, m_d)$
		$\mathbf{2}$, $\mathbf{1_1}$	$\mathbf{1_1}$	0	$(0, 0, m_{u,d})$
		$\mathbf{2}$, $\mathbf{1_1}$	$\mathbf{2}$, $\mathbf{1_1}$	0	$(0, 0, m_{u,d})$
		$\mathbf{2}$, $\mathbf{1_1}$	$\mathbf{3_i}$	0	$(0, 0, m_u)$
$(1, 0, 0)$					$(0, m_d, m_d)$
		3i	$\mathbf{1_1}$	0	$(0, m_u, m_u)$
					$(0, 0, m_d)$
		$\mathbf{3_i}$	$\mathbf{2}$, $\mathbf{1_1}$	0	$(0, m_u, m_u)$
					$(0, 0, m_d)$
		$\mathbf{3_i}$	$\mathbf{3_j}$	0	$(0, m_{u,d}, m_{u,d})$
	$\mathbf{2}$, $\mathbf{1_i}$	$\mathbf{3_i}$	$\mathbf{3_i}$	1	$(0, 0, m_{u,d})$
	$\mathbf{1_i}$	$\mathbf{3_i}$	$\mathbf{3_i}$	2	$(0, 0, m_{u,d})$

TABLE VII. As in Table VI; for the VEV $v(1, 1, 1)$ and $v(1, \eta, \eta^*)$.

VEV	Q_L	u_R	d_R	Number of PDMA	Mass spectrum
	$\mathbf{3_1}$	$\mathbf{1_1}$	$\mathbf{1_1}$	0	$(0, 0, m_{u,d})$
		$\mathbf{1_1}$	$\mathbf{2}, \mathbf{1_1}$	0	$(0, 0, m_u)$ (m_d, m_d, m'_d)
		$\mathbf{1_1}$	$\mathbf{3_i}$	0	$(0, 0, m_u)$ $(m_d, m_d, 2m_d\delta_{1i})$
		$\mathbf{2}, \mathbf{1_1}$	$\mathbf{1_1}$	0	(m_u, m_u, m'_u) $(0, 0, m_d)$
		$\mathbf{2}, \mathbf{1_1}$	$\mathbf{2}, \mathbf{1_1}$	0	$(m_{u,d}, m_{u,d}, m'_{u,d})$
$(1, 1, 1), (\pm 1, \eta, \eta^*)$		$\mathbf{2}, \mathbf{1_1}$	$\mathbf{3_i}$	0	(m_u, m_u, m'_u) $(m_d, m_d, 2m_d\delta_{1i})$
		$\mathbf{3_i}$	$\mathbf{1_1}$	0	$(m_u, m_u, 2m_u\delta_{1i})$ $(0, 0, m_d)$
		$\mathbf{3_i}$	$\mathbf{2}, \mathbf{1_1}$	0	$(m_u, m_u, 2m_u\delta_{1i})$ (m_d, m_d, m'_d)
		$\mathbf{3_i}$	$\mathbf{3_j}$	0	$(m_u, m_u, 2m_u\delta_{1i})$ $(m_d, m_d, 2m_d\delta_{1j})$
	$\mathbf{2}, \mathbf{1_i}$	$\mathbf{3_i}$	$\mathbf{3_i}$	0	$(m_{u,d}, m_{u,d}, m'_{u,d})$
	$\mathbf{1_i}$	$\mathbf{3_i}$	$\mathbf{3_i}$	2	$(0, 0, m_{u,d})$

$$M_u = \begin{pmatrix} \beta_1 v_1^* & \omega\beta_1 v_2^* & \omega^2\beta_1 v_3^* \\ \beta_1 v_1^* & \omega^2\beta_1 v_2^* & \omega\beta_1 v_3^* \\ \beta_2 v_1^* & \beta_2 v_2^* & \beta_2 v_3^* \end{pmatrix}. \tag{29}$$

The predictions for the physical observables should now be found for all the possible global minima presented in Eq. (2). Let us test the case with the VEV alignment $v(1, 1, 1)$. We find

TABLE VIII. As in Table VI; for the VEV $v(1, i, 0)$.

