

Horizontal symmetries $\Delta(150)$ and $\Delta(600)$

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(Received 9 January 2013; published 18 March 2013)

Using the group theory of mixing to examine all finite subgroups of $SU(3)$ with an order less than 512, we found recently that only the group $\Delta(150)$ can give rise to a correct reactor angle θ_{13} of neutrino mixing without any free parameters. It predicts $\sin^2 2\theta_{13} = 0.11$ and a submaximal atmospheric angle with $\sin^2 2\theta_{23} = 0.94$, in good agreement with experiment. The solar angle θ_{12} , the charge conjugation-parity phase δ , and the neutrino masses m_i are left as free parameters. In this article we provide more details of this case and discuss possible gains and losses by introducing right-handed symmetries and/or valons to construct dynamical models. A simple model is discussed where the solar angle agrees with experiment and all its mixing parameters can be obtained from the group $\Delta(600)$ by symmetry alone. The promotion of $\Delta(150)$ to $\Delta(600)$ is, on one hand, analogous to the promotion of S_3 to S_4 in the presence of tribimaximal mixing, and on the other hand, it is similar to the extension from A_4 to S_4 in that case.

DOI: [10.1103/PhysRevD.87.053012](https://doi.org/10.1103/PhysRevD.87.053012)

PACS numbers: 14.60.Pq

I. INTRODUCTION

Before the reactor angle θ_{13} was successfully measured in the recent past, the neutrino mixing data were consistent with a zero θ_{13} and a maximal atmospheric angle θ_{23} . These two are explainable by the leptonic symmetry S_3 . This symmetry group is generated by two unitary matrices, $F = \text{diag}(1, \omega, \omega^2)$, where $\omega = \exp(2\pi i/3)$, and G with $G^2 = 1$, $\det(G) = 1$, which possesses the eigenvector $(0, 1, -1)^T$ with eigenvalue $+1$. This symmetry predicts a zero reactor angle and a maximal atmospheric angle, but it leaves the solar angle θ_{12} undetermined.

The solar angle can also be explained by symmetry if we enlarge the group S_3 by incorporating another unitary generator G' which commutes with G , satisfying $G'^2 = 1$, $\det(G') = 1$, and possessing an eigenvector $(1, 1, 1)^T$ with eigenvalue $+1$. The order of the group \mathcal{G} generated by F , G , G' must then be an even multiple of 6, the order of S_3 .

The simplest non-Abelian group with order 12 is A_4 , but it contains only even permutations so it does not contain S_3 as a subgroup. The simplest nontrivial group that contains S_3 is S_4 , with order 24. Moreover, it also contains the G' above, so it predicts the correct solar angle. The resulting neutrino matrix has the well-known tribimaximal form.

Now that the reactor angle is no longer zero and there are indications that the atmospheric angle may not be maximal, the question is whether we can still find a symmetry to explain the mixing data. In a recent work, we searched all the finite subgroups of $SU(3)$ with an order less than 512 and discovered that the only group that can explain the reactor and atmospheric angles by symmetry alone is $\Delta(150)$ [1]. It is generated by the same F as S_3 , but a different G . It also leaves the solar angle free. To explain the solar angle by symmetry as well, we need to enlarge the group by

incorporating another order-2 generator G' which commutes with G . As before, the extended group \mathcal{G} must have an order which is an even multiple of 150, the order of $\Delta(150)$. Again no nontrivial order-300 group contains a $\Delta(150)$ subgroup, but the order-600 group $\Delta(600)$ does, and in fact that is the only order-600 group that does. The symmetry of $\Delta(600)$ also contains the same G' as before, which gives rise to a solar angle that agrees with experiment. In this way the pair $(\Delta(150), \Delta(600))$ is analogous to the previous pair (S_3, S_4) before the reactor angle was measured.

It is interesting that $\Delta(600)$ is the smallest of the three groups that yield a full $Z_2 \times Z_2$ residual symmetry in the neutrino sector, among all the finite $SU(3)$ subgroups of orders < 1536 [2]. It also shows up in the method using von Dyck groups [3,4].

In this article we discuss the groups $\Delta(150)$ and $\Delta(600)$ and their predictions [5]. The mathematical properties of these groups are summarized in Sec. II. Given a symmetry group, there are three ways to apply its symmetries to neutrino physics, each with an additional assumption over the previous, but potentially (though not necessarily) also an additional gain. We will discuss these three in turn. The simplest is to apply symmetry only to the left-handed fermions, to obtain information on the effective left-handed mass matrices. This is the method used in the recent survey [1] and shall be referred to as the “left-handed symmetry” method. The next simplest is to apply symmetry to both the left-handed and the right-handed fermions, to obtain information also on the Dirac and Majorana mass matrices. We shall refer to this method as “both-handed symmetry”. Without imposing the additional right-handed symmetry there is no way to get the Dirac or the Majorana mass matrices. The most involved but the most well-known method is to construct dynamical models. This requires the introduction of additional scalar fields called valons to account for the spontaneous

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breaking of symmetry. These three methods will be applied to $\Delta(150)$ and their relative merits will be discussed in Secs. III, IV, and V, respectively.

No matter which of these three methods is used, we can account for the correct reactor and atmospheric angles as long as $\Delta(150)$ is the symmetry. The other neutrino mixing parameters, as well as the fermion masses, are hidden in the adjustable parameters in the mass matrices. For left-handed symmetry, there are just enough parameters to fit all the experimental quantities. For both-handed symmetry, the additional freedom in assigning right-handed symmetry increases the number of parameters, but except for special cases such as the $c = 0$ example in the text, the additional parameters do not yield additional information on the mixing parameters or the masses. Nevertheless, when dynamical models are built, most of these parameters in both-handed symmetry will acquire a physical meaning in terms of Yukawa coupling constants. The groups $\Delta(6n^2)$ can also be incorporated into grand-unified-theory models [6].

In principle, a fit to the experimental data can determine the parameters in left-handed symmetry, but in practice this is impossible because the CP phase and one of the neutrino masses are unknown. The best one can do then is to produce simple models, hopefully with simple choices of parameters, that can account for all the known experimental quantities. Such a model will be given in the text, first in Sec. III, and then subsequently discussed and refined in Secs. IV and V.

In Sec. VI, the left-handed symmetry will be applied to $\Delta(600)$, and in Sec. VII, a summary of the results will be given. There is also an Appendix to explain real representations for the group $\Delta(150)$.

II. $\Delta(6n^2)$

This series of finite subgroups of $SU(3)$ have been studied in various places [7]. In this article we will follow the information given in the mathematical software GAP [8].

$\Delta(6n^2)$ can be described by the semidirect products of a direct product of cyclic groups,

$$\Delta(6n^2) = (Z_n \times Z_n) \rtimes Z_3 \rtimes Z_2. \quad (1)$$

The generators of the two cyclic groups Z_n are f_4 and f_3 , respectively; the generator of Z_3 is f_2 , and the generator of Z_2 is f_1 . The orders of f_4 , f_3 , f_2 , and f_1 are, respectively, n , n , 3, and 2.

The direct product of the two Z_n 's tells us that f_4 and f_3 commute. The semidirect product with Z_3 implies that $f_2 f_3 f_2^{-1}$ and $f_2 f_4 f_2^{-1}$ are monomials of f_3 and f_4 . The semidirect product with Z_2 implies that $f_1 f_3 f_1^{-1}$ and $f_1 f_4 f_1^{-1}$ are monomials of f_2 , f_3 , and f_4 , while $f_1 f_2 f_1^{-1}$ is a monomial of f_2 .

With these commutation relations, one can rearrange any monomial of the four f_i 's, namely, any group element of $\Delta(6n^2)$, into the canonical form $f_4^{e_4} f_3^{e_3} f_2^{e_2} f_1^{e_1}$, with e_4 and e_3 ranging between 0 and $n - 1$, e_2 between 0 and 2,

and e_1 as either 0 or 1. The total number of independent elements of the group can easily be counted from the canonical form to be $6n^2$.

In this article we are interested in the case $n = 5$ and $n = 10$. We will discuss the details of these two groups separately below.

A. $\Delta(150) = (Z_5 \times Z_5) \rtimes Z_3 \rtimes Z_2$

The nontrivial commutation relations of the generators are

$$\begin{aligned} f_2 f_4 f_2^{-1} &= f_4 f_3, & f_2 f_3 f_2^{-1} &= f_4^2 f_3^3, & f_1 f_4 f_1^{-1} &= f_4^4 f_3^4, \\ f_1 f_3 f_1^{-1} &= f_3, & f_1 f_2 f_1^{-1} &= f_2^2. \end{aligned} \quad (2)$$

This group has 13 conjugacy classes, \mathcal{C}_1 to \mathcal{C}_{13} , and 13 irreducible representations, IR1 to IR13. Its character table is given in Table I, with various notations explained in Table II and below. The GAP notation $/A$ in Table I means the complex conjugate of A , and likewise for $/B$, $/F$, and $/G$. The notation $*C$ is obtained from C by changing $\sqrt{5}$ to $-\sqrt{5}$, and likewise for $*D$ and $*E$.

The designations of the IRs in Table I are those in the IrreducibleRepresentations command of GAP. They differ from those used in the command CharacterTable, which are denoted as $X.n$ in GAP, with n varying from 1 to 13.

The generators f_1, f_2, f_3, f_4 belong to classes $\mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4, \mathcal{C}_5$, respectively. The only class with order 2 is \mathcal{C}_2 . No two elements within this class mutually commute.

As usual, the dimension of an IR is its character for the identity class \mathcal{C}_1 . The characters χ of the three-dimensional representations IR4 to IR11 are complex, with the pairs (4,10), (5,11), (6,8), (7,9) being complex conjugates of one another. The other representations, IR1 to IR3, IR12, and IR13, have real characters, so they are either real or quaternionic representations. Their Frobenius-Schur indices $\sum_{g \in \Delta(150)} \chi(g^2)/150$ are all +1, so all of them are real representations. The representations given by GAP for IR1 and IR2 are explicitly real, but not those for IR3, IR12, or IR13. In the Appendix we show how to convert the GAP representations into representations that are explicitly real.

Using Table I, we can explicitly work out the Clebsch-Gordan series. The result is given in Table III. Since $\text{IR}_i \times \text{IR}_j$ is the same as $\text{IR}_j \times \text{IR}_i$, only half of the table is explicitly shown. For example, the products of IR12 and IR12 are given by (with the prefix IR omitted)

$$12 \times 12 = 1 + 2 + 3 + 3 + 6 + 7 + 8 + 9 + 12 + 12 + 13. \quad (3)$$

From Table II, it can be checked that there are 36 states both on the left-hand side and the right-hand side of (3).

B. $\Delta(600) = (Z_{10} \times Z_{10}) \rtimes Z_3 \rtimes Z_2$

Since $Z_{10} = Z_5 \times Z_2'$ (a prime is used to distinguish this Z_2 from the one in the semidirect product), its generator f_4 is related to the Z_5 generator c_4 and the Z_2' generator d_4 by

TABLE I. Character table of $\Delta(150)$.

| | | | | | | | | | | | | | |
|-------|---|----|----|------|------|-------|-------|------|------|-------|-------|------|------|
| C | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
| Order | 1 | 2 | 3 | 5 | 5 | 10 | 10 | 5 | 5 | 10 | 10 | 5 | 5 |
| $ C $ | 1 | 15 | 50 | 3 | 6 | 15 | 15 | 3 | 6 | 15 | 15 | 3 | 3 |
| IR1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| IR2 | 1 | -1 | 1 | 1 | 1 | -1 | -1 | 1 | 1 | -1 | -1 | 1 | 1 |
| IR3 | 2 | 0 | -1 | 2 | 2 | 0 | 0 | 2 | 2 | 0 | 0 | 2 | 2 |
| IR4 | 3 | 1 | 0 | $/B$ | $*D$ | $-/G$ | $-/F$ | $/A$ | D | $-F$ | $-G$ | A | B |
| IR5 | 3 | -1 | 0 | $/B$ | $*D$ | $/G$ | $/F$ | $/A$ | D | F | G | A | B |
| IR6 | 3 | 1 | 0 | $/A$ | D | $-/F$ | $-G$ | B | $*D$ | $-/G$ | $-F$ | $/B$ | A |
| IR7 | 3 | -1 | 0 | $/A$ | D | $/F$ | G | B | $*D$ | $/G$ | F | $/B$ | A |
| IR8 | 3 | 1 | 0 | A | D | $-F$ | $-/G$ | $/B$ | $*D$ | $-G$ | $-/F$ | B | $/A$ |
| IR9 | 3 | -1 | 0 | A | D | F | $/G$ | $/B$ | $*D$ | G | $/F$ | B | $/A$ |
| IR10 | 3 | 1 | 0 | B | $*D$ | $-G$ | $-F$ | A | D | $-/F$ | $-/G$ | $/A$ | $/B$ |
| IR11 | 3 | -1 | 0 | B | $*D$ | G | F | A | D | $/F$ | $/G$ | $/A$ | $/B$ |
| IR12 | 6 | 0 | 0 | C | E | 0 | 0 | $*C$ | $*E$ | 0 | 0 | $*C$ | C |
| IR13 | 6 | 0 | 0 | $*C$ | $*E$ | 0 | 0 | C | E | 0 | 0 | C | $*C$ |

TABLE II. Abbreviations used in Table I, where $\eta = e^{2\pi i/5}$.

| | | | | | | |
|--------------------|--------------------|----------------|--------------------|---------------------|-----------|-----------|
| A | B | C | D | E | F | G |
| $2\eta^3 + \eta^4$ | $\eta^2 + 2\eta^4$ | $1 - \sqrt{5}$ | $(1 - \sqrt{5})/2$ | $(-3 + \sqrt{5})/2$ | $-\eta^4$ | $-\eta^2$ |

$$c_4 = f_4^2, \quad d_4 = f_4^5, \quad f_4 = c_4^3 d_4. \quad (4)$$

Similarly, for the other Z_{10} in the direct product,

$$c_3 = f_3^2, \quad d_3 = f_3^5, \quad f_3 = c_3^3 d_3. \quad (5)$$

The nontrivial commutation relations between f_1, f_2, c_3, c_4 are the same as in (2), namely,

$$f_2 c_4 f_2^{-1} = c_4 c_3, \quad f_2 c_3 f_2^{-1} = c_4^2 c_3^3, \quad f_1 c_4 f_1^{-1} = c_4^4 c_4^4, \\ f_1 c_3 f_1^{-1} = c_3, \quad f_1 f_2 f_1^{-1} = f_2^2. \quad (6)$$

Together they generate the subgroup $\Delta(150)$ of $\Delta(600)$.

The other nontrivial commutation relations of $\Delta(600)$ are

$$f_2 d_4 f_2^{-1} = d_3, \quad f_2 d_3 f_2^{-1} = d_4 d_3, \\ f_1 d_4 f_1^{-1} = d_4 d_3, \quad f_1 d_3 f_1^{-1} = d_3. \quad (7)$$

The group formed by $\{f_1, f_2, d_3\}$ (or $\{f_1, f_2, d_4\}$) is S_4 ; thus, $\Delta(600)$ contains S_4 as a subgroup. The subgroup formed by $\{f_2, d_3\}$ (or $\{f_2, d_4\}$) is A_4 , and the subgroup formed by $\{f_1, d_3\}$ (or $\{f_1, d_4\}$) is the Klein four-group $Z_2 \times Z_2$ [9].

TABLE III. Clebsch-Gordan series of $\Delta(150)$.

| | | | | | | | | | | | | | |
|----|---|---|-------|---------|---------|--------|--------|--------|--------|--------|--------|--------------------------|--------------------------|
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
| 1 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
| 2 | | 1 | 3 | 5 | 4 | 7 | 6 | 9 | 8 | 11 | 10 | 12 | 13 |
| 3 | | | 1,2,3 | 4,5 | 4,5 | 6,7 | 6,7 | 8,9 | 8,9 | 10,11 | 10,11 | 12 ² | 13 ² |
| 4 | | | | 6,10,11 | 7,10,11 | 8,12 | 9,12 | 10,13 | 11,13 | 1,3,12 | 2,3,12 | 4,5,8,9,13 | 6,7,12,13 |
| 5 | | | | | 6,10,11 | 9,12 | 8,12 | 11,13 | 10,13 | 2,3,12 | 1,3,12 | 4,5,8,9,13 | 6,7,12,13 |
| 6 | | | | | | 8,9,10 | 8,9,11 | 1,3,13 | 2,3,13 | 4,13 | 5,13 | 10,11,12,13 | 4,5,6,7,12 |
| 7 | | | | | | | 8,9,10 | 2,3,13 | 1,3,13 | 5,13 | 4,13 | 10,11,12,13 | 4,5,6,7,12 |
| 8 | | | | | | | | 4,6,7 | 5,6,7 | 6,12 | 7,12 | 4,5,12,13 | 8,9,10,11,12 |
| 9 | | | | | | | | | 4,6,7 | 7,12 | 6,12 | 4,5,12,13 | 8,9,10,11,12 |
| 10 | | | | | | | | | | 4,5,8 | 4,5,9 | 6,7,10,11,13 | 8,9,12,13 |
| 11 | | | | