Taste non-Goldstone, flavor-charged pseudo-Goldstone boson decay constants in staggered chiral perturbation theory

Jon A. Bailey, Weonjong Lee, and Boram Yoon

(SWME Collaboration)

Lattice Gauge Theory Research Center, FPRD, and CTP, Department of Physics and Astronomy, Seoul National University, Seoul 151-747, South Korea
(Received 28 December 2012; published 28 March 2013)

We calculate the axial current decay constants of taste non-Goldstone pions and kaons in staggered chiral perturbation theory through next-to-leading order. The results are a simple generalization of the results for the taste Goldstone case. New low-energy couplings are limited to analytic corrections that vanish in the continuum limit; certain coefficients of the chiral logarithms are modified, but they contain no new couplings. We report results for quenched, fully dynamical, and partially quenched cases of interest in the chiral SU(3) and SU(2) theories.

DOI: 10.1103/PhysRevD.87.054508 PACS numbers: 12.38.Gc, 11.30.Rd, 12.39.Fe

I. INTRODUCTION

The decay constants f_π and f_K parametrize hadronic matrix elements entering the leptonic decays of π and K mesons. The values of the decay constants can be combined with the leptonic decay rates from experiment to extract the CKM matrix elements $|V_{ud}|$ and $|V_{us}|$ and test first-row CKM unitarity. Tighter constraints on new physics are obtained by taking the ratio f_K/f_π and the form factor for the semileptonic decay $K \to \pi \ell \nu$ as theoretical inputs; doing so has led to impressive agreement between the Standard Model and experiment [1,2].

Staggered quarks have four tastes per flavor by construction [3,4]. The full taste symmetry group for a single massless flavor is $SU(4)_L \times SU(4)_R$ in the continuum limit ($a = 0$). At finite lattice spacing, lattice artifacts of $\mathcal{O}(a^2)$ break the taste symmetry, and the remaining exact chiral symmetry is $U(1)_A$, which is enough to prevent the staggered quark mass from being additively renormalized. Hence, staggered fermions have an exact chiral symmetry at nonzero lattice spacing. In addition, lattice calculations with staggered fermions are comparatively fast. Staggered chiral perturbation theory (SChPT) was first developed to describe the lattice artifacts and light quark mass dependence of lattice data for the pseudo-Goldstone boson (PGB) masses [5–8]. Lattice data were extrapolated to the continuum limit and physical quark masses to determine the light quark masses, tree-level PGB mass splittings, and low-energy couplings (LECs); these served as inputs to lattice calculations of the decay constants, semileptonic form factors, mixing parameters, and other quantities [9–27]. Lattice calculations of f_π have become precise enough to use it to determine the lattice spacing [28].

While there have been a few attempts to calculate the decay constants for the taste non-Goldstone sectors [29,30], most lattice calculations of the decay constants have been concentrated on the taste Goldstone sector associated with the exact chiral symmetry of the staggered action. In Ref. [31], Aubin and Bernard calculated the decay constants of the taste Goldstone pions and kaons through next-to-leading order (NLO) in SChPT. Here we extend their calculation to the taste non-Goldstone pions and kaons; we find that the general results are simply related to those in the taste Goldstone case.

In Sec. II we recall the staggered chiral Lagrangian and the tree-level propagators. In Sec. III we consider the definition of the decay constants, recall the various contributions through NLO in SChPT, and write down the general results in the $4 + 4 + 4$ theory. Section IV contains the results for specific cases of interest in the $1 + 1 + 1$ theory, and we conclude in Sec. V. We use the notation of Ref. [32] throughout.

II. CHIRAL LAGRANGIAN FOR STAGGERED QUARKS

In this section, we write down the chiral Lagrangian for staggered quarks. The single-flavor Lagrangian was formulated by Lee and Sharpe [5] and generalized to multiple flavors by Aubin and Bernard [7]. Here, we consider the $4 + 4 + 4$ theory, in which there are three flavors and four tastes per flavor. The exponential parameterization of the PGB fields is a 12×12 unitary matrix,

$$\Sigma = e^{i\phi/f} \in \mathrm{U}(12), \tag{1}$$

where the PGB fields are

$$\phi = \sum_a \phi^a \otimes T^a, \tag{2}$$

$$\phi^a = \begin{pmatrix} U_a & \pi_a^+ & K_a^+ \\ \pi_a^- & D_a & K_a^0 \\ K_a^- & \bar{K}_a^0 & S_a \end{pmatrix}, \tag{3}$$

$$T^a \in \{\xi_5, i\xi_{\mu 5}, i\xi_{\mu\nu}(\mu < \nu), \xi_\mu, \xi_I\}. \tag{4}$$

Here a runs over the 16 PGB tastes, and the T^a are 4×4 generators of $U(4)_T$; ξ_I is the identity matrix. Under a chiral transformation, Σ transforms as

$$\mathrm{SU}(12)_L \times \mathrm{SU}(12)_R : \Sigma \to L\Sigma R^\dagger, \tag{5}$$

where $L, R \in \mathrm{SU}(12)_{L,R}$.

In the standard power counting,

$$\mathcal{O}(p^2/\Lambda_\chi^2) \approx \mathcal{O}(m_q/\Lambda_\chi) \approx \mathcal{O}(a^2\Lambda_\chi^2). \tag{6}$$

The order of a Lagrangian operator is defined as the sum of n_{p^2}, n_m, and n_{a^2}, which are the number of derivative pairs, powers of (light) quark masses, and powers of the squared lattice spacing, respectively, in the operator. At leading order, the Lagrangian operators fall into three classes: $(n_{p^2}, n_m, n_{a^2}) = (1, 0, 0)$, (0, 1, 0), and (0, 0, 1), and we have

$$\mathcal{L}_{\mathrm{LO}} = \frac{f^2}{8}\mathrm{Tr}(\partial_\mu\Sigma\partial_\mu\Sigma^\dagger) - \frac{1}{4}\mu f^2\,\mathrm{Tr}(M\Sigma + M\Sigma^\dagger) + \frac{2m_0^2}{3}(U_I + D_I + S_I)^2 + a^2(\mathcal{U} + \mathcal{U}'), \tag{7}$$

where f is the decay constant at leading order (LO), μ is the condensate parameter, and M is the mass matrix,

$$M = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix} \otimes \xi_I. \tag{8}$$

The term multiplied by m_0^2 is the anomaly contribution [33], and the potentials $\mathcal{U}$ and $\mathcal{U}'$ are the taste symmetry breaking potentials of Ref. [7].

At NLO, there are six classes of operators satisfying $n_{p^2} + n_m + n_{a^2} = 2$, but only two classes contribute to the decay constants: $(n_{p^2}, n_m, n_{a^2}) = (1, 1, 0)$ and (1, 0, 1). Contributing operators in the former are Gasser-Leutwyler terms [34],

$$\mathcal{L}_{\mathrm{GL}} = L_4\,\mathrm{Tr}(\partial_\mu\Sigma^\dagger\partial_\mu\Sigma)\mathrm{Tr}(\chi^\dagger\Sigma + \chi\Sigma^\dagger) + L_5\,\mathrm{Tr}(\partial_\mu\Sigma^\dagger\partial_\mu\Sigma(\chi^\dagger\Sigma + \Sigma^\dagger\chi)), \tag{9}$$

where $\chi = 2\mu M$, and contributing operators in the latter are $\mathcal{O}(p^2a^2)$ terms enumerated by Sharpe and Van de Water [35].

III. DECAY CONSTANTS OF FLAVOR-CHARGED PSEUDO-GOLDSTONE BOSONS

For a flavor-charged PGB with taste t, P_t^+, the decay constant $f_{P_t^+}$ is defined by the matrix element of the axial current, $j_{\mu 5,t}^{P^+}$, between the single-particle state and the vacuum,

$$\langle 0|j_{\mu 5,t}^{P^+}|P_t^+(p)\rangle = -if_{P_t^+}p_\mu. \tag{10}$$

From the LO Lagrangian, the LO axial current is

$$j_{\mu 5,t}^{P^+} = -i\frac{f^2}{8}\mathrm{Tr}[T^{t(3)}\mathcal{P}^{P^+}(\partial_\mu\Sigma\Sigma^\dagger + \Sigma^\dagger\partial_\mu\Sigma)], \tag{11}$$

where $T^{a(3)} \equiv I_3 \otimes T^a$, I_3 is the identity matrix in flavor space, and $\mathcal{P}^{P^+}$ is a projection operator that chooses the P^+ from the Σ field. For example, for π^+ it is $\mathcal{P}_{ij}^{\pi^+} = \delta_{i1}\delta_{j2}$. In general, $\mathcal{P}_{ij}^{P^+} = \delta_{ix}\delta_{jy}$, where x and y are the light quarks in P^+. For flavor-charged states, $x \neq y$, by definition. Note that $\mathcal{P}^{P^+}$ and $T^{a(3)}$ commute with each other.

Expanding the exponentials $\Sigma = e^{i\phi/f}$ in the LO current gives

$$\partial_\mu\Sigma\Sigma^\dagger + \Sigma^\dagger\partial_\mu\Sigma = \frac{2i}{f}\partial_\mu\phi - \frac{i}{3f^3}(\partial_\mu\phi\phi^2 - 2\phi\partial_\mu\phi\phi + \phi^2\partial_\mu\phi) + \cdots. \tag{12}$$

The $\mathcal{O}(\phi)$ term of the axial current gives the LO term of the decay constants, f, and NLO corrections from the wave function renormalization. The wave function renormalization consists of NLO analytic terms and one-loop chiral logarithms at NLO; we denote the former by $\delta f_{P_t^+}^{\mathrm{anal},Z}$ and the latter by $\delta f_{P_t^+}^{Z}$. The $\mathcal{O}(\phi^3)$ term of the axial current also gives one-loop chiral logarithms at NLO, $\delta f_{P_t^+}^{\mathrm{current}}$. Figures 1(a) and 1(b) show the one-loop corrections to the decay constant. In addition, there is an analytic contribution to the decay constants from the NLO current. We denote the total of the NLO analytic terms by $\delta f_{P_t^+}^{\mathrm{anal}}$, which consists of $\delta f_{P_t^+}^{\mathrm{anal},Z}$ and analytic terms from the NLO current. Combining $\delta f_{P_t^+}^{\mathrm{anal}}$ with the one-loop corrections, we write the decay constants up to NLO,

$$f_{P_t^+} = f\left[1 + \frac{1}{16\pi^2f^2}(\delta f_{P_t^+}^{Z} + \delta f_{P_t^+}^{\mathrm{current}}) + \delta f_{P_t^+}^{\mathrm{anal}}\right]. \tag{13}$$

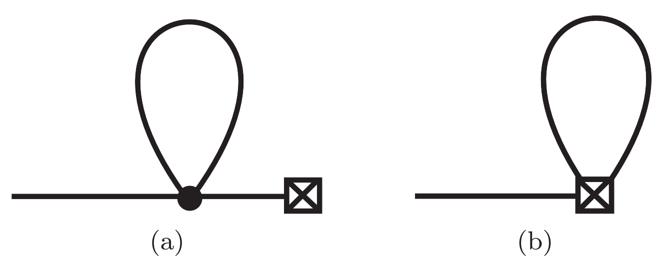

FIG. 1. One-loop diagrams contributing to the decay constants at NLO. (a) is the wave function renormalization correction and (b) is the current correction. The propagators include all insertions of hairpin vertices.

In this section we outline the calculation of $\delta f^Z_{P_t^+}$, $\delta f^{\rm current}_{P_t^+}$, and $\delta f^{\rm anal}_{P_t^+}$ and present results for the 4 + 4 + 4 theory.

A. Wave function renormalization correction

At $\mathcal{O}(\phi)$ the axial current, Eq. (11), is

$$j^{P^+,\phi}_{\mu 5,t} = f(\partial_\mu \phi^t_{yx}), \tag{14}$$

where we used $\tau_{ta} = 4\delta_{ta}$, $\mathcal{P}^{P^+}_{ij} = \delta_{ix}\delta_{jy}$ and performed the trace over taste indices. Here $\tau_{abcd\ldots}$ is defined by

$$\tau_{abcd\ldots} \equiv {\rm Tr}(T^a T^b T^c T^d \ldots). \tag{15}$$

The contributions of the $\mathcal{O}(\phi)$ current to the decay constants are defined by the matrix element

$$\langle 0| j^{P^+,\phi}_{\mu 5,t} |P^+_t(p)\rangle = f(-ip_\mu)\langle 0|\phi^t_{yx}|P^+_t(p)\rangle \tag{16}$$

$$= f(-ip_\mu)\sqrt{Z_{P_t^+}}, \tag{17}$$

where $Z_{P_t^+} = 1 + \delta Z_{P_t^+}$ is the wave function renormalization constant of the ϕ^t_{xy} field. At NLO the wave function renormalization corrections are

$$\frac{1}{16\pi^2 f^2}\delta f^Z_{P_t^+} + \delta f^{{\rm anal},Z}_{P_t^+} = \frac{1}{2}\delta Z_{P_t^+} = -\frac{1}{2}\frac{d\Sigma}{dp^2}\bigg|_{p^2=-m^2_{P_t^+}}, \tag{18}$$

where Σ is the self-energy of P_t^+. Using the self-energy from Ref. [32], we find the one-loop corrections

$$\delta f^Z_{P_t^+} = \frac{1}{24}\sum_a\bigg[\sum_Q l(Q_a) + 16\pi^2\int\frac{d^4q}{(2\pi)^4}(D^a_{xx} + D^a_{yy} - 2\theta^{at}D^a_{xy})\bigg], \tag{19}$$

where Q runs over the six flavor combinations xi and yi for $i \in \{u, d, s\}$, a runs over the 16 PGB tastes in the **15** and **1** of $SU(4)_T$, Q_a is the squared tree-level pseudoscalar meson mass with flavor Q and taste a, and $\theta^{ab} \equiv \frac{1}{4}\tau_{abab}$. In Eq. (19), $l(Q_a)$ and D^a_{ij} are chiral logarithms and the disconnected piece of the tree-level propagator, respectively, [7,32]

$$l(X) \equiv X(\ln X/\Lambda^2 + \delta_1(\sqrt{X}L)), \tag{20}$$

where $\delta_1(\sqrt{X}L)$ is the finite volume correction of Ref. [6], and

$$D^a_{ij} = -\frac{\delta_a}{(q^2+I_a)(q^2+J_a)} \times \frac{(q^2+U_a)(q^2+D_a)(q^2+S_a)}{(q^2+\pi^0_a)(q^2+\eta_a)(q^2+\eta'_a)}. \tag{21}$$

Here the names of mesons denote the squares of their tree-level masses, and

$$\delta_I = 4m_0^2/3, \qquad \delta_{\mu\nu} = 0, \qquad \delta_5 = 0 \tag{22}$$

$$\delta_\mu = a^2\delta'_V, \qquad \delta_{\mu 5} = a^2\delta'_A. \tag{23}$$

For $X \in \{I, J, U, D, S\}$,

$$X_a \equiv m^2_{X_a} = 2\mu m_x + a^2\Delta_a, \tag{24}$$

where m_x is the mass of the quark of flavor $x \in \{i, j, u, d, s\}$, while for $X \in \{\pi^0, \eta, \eta'\}$, the squares of the tree-level meson masses are the eigenvalues of the matrix,

$$\begin{pmatrix} U_a+\delta_a & \delta_a & \delta_a \\ \delta_a & D_a+\delta_a & \delta_a \\ \delta_a & \delta_a & S_a+\delta_a \end{pmatrix}. \tag{25}$$

The squared tree-level mass of a flavor-charged meson ($\pi^\pm$, $K^\pm$, K^0, $\bar{K}^0$) is

$$P_t^+ \equiv \frac{1}{2}(X_t + Y_t) = \mu(m_x + m_y) + a^2\Delta_t, \tag{26}$$

where $X \neq Y \in \{U, D, S\}$ and $x \neq y \in \{u, d, s\}$. The hairpin couplings $\delta'_{V,A}$ and taste splittings Δ_a are combinations of the couplings of the LO Lagrangian [7].

We defer discussing the analytic corrections $\delta f^{{\rm anal},Z}_{P_t^+}$ to Sec. III C.

B. Current correction

The $\mathcal{O}(\phi^3)$ terms of the axial current are

$$j^{P^+,\phi^3}_{\mu 5,t} = -\frac{1}{24f}\tau_{tabc}(\partial_\mu\phi^a_{yk}\phi^b_{kl}\phi^c_{lx} - 2\phi^a_{yk}\partial_\mu\phi^b_{kl}\phi^c_{lx} + \phi^a_{yk}\phi^b_{kl}\partial_\mu\phi^c_{lx}). \tag{27}$$

In the calculation of the matrix element defined in Eq. (10), each term of Eq. (27) contributes only one contraction because the derivatively coupled fields in the current must contract with the external field to obtain a nonzero result. For example, the first term gives

$$\partial_\mu\phi^a_{yk}\overbrace{\phi^b_{kl}\phi^c_{lx}} \quad\to\quad -ip_\mu\delta^{ta}\delta^{bc}K^b_{xl,lx}, \tag{28}$$

where [7,32,36]

$$K^a_{ij,kl} \equiv \int\frac{d^4q}{(2\pi)^4}\langle\phi^a_{ij}\phi^a_{kl}\rangle \qquad \text{(no sum)}, \tag{29}$$

$$\langle\phi^a_{ij}\phi^b_{kl}\rangle = \delta^{ab}\bigg(\delta_{il}\delta_{jk}\frac{1}{q^2+\frac{1}{2}(I_a+J_a)} + \delta_{ij}\delta_{kl}D^a_{il}\bigg). \tag{30}$$

Collecting the three contributions from Eq. (27), we find

$$i\frac{p_\mu}{6f}\sum_a\bigg[\sum_j(K^a_{xj,jx} + K^a_{yj,jy}) - 2\theta^{at}K^a_{xx,yy}\bigg], \tag{31}$$

where j runs over $\{u, d, s\}$. Performing the integrals over the loop momenta gives the one-loop current corrections to the decay constants,

$$\delta f^{\text{current}}_{P_t^+} \equiv -\frac{1}{6}\sum_a\bigg[\sum_Q l(Q_a) + 16\pi^2\int\frac{d^4q}{(2\pi)^4}(D^a_{xx}+D^a_{yy}-2\theta^{at}D^a_{xy})\bigg]. \quad (32)$$

Note that $\delta f^{\text{current}}_{P_t^+}$ is proportional to the one-loop wave function renormalization correction, $\delta f^{Z}_{P_t^+}$. This was shown in the taste Goldstone case in Ref. [31].

C. Next-to-leading order analytic contributions

Now we consider the NLO analytic contributions to the decay constants. They come from the $\mathcal{O}(p^2m)$ Gasser-Leutwyler Lagrangian in Eq. (9) and the $\mathcal{O}(p^2a^2)$ Sharpe-Van de Water Lagrangian of Ref. [35]. Both Lagrangians contribute to wave function renormalization and the current.

The analytic corrections to the self-energy [32] give the wave function renormalization correction,

$$\delta f^{\text{anal},Z}_{P_t^+} = -\frac{64}{f^2}L_4\mu(m_u+m_d+m_s) - \frac{8}{f^2}L_5\mu(m_x+m_y) - \frac{8}{f^2}a^2\mathcal{C}_t, \quad (33)$$

while the NLO current from the Gasser-Leutwyler terms gives the current correction,

$$\delta f^{\text{current,GL}}_{P_t^+} = \frac{128}{f^2}L_4\mu(m_u+m_d+m_s)+\frac{16}{f^2}L_5\mu(m_x+m_y). \quad (34)$$

The contributions of the $\mathcal{O}(p^2a^2)$ operators coming from the Sharpe-Van de Water Lagrangian in Ref. [35] give the current correction,

$$\delta f^{\text{current,SV}}_{P_t^+} = a^2\mathcal{C}'_t. \quad (35)$$

The LECs $\mathcal{C}_t$ and $\mathcal{C}'_t$ are degenerate within the irreducible representations (irreps) of the lattice symmetry group. Sharpe and Van de Water observed that contributions from the $\mathcal{O}(p^2a^2)$ source operators destroy would-be relations between the SO(4) violations in the PGB masses and the (axial current) decay constants [35].

Collecting the analytic corrections, we have (in the $4+4+4$ theory)

$$\delta f^{\text{anal}}_{P_t^+} = \frac{64}{f^2}L_4\mu(m_u+m_d+m_s) + \frac{8}{f^2}L_5\mu(m_x+m_y) + a^2\mathcal{F}_t, \quad (36)$$

where the constants $\mathcal{F}_t$ subsume the constants $\mathcal{C}^{(\prime)}_t$. Examining Eqs. (19), (32), and (36), we see that the constants $\mathcal{F}_t$ (for $t\neq 5$) are the only new LECs entering the (NLO) expressions for the decay constants, in the sense that the others are present also in the taste Goldstone case.

IV. RESULTS

To formulate the full QCD and (partially) quenched results in rooted SChPT, we employ the replica method [37–39]. Rooting introduces a factor of $1/4$ in front of the explicit chiral logarithms $l(Q_a)$ in Eqs. (19) and (32) and in the L_4 term in Eq. (36). We must also replace the eigenvalues of the mass matrix (25) with those of the matrix obtained by sending $\delta_a \to \delta_a/4$ there. We have

$$\delta f_{P_F^+} = \delta f^{Z}_{P_F^+} + \delta f^{\text{current}}_{P_F^+} = \delta f^{\text{con}}_{P_F^+} + \delta f^{\text{disc}}_{P_F^+}, \quad (37)$$

$$\delta f^{\text{anal}}_{P_t^+} = \frac{16}{f^2}L_4\mu(m_u+m_d+m_s) + \frac{8}{f^2}L_5\mu(m_x+m_y) + a^2\mathcal{F}_t, \quad (38)$$

where

$$\delta f^{\text{con}}_{P_F^+} \equiv -\frac{1}{32}\sum_{Q,B}g_B l(Q_B), \quad (39)$$

$$\delta f^{\text{disc}}_{P_F^+} \equiv -2\pi^2\int\frac{d^4q}{(2\pi)^4}(D^I_{xx}+D^I_{yy}-2D^I_{xy} + 4D^V_{xx}+4D^V_{yy}-2\Theta^{VF}D^V_{xy}+4D^A_{xx} + 4D^A_{yy}-2\Theta^{AF}D^A_{xy}). \quad (40)$$

In Eq. (40), the flavor-neutral, tree-level masses $(\pi^0_a, \eta_a, \eta'_a)$ appearing in D^a_{ij} have been replaced with the masses obtained by sending $\delta_a \to \delta_a/4$ in the flavor-neutral meson mass matrix. In Eqs. (39) and (40), we summed over a within each SO(4) irrep in Eqs. (19) and (32), B and F represent (taste) SO(4) irreps,

$$B, F \in \{I, V, T, A, P\}, \quad (41)$$

$t\in F$, and

$$\Theta^{BF} \equiv \sum_{a\in B}\theta^{at}, \qquad g_B \equiv \sum_{a\in B}1. \quad (42)$$

The coefficients Θ^{BF} are given in Table I. The loop corrections differ from those in the taste Goldstone case only in the values of the coefficients Θ^{BF}.

TABLE I. The coefficient Θ^{BF} defined in Eq. (42) is in row B and column F.

$B\backslash F$	P	A	T	V	I
V	-4	2	0	-2	4
A	-4	-2	0	2	4

Equation (38) subsumes the NLO analytic corrections in fully dynamical and partially quenched SU(3) SChPT; in the former case, $m_x \neq m_y$ are chosen from m_u, m_d, and m_s. In the quenched case, the L_4 term is dropped. To obtain the NLO analytic corrections in SU(2) SChPT, we drop terms with the heavy quark mass(es), and the LECs become heavy quark mass dependent [40].

Below we give the one-loop contributions to the decay constants for each of these cases. In Sec. IVA 1 and IVA 2, fully dynamical and partially quenched results for the 1 + 1 + 1 and 2 + 1 flavor cases in SU(3) chiral perturbation theory are given. The analogous results in SU(2) chiral perturbation theory are presented in Sec. IV B. In Sec. IVA 3, we write down the results in the quenched case.

A. SU(3) chiral perturbation theory

1. Fully dynamical case

In Eq. (39) Q runs over six flavor combinations, xi and yi for $i \in \{u, d, s\}$. Setting $xy = ud$, us, ds gives the results for the π^+, K^+, and K^0 in the fully dynamical 1 + 1 + 1 flavor case,

$$\delta f^{\rm con}_{\pi^+_F} = -\frac{1}{32}\sum_B g_B(l(U_B) + 2l(\pi^+_B) + l(K^+_B) + l(D_B) + l(K^0_B)), \tag{43}$$

$$\delta f^{\rm con}_{K^+_F} = -\frac{1}{32}\sum_B g_B(l(U_B) + l(\pi^+_B) + 2l(K^+_B) + l(K^0_B) + l(S_B)). \tag{44}$$

$$\delta f^{\rm con}_{K^0_F} = -\frac{1}{32}\sum_B g_B(l(\pi^+_B) + l(D_B) + 2l(K^0_B) + l(K^+_B) + l(S_B)). \tag{45}$$

In the disconnected parts, Eq. (40), the integrals can be performed as explained in Ref. [7]. After performing the integrals and decoupling the η'_I by taking $m_0^2 \to \infty$ [33], we find

$$\begin{aligned}\delta f^{\rm disc}_{\pi^+_F} = \sum_X\Big[&\frac{1}{6}\{R^{DS}_{U\pi^0\eta}(X_I)l(X_I) + R^{US}_{D\pi^0\eta}(X_I)l(X_I) \\ &- 2R^{S}_{\pi^0\eta}(X_I)l(X_I)\} + \frac{1}{4}a^2\delta'_V\{2R^{DS}_{U\pi^0\eta\eta'}(X_V)l(X_V) \\ &+ 2R^{US}_{D\pi^0\eta\eta'}(X_V)l(X_V) - \Theta^{VF}R^{S}_{\pi^0\eta\eta'}(X_V)l(X_V)\} \\ &+ (V\to A)\Big],\end{aligned} \tag{46}$$

$$\begin{aligned}\delta f^{\rm disc}_{K^+_F} = \sum_X\Big[&\frac{1}{6}\{R^{DS}_{U\pi^0\eta}(X_I)l(X_I) + R^{UD}_{S\pi^0\eta}(X_I)l(X_I) \\ &- 2R^{D}_{\pi^0\eta}(X_I)l(X_I)\} + \frac{1}{4}a^2\delta'_V\{2R^{DS}_{U\pi^0\eta\eta'}(X_V)l(X_V) \\ &+ 2R^{UD}_{S\pi^0\eta\eta'}(X_V)l(X_V) - \Theta^{VF}R^{D}_{\pi^0\eta\eta'}(X_V)l(X_V)\} \\ &+ (V\to A)\Big],\end{aligned} \tag{47}$$

$$\begin{aligned}\delta f^{\rm disc}_{K^0_F} = \sum_X\Big[&\frac{1}{6}\{R^{US}_{D\pi^0\eta}(X_I)l(X_I) + R^{UD}_{S\pi^0\eta}(X_I)l(X_I) \\ &- 2R^{U}_{\pi^0\eta}(X_I)l(X_I)\} + \frac{1}{4}a^2\delta'_V\{2R^{US}_{D\pi^0\eta\eta'}(X_V)l(X_V) \\ &+ 2R^{UD}_{S\pi^0\eta\eta'}(X_V)l(X_V) - \Theta^{VF}R^{U}_{\pi^0\eta\eta'}(X_V)l(X_V)\} \\ &+ (V\to A)\Big].\end{aligned} \tag{48}$$

In the sum, X runs over the subscripts of the residue, R, where the residues are defined by

$$R^{A_1A_2\cdots A_k}_{B_1B_2\cdots B_n}(X_F) \equiv \frac{\prod_{A_j}(A_{jF} - X_F)}{\prod_{B_i\neq X}(B_{iF} - X_F)}, \tag{49}$$

where $X \in \{B_1, B_2, \ldots, B_n\}$ and $F \in \{V, A, I\}$ is the $SO(4)_T$ irrep.

The results in the 2 + 1 flavor case are easily obtained by setting $xy = ud$, us and $m_u = m_d$. Equation (39) gives connected contributions for the π and K,

$$\delta f^{\rm con}_{\pi_F} = -\frac{1}{16}\sum_B g_B(2l(\pi_B) + l(K_B)), \tag{50}$$

$$\delta f^{\rm con}_{K_F} = -\frac{1}{32}\sum_B g_B(2l(\pi_B) + 3l(K_B) + l(S_B)). \tag{51}$$

Setting $xy = ud$, us and $m_u = m_d$ in Eq. (40) gives

$$\delta f^{\rm disc}_{\pi_F} = \frac{1}{4}a^2\delta'_V(4 - \Theta^{VF})\sum_X R^{S}_{\pi\eta\eta'}(X_V)l(X_V) + (V\to A) \tag{52}$$

and

$$\begin{aligned}\delta f^{\rm disc}_{K_F} = &\frac{1}{6}\sum_X\{R^{S}_{\pi\eta}(X_I)l(X_I) + R^{\pi}_{S\eta}(X_I)l(X_I)\} - 2l(\eta_I) \\ &+ \frac{1}{4}a^2\delta'_V\sum_X\{2R^{S}_{\pi\eta\eta'}(X_V)l(X_V) \\ &+ 2R^{\pi}_{S\eta\eta'}(X_V)l(X_V) - \Theta^{VF}R_{\eta\eta'}(X_V)l(X_V)\} \\ &+ (V\to A),\end{aligned} \tag{53}$$

where $R_{B_1B_2}(X_F)$ is defined by

$$R_{B_1B_2}(X_F) = \begin{cases} \frac{1}{B_2-B_1} & (X_F = B_1) \\ \frac{1}{B_1-B_2} & (X_F = B_2) \end{cases}. \tag{54}$$

Using the tree-level masses of the taste singlet channel, one finds

$$R^S_{\pi\eta}(\pi_I) = \frac{3}{2}, \qquad R^S_{\pi\eta}(\eta_I) = -\frac{1}{2}, \tag{55}$$

$$R^{\pi}_{S\eta}(S_I) = 3, \qquad R^{\pi}_{S\eta}(\eta_I) = -2. \tag{56}$$

They simplify the results, Eqs. (52) and (53),

$$\delta f^{\rm disc}_{\pi_F} = \frac{1}{4}a^2\delta'_V(4-\Theta^{VF})\left[\frac{S_V-\pi_V}{(\eta_V-\pi_V)(\eta'_V-\pi_V)}l(\pi_V) + \frac{S_V-\eta_V}{(\pi_V-\eta_V)(\eta'_V-\eta_V)}l(\eta_V) + \frac{S_V-\eta'_V}{(\pi_V-\eta'_V)(\eta_V-\eta'_V)}l(\eta'_V)\right] + (V\to A), \tag{57}$$

$$\delta f^{\rm disc}_{K_F} = \frac{1}{12}[3l(\pi_I) - 5l(\eta_I) + 6l(S_I) - 4l(\eta_I)] + \frac{1}{2}a^2\delta'_V\left[\frac{S_V-\pi_V}{(\eta_V-\pi_V)(\eta'_V-\pi_V)}l(\pi_V) + \frac{(\pi_V-\eta_V)^2+(S_V-\eta_V)^2}{(\pi_V-\eta_V)(\eta'_V-\eta_V)(S_V-\eta_V)}l(\eta_V) + \frac{(\pi_V-\eta'_V)^2+(S_V-\eta'_V)^2}{(\pi_V-\eta'_V)(\eta_V-\eta'_V)(S_V-\eta'_V)}l(\eta'_V) + \frac{\pi_V-S_V}{(\eta_V-S_V)(\eta'_V-S_V)}l(S_V) - \frac{1}{2}\Theta^{VF}\frac{1}{\eta_V-\eta'_V}\{l(\eta'_V)-l(\eta_V)\}\right] + (V\to A). \tag{58}$$

2. Partially quenched case

In the partially quenched case, the valence quark masses, m_x and m_y, are not degenerate with the sea quark masses, m_u, m_d, and m_s. The connected contributions to the decay constants in the partially quenched 1 + 1 + 1 flavor case are

$$\delta f^{\rm con}_{P_F^+} = -\frac{1}{32}\sum_{Q,B} g_B l(Q_B). \tag{59}$$

Performing the integrals in Eq. (40) keeping all quark masses distinct gives the disconnected contributions for the partially quenched 1 + 1 + 1 flavor case,

$$\delta f^{\rm disc}_{P_F^+,m_x\neq m_y} = \sum_Z\Big[\frac{1}{6}\{D^{UDS}_{X\pi^0\eta,X}(Z_I)l(Z_I) + D^{UDS}_{Y\pi^0\eta,Y}(Z_I)l(Z_I) - 2R^{UDS}_{XY\pi^0\eta}(Z_I)l(Z_I)\} + \frac{1}{4}a^2\delta'_V\{2D^{UDS}_{X\pi^0\eta\eta',X}(Z_V)l(Z_V) + 2D^{UDS}_{Y\pi^0\eta\eta',Y}(Z_V)l(Z_V) - \Theta^{VF}R^{UDS}_{XY\pi^0\eta\eta'}(Z_V)l(Z_V)\} + (V\to A)\Big] + \frac{1}{6}\{R^{UDS}_{X\pi^0\eta}(X_I)\tilde{l}(X_I) + R^{UDS}_{Y\pi^0\eta}(Y_I)\tilde{l}(Y_I)\} + \frac{1}{2}a^2\delta'_V\{R^{UDS}_{X\pi^0\eta\eta'}(X_V)\tilde{l}(X_V) + R^{UDS}_{Y\pi^0\eta\eta'}(Y_V)\tilde{l}(Y_V)\} + (V\to A), \tag{60}$$

where

$$D^{A_1A_2\cdots A_k}_{B_1B_2\cdots B_n,B_i}(X_F) \equiv -\frac{\partial}{\partial B_{iF}}R^{A_1A_2\cdots A_k}_{B_1B_2\cdots B_n}(X_F) \tag{61}$$

and

$$\tilde{l}(X) \equiv -(\ln X/\Lambda^2 + 1) + \delta_3(\sqrt{X}L). \tag{62}$$

Here $\delta_3(\sqrt{X}L)$ is the finite volume correction defined in Ref. [6], and X and Y represent the squared tree-level masses of $x\bar{x}$ and $y\bar{y}$ PGBs, respectively.

For $m_x = m_y$, we find

$$\delta f^{\rm disc}_{P_F^+,m_x=m_y} = \frac{1}{4}a^2\delta'_V(4-\Theta^{VF})\Big[R^{UDS}_{X\pi^0\eta\eta'}(X_V)\tilde{l}(X_V) + \sum_Z D^{UDS}_{X\pi^0\eta\eta',X}(Z_V)l(Z_V)\Big] + (V\to A). \tag{63}$$

The connected contributions in the 2 + 1 flavor case are obtained by setting $m_u = m_d$ in Eq. (59). To obtain the disconnected contributions, we perform the integrals in Eq. (40) setting $m_u = m_d$. For $m_x \neq m_y$, we find

$$\delta f^{\rm disc}_{P_F^+,m_x\neq m_y} = \sum_Z\bigg[\frac{1}{6}\{D^{\pi S}_{X\eta,X}(Z_I)l(Z_I) + D^{\pi S}_{Y\eta,Y}(Z_I)l(Z_I) - 2R^{\pi S}_{XY\eta}(Z_I)l(Z_I)\} + \frac{1}{4}a^2\delta'_V\{2D^{\pi S}_{X\eta\eta',X}(Z_V)l(Z_V) + 2D^{\pi S}_{Y\eta\eta',Y}(Z_V)l(Z_V) - \Theta^{VF}R^{\pi S}_{XY\eta\eta'}(Z_V)l(Z_V)\} + (V\to A)\bigg] + \frac{1}{6}\{R^{\pi S}_{X\eta}(X_I)\tilde{l}(X_I) + R^{\pi S}_{Y\eta}(Y_I)\tilde{l}(Y_I)\} + \frac{1}{2}a^2\delta'_V\{R^{\pi S}_{X\eta\eta'}(X_V)\tilde{l}(X_V) + R^{\pi S}_{Y\eta\eta'}(Y_V)\tilde{l}(Y_V)\} + (V\to A). \quad (64)$$

For $m_x = m_y$, we find

$$\delta f^{\rm disc}_{P_F^+,m_x=m_y} = \frac{1}{4}a^2\delta_V(4-\Theta^{VF})\bigg[R^{\pi S}_{X\eta\eta'}(X_V)\tilde{l}(X_V) + \sum_Z D^{\pi S}_{X\eta\eta',X}(Z_V)l(Z_V)\bigg] + (V\to A). \quad (65)$$

3. Quenched case

In the quenched case, there is no connected contribution, Eq. (39). As explained in Refs. [6,7,41], quenching the sea quarks in the disconnected part can be done by replacing the disconnected propagator with

$$D^{a,\rm quench}_{il} = -\frac{\delta^{\rm quench}_a}{(q^2+I_a)(q^2+L_a)}, \quad (66)$$

where

$$\delta^{\rm quench}_a = \begin{cases} 4(m_0^2+\alpha q^2)/3 & \text{if } a = I \\ \delta_a & \text{if } a \neq I \end{cases}. \quad (67)$$

Here, note that I_a and L_a represent the squared tree-level masses of $i\bar{i}$ and $l\bar{l}$ PGBs, respectively, while I represents the taste-singlet irrep.

Replacing D^a_{il} with the quenched disconnected propagator in Eq. (40) for $m_x \neq m_y$ gives

$$\delta f^{\rm disc}_{P_F^+,m_x\neq m_y} = \frac{\alpha}{6}\bigg\{\frac{Y_I+X_I}{Y_I-X_I}(l(X_I)-l(Y_I)) - X_I\tilde{l}(X_I) - Y_I\tilde{l}(Y_I)\bigg\} + \frac{m_0^2}{6}\bigg\{\tilde{l}(X_I)+\tilde{l}(Y_I) - 2\frac{l(X_I)-l(Y_I)}{Y_I-X_I}\bigg\} + \frac{1}{4}a^2\delta'_V\bigg\{2\tilde{l}(X_V)+2\tilde{l}(Y_V) - \Theta^{VF}\frac{l(X_V)-l(Y_V)}{Y_V-X_V}\bigg\} + (V\to A), \quad (68)$$

and for $m_x = m_y$,

$$\delta f^{\rm disc}_{P_F^+,m_x=m_y} = \frac{1}{4}a^2\delta'_V(4-\Theta^{VF})\tilde{l}(X_V) + (V\to A). \quad (69)$$

Quenching the sea quarks also affects the analytic terms. In the quenched version of Eq. (36), there is no L_4 term, which is coming from the sea quarks.

B. SU(2) chiral perturbation theory

We obtain the SU(2) SChPT results from the SU(3) SChPT results by using the prescription of Ref. [40]. SU(2) chiral perturbation theory was developed in Ref. [42] and applied to simulation data for the taste Goldstone decay constants in Refs. [43–45]. The results of this section extend the results of Refs. [45,46] to the taste non-Goldstone case.

1. Fully dynamical case

From Eqs. (43)–(45), we obtain the connected contributions for the fully dynamical 1 + 1 + 1 flavor case ($m_u \neq m_d \ll m_s$),

$$\delta f^{\rm con}_{\pi_F^+} = -\frac{1}{32}\sum_B g_B(l(U_B) + 2l(\pi_B^+) + l(D_B)), \quad (70)$$

$$\delta f^{\rm con}_{K_F^+} = -\frac{1}{32}\sum_B g_B(l(U_B) + l(\pi_B^+)). \quad (71)$$

$$\delta f^{\rm con}_{K_F^0} = -\frac{1}{32}\sum_B g_B(l(D_B) + l(\pi_B^+)). \quad (72)$$

For the disconnected contributions, we find from Eqs. (46)–(48),

$$\delta f^{\rm disc}_{\pi_F^+} = \frac{1}{2}(l(U_I)+l(D_I)) - l(\pi_I^0) + \frac{1}{4}a^2\delta'_V\sum_X\{2R^D_{U\pi^0\eta}(X_V)l(X_V) + 2R^U_{D\pi^0\eta}(X_V)l(X_V)\} + \frac{1}{4}a^2\delta'_V\Theta^{VF}\frac{l(\eta_V)-l(\pi_V^0)}{\eta_V-\pi_V^0} + (V\to A), \quad (73)$$

$$\delta f^{\rm disc}_{K_F^+} = \frac{1}{2}l(U_I) - \frac{1}{4}l(\pi_I^0) + \frac{1}{2}a^2\delta'_V\sum_X R^D_{U\pi^0\eta}(X_V)l(X_V) + (V\to A), \quad (74)$$

and

$$\delta f^{\rm disc}_{K^0_F} = \frac{1}{2} l(D_I) - \frac{1}{4} l(\pi^0_I) + \frac{1}{2} a^2 \delta'_V \sum_X R^U_{D\pi^0\eta}(X_V) l(X_V) + (V \to A). \tag{75}$$

The connected contributions in the fully dynamical $2+1$ flavor case ($m_u = m_d \ll m_s$) are

$$\delta f^{\rm con}_{\pi_F} = -\frac{1}{8} \sum_B g_B l(\pi_B), \tag{76}$$

$$\delta f^{\rm con}_{K_F} = -\frac{1}{16} \sum_B g_B l(\pi_B). \tag{77}$$

For the disconnected contributions in the fully dynamical $2+1$ flavor case, we find

$$\delta f^{\rm disc}_{\pi_F} = \frac{1}{2}(4 - \Theta^{VF})\{l(\pi_V) - l(\eta_V)\} + (V \to A), \tag{78}$$

$$\delta f^{\rm disc}_{K_F} = \frac{1}{4} l(\pi_I) + l(\pi_V) - l(\eta_V) + (V \to A). \tag{79}$$

2. Partially quenched case

Considering x and y to be light quarks ($m_s \gg m_u, m_d, m_x, m_y$), the connected contributions to the decay constants in the partially quenched $1+1+1$ flavor case can be obtained by dropping terms corresponding to strange sea quark loops from Eq. (59). Equations (60), (63), and (40) give the disconnected contributions,

$$\begin{aligned}\delta f^{\rm disc}_{P^+_F, m_x \neq m_y} = & \sum_Z \Big[\frac{1}{4}\{D^{UD}_{X\pi^0,X}(Z_I) l(Z_I) + D^{UD}_{Y\pi^0,Y}(Z_I) l(Z_I) - 2R^{UD}_{XY\pi^0}(Z_I) l(Z_I)\} \\ & + \frac{1}{4} a^2 \delta'_V \{2D^{UD}_{X\pi^0\eta,X}(Z_V) l(Z_V) + 2D^{UD}_{Y\pi^0\eta,Y}(Z_V) l(Z_V) - \Theta^{VF} R^{UD}_{XY\pi^0\eta}(Z_V) l(Z_V)\} + (V \to A)\Big] \\ & + \frac{1}{4}\{R^{UD}_{X\pi^0}(X_I)\tilde{l}(X_I) + R^{UD}_{Y\pi^0}(Y_I)\tilde{l}(Y_I)\} + \frac{1}{2} a^2 \delta'_V \{R^{UD}_{X\pi^0\eta}(X_V)\tilde{l}(X_V) + R^{UD}_{Y\pi^0\eta}(Y_V)\tilde{l}(Y_V)\} + (V \to A)\end{aligned} \tag{80}$$

and

$$\delta f^{\rm disc}_{P^+_F, m_x = m_y} = \frac{1}{4} a^2 \delta'_V (4 - \Theta^{VF}) \Big[R^{UD}_{X\pi^0\eta}(X_V)\tilde{l}(X_V) + \sum_Z D^{UD}_{X\pi^0\eta,X}(Z_V) l(Z_V)\Big] + (V \to A). \tag{81}$$

The connected contributions to the decay constants in the partially quenched $2+1$ flavor case can be obtained by setting $m_u = m_d$ and decoupling the strange quark in the $1+1+1$ flavor case, Eq. (59). From Eqs. (60) and (63), we find the disconnected contributions in the $2+1$ flavor case,

$$\begin{aligned}\delta f^{\rm disc}_{P^+_F, m_x \neq m_y} = & \sum_Z \Big[-\frac{1}{2} R^\pi_{XY}(Z_I) l(Z_I) + \frac{1}{4} a^2 \delta'_V \{2D^\pi_{X\eta,X}(Z_V) l(Z_V) + 2D^\pi_{Y\eta,Y}(Z_V) l(Z_V) - \Theta^{VF} R^\pi_{XY\eta}(Z_V) l(Z_V)\} + (V \to A)\Big] \\ & + \frac{1}{4}\{l(X_I) + (\pi_I - X_I)\tilde{l}(X_I) + l(Y_I) + (\pi_I - Y_I)\tilde{l}(Y_I)\} + \frac{1}{2} a^2 \delta'_V \{R^\pi_{X\eta}(X_V)\tilde{l}(X_V) + R^\pi_{Y\eta}(Y_V)\tilde{l}(Y_V)\} + (V \to A)\end{aligned} \tag{82}$$

and

$$\delta f^{\rm disc}_{P^+_F, m_x = m_y} = \frac{1}{4} a^2 \delta'_V (4 - \Theta^{VF}) \Big[R^\pi_{X\eta}(X_V)\tilde{l}(X_V) + \sum_Z D^\pi_{X\eta,X}(Z_V) l(Z_V)\Big] + (V \to A). \tag{83}$$

Considering x to be a light quark and y to be a heavy quark ($m_s, m_y \gg m_u, m_d, m_x$), the connected contributions to the decay constants can be obtained by dropping terms from Eq. (59) corresponding to strange sea quarks and y valence quarks circulating in loops; i.e., only the xu and xd terms survive in the sum over Q. From Eqs. (80), (82), and (40) [or alternatively, Eqs. (60) and (64)], we find the disconnected contribution for the partially quenched $1+1+1$ flavor case,

$$\begin{aligned}\delta f^{\rm disc}_{P^+_F} = & \frac{1}{4}\sum_Z [D^{UD}_{X\pi^0,X}(Z_I) l(Z_I) + 2a^2 \delta'_V D^{UD}_{X\pi^0\eta,X}(Z_V) l(Z_V) \\ & + (V \to A)] + \frac{1}{4} R^{UD}_{X\pi^0}(X_I)\tilde{l}(X_I) \\ & + \frac{1}{2} a^2 \delta'_V R^{UD}_{X\pi^0\eta}(X_V)\tilde{l}(X_V) + (V \to A).\end{aligned} \tag{84}$$

For the $2+1$ flavor case, we find

$$\begin{aligned}\delta f^{\rm disc}_{P^+_F} = & \sum_Z \Big[\frac{1}{2} a^2 \delta'_V D^\pi_{X\eta,X}(Z_V) l(Z_V) + (V \to A)\Big] \\ & + \frac{1}{4}\{l(X_I) + (\pi_I - X_I)\tilde{l}(X_I)\} \\ & + \frac{1}{2} a^2 \delta'_V R^\pi_{X\eta}(X_V)\tilde{l}(X_V) + (V \to A).\end{aligned} \tag{85}$$

V. CONCLUSION

Our results for the decay constants are given compactly by Eq. (13) with Eqs. (37) through (38); they reduce to those of Ref. [31] in the taste Goldstone sector. The only new LECs are those parametrizing the analytic corrections proportional to a^2; the SO(4)-violating contributions are independent of those in the masses. As shown in Table I, the factors Θ^{BF} multiplying the disconnected pieces of the propagators $D^{V,A}_{xy}$ differ from the coefficients in the taste Goldstone case, but no new LECs arise in the loop diagrams. In SU(2) chiral perturbation theory with a heavy valence quark, the chiral logarithms are the same in all taste channels; only the analytic $\mathcal{O}(a^2)$ corrections differ.

Results for special cases of interest can be obtained by expanding the disconnected pieces of the propagators in Eq. (40). For the fully dynamical case with three nondegenerate quarks, the loop corrections in the SU(3) chiral theory are in Eqs. (43)–(48). Results in the isospin limit are in Eqs. (50)–(58). For the partially quenched case with three nondegenerate sea quarks, loop corrections in the SU(3) chiral theory are in Eqs. (59)–(63). Results in the isospin limit are in Eqs. (64) and (65). For the quenched case the results are in Eqs. (68) and (69). Results in SU(2) chiral perturbation theory are in Eqs. (70)–(85). These results can be used to improve determinations of the decay constants, quark masses, and the Gasser-Leutwyler LECs by analyzing lattice data from taste non-Goldstone channels.

ACKNOWLEDGMENTS

W. Lee is supported by the Creative Research Initiatives program (Grant No. 2012-0000241) of the NRF grant funded by the Korean government (MEST). W. Lee acknowledges support from the KISTI supercomputing center through the Strategic Support program [Grant No. KSC-2011-G2-06].

[1] G. Colangelo, S. Durr, A. Juttner, L. Lellouch, H. Leutwyler *et al.*, Eur. Phys. J. C **71**, 1695 (2011).
[2] J. Beringer *et al.* (Particle Data Group), Phys. Rev. D **86**, 010001 (2012).
[3] M. F. Golterman and J. Smit, Nucl. Phys. **B245**, 61 (1984).
[4] M. F. Golterman, Nucl. Phys. **B273**, 663 (1986).
[5] W.-J. Lee and S. R. Sharpe, Phys. Rev. D **60**, 114503 (1999).
[6] C. Bernard (MILC Collaboration), Phys. Rev. D **65**, 054031 (2002).
[7] C. Aubin and C. Bernard, Phys. Rev. D **68**, 034014 (2003).
[8] C. Aubin and C. Bernard, Nucl. Phys. B, Proc. Suppl. **129–130**, 182 (2004).
[9] C. Aubin *et al.* (HPQCD Collaboration, MILC Collaboration, and UKQCD Collaboration), Phys. Rev. D **70**, 031504 (2004).
[10] C. Aubin, C. Bernard, C. DeTar, J. Osborn, S. Gottlieb, E. Gregory, D. Toussaint, U. Heller, J. Hetrick, and R. Sugar (MILC Collaboration), Phys. Rev. D **70**, 114501 (2004).
[11] C. Aubin *et al.* (Fermilab Lattice Collaboration, MILC Collaboration, and HPQCD Collaboration), Phys. Rev. Lett. **94**, 011601 (2005).
[12] C. Aubin, C. Bernard, C. E. DeTar, M. Di Pierro, E. D. Freeland *et al.*, Phys. Rev. Lett. **95**, 122002 (2005).
[13] A. Gray, M. Wingate, C. Davies, E. Gulez, G. Lepage, Q. Mason, M. Nobes, and J. Shigemitsu (HPQCD Collaboration), Phys. Rev. Lett. **95**, 212001 (2005).
[14] C. Aubin and T. Blum, Proc. Sci., LAT (2005) 089.
[15] M. Okamoto, Proc. Sci., LAT (2005) 013.
[16] E. Gulez, A. Gray, M. Wingate, C. T. Davies, G. Lepage, and J. Shigemitsu, Phys. Rev. D **73**, 074502 (2006).
[17] C. Aubin and T. Blum, Phys. Rev. D **75**, 114502 (2007).
[18] C. Bernard, C. E. DeTar, M. Di Pierro, A. El-Khadra, R. Evans *et al.*, Phys. Rev. D **79**, 014506 (2009).
[19] J. A. Bailey, C. Bernard, C. E. DeTar, M. Di Pierro, A. El-Khadra *et al.*, Phys. Rev. D **79**, 054507 (2009).
[20] E. Gamiz, C. T. Davies, G. Lepage, J. Shigemitsu, and M. Wingate (HPQCD Collaboration), Phys. Rev. D **80**, 014503 (2009).
[21] A. Bazavov, D. Toussaint, C. Bernard, J. Laiho, C. DeTar *et al.*, Rev. Mod. Phys. **82**, 1349 (2010).
[22] T. Bae, Y.-C. Jang, C. Jung, H.-J. Kim, Jangho Kim, Jongjeong Kim, K. Kim, W. Lee, S. R. Sharpe, and B. Yoon, Phys. Rev. D **82**, 114509 (2010).
[23] J. Kim, C. Jung, H.-J. Kim, W. Lee, and S. R. Sharpe, Phys. Rev. D **83**, 117501 (2011).
[24] T. Bae, Y.-C. Jang, C. Jung, H.-J. Kim, J. Kim *et al.* Phys. Rev. Lett. **109**, 041601 (2012).
[25] A. Bazavov *et al.* (Fermilab Lattice and MILC Collaborations), Phys. Rev. D **85**, 114506 (2012).
[26] J. A. Bailey, A. Bazavov, C. Bernard, C. Bouchard, C. DeTar *et al.*, Phys. Rev. D **85**, 114502 (2012); **86**, 039904 (E) (2012).
[27] A. Bazavov, C. Bernard, C. Bouchard, C. DeTar, M. Di Pierro *et al.* Phys. Rev. D **86**, 034503 (2012).
[28] A. Bazavov *et al.* (MILC Collaboration), Proc. Sci., LAT (2009) 079.
[29] S. Aoki *et al.* (JLQCD Collaboration), Phys. Rev. D **62**, 094501 (2000).
[30] C. Bernard (private communication).
[31] C. Aubin and C. Bernard, Phys. Rev. D **68**, 074011 (2003).
[32] J. A. Bailey, H.-J. Kim, and W. Lee (SWME Collaboration) Phys. Rev. D **85**, 094503 (2012).

[33] S. R. Sharpe and N. Shoresh, Phys. Rev. D **64**, 114510 (2001).
[34] J. Gasser and H. Leutwyler, Nucl. Phys. **B250**, 465 (1985).
[35] S. R. Sharpe and R. S. Van de Water, Phys. Rev. D**71**, 114505 (2005).
[36] S. R. Sharpe and N. Shoresh, Phys. Rev. D **62**, 094503 (2000).
[37] C. W. Bernard and M. F. Golterman, Phys. Rev. D **49**, 486 (1994).
[38] P. Damgaard and K. Splittorff, Phys. Rev. D **62**, 054509 (2000).
[39] C. Bernard, M. Golterman, and Y. Shamir, Phys. Rev. D **77**, 074505 (2008).
[40] J. A. Bailey, H.-J. Kim, W. Lee, and S. R. Sharpe, Phys. Rev. D **85**, 074507 (2012).
[41] C. W. Bernard and M. F. Golterman, Phys. Rev. D **46**, 853 (1992).
[42] J. Gasser and H. Leutwyler, Ann. Phys. (N.Y.) **158**, 142 (1984).
[43] A. Bazavov *et al.* (MILC Collaboration), Proc. Sci., CD (2009) 007.
[44] A. Bazavov *et al.* (The MILC Collaboration), Proc. Sci., LAT (2009) 077.
[45] A. Bazavov, C. Bernard, C. DeTar, X. Du, W. Freeman *et al.*, Proc. Sci., LATTICE (2010) 083.
[46] X. Du, Phys. Rev. D **82**, 014016 (2010).