VEV	Q_L	u_R	d_R	Number of PDMA	Mass spectrum
	$\mathbf{3_1}$	$\mathbf{1_1}$	$\mathbf{1_1}$	0	$(0, 0, m_{u,d})$
		$\mathbf{1_1}$	$\mathbf{2}, \mathbf{1_1}$	0	$(0, 0, m_u)$ $(0, m_d, m'_d)$
		$\mathbf{1_1}$	$\mathbf{3_i}$	0	$(0, 0, m_u)$ $(0, m_d, m_d)$
		$\mathbf{2}, \mathbf{1_1}$	$\mathbf{1_1}$	0	$(0, m_u, m'_u)$ $(0, 0, m_d)$
		$\mathbf{2}, \mathbf{1_1}$	$\mathbf{2}, \mathbf{1_1}$	0	$(0, m_{u,d}, m'_{u,d})$
$(1, i, 0)$		$\mathbf{2}, \mathbf{1_1}$	$\mathbf{3_i}$	0	$(0, m_u, m'_u)$ $(0, m_d, m_d)$
		$\mathbf{3_i}$	$\mathbf{1_1}$	0	$(0, m_u, m_u)$ $(0, 0, m_d)$
		$\mathbf{3_i}$	$\mathbf{2}, \mathbf{1_1}$	0	$(0, m_u, m_u)$ $(0, m_d, m'_d)$
		$\mathbf{3_i}$	$\mathbf{3_j}$	0	$(0, m_{u,d}, m_{u,d})$
	$\mathbf{2}, \mathbf{1_i}$	$\mathbf{3_i}$	$\mathbf{3_i}$	1	$(0, m_{u,d}, m'_{u,d})$
	$\mathbf{1_i}$	$\mathbf{3_i}$	$\mathbf{3_i}$	2	$(0, 0, m_{u,d})$

$$H_d = 3v^2 \begin{pmatrix} |\alpha_1|^2 & 0 & 0 \\ 0 & |\alpha_1|^2 & 0 \\ 0 & 0 & |\alpha_2|^2 \end{pmatrix}, \quad H_u = 3v^2 \begin{pmatrix} |\beta_1|^2 & 0 & 0 \\ 0 & |\beta_1|^2 & 0 \\ 0 & 0 & |\beta_2|^2 \end{pmatrix}. \tag{30}$$

Although this case does not exhibit massless quarks, it has a pair of degenerate quarks in each sector, the CKM is the unit matrix, and of course there is no *CP* violation.

The analysis for $\Phi \sim \mathbf{3_2}$ leads to a new set of cases obtained trivially from Tables III, IV, and V by noting that $\mathbf{3_2} = \mathbf{3_1} \otimes \mathbf{1_2}$. As we did for A_4, we have also built a program to test all S_4 possibilities automatically. In all cases, there is no *CP* violation in the CKM matrix ($J = 0$) and, in the absence of massless quarks, there will always be one pair of degenerate quarks in each sector. The restrictions on the physical parameters obtained for each choice of representations and for each VEV alignment in Eq. (2) can be found in Tables VI, VII, and VIII. This may help model builders in identifying what features need to be corrected when adding extra fields to the theory.

IV. CONCLUSIONS

We have studied the possibility of generating the quark masses and CKM mixing in the context of three Higgs doublet models extended by a discrete A_4 or S_4 symmetry. Assuming that the Higgs fields are in the triplet (faithful) representation of the discrete group, we have shown that none of the possible VEV alignments that corresponds to a global minimum of the scalar potential leads to phenomenologically viable mass matrices for the three generations of quarks of the Standard Model and, simultaneously, to a nonvanishing CKM phase. Clearly, these conclusions can be evaded by extending the field content with extra scalars and/or fermions.

Our analysis can be applied straightforwardly to the leptonic sector of the theory, if neutrinos are Dirac particles. In that case, one massless neutrino or lack of leptonic *CP* violation would not contradict current experiments.

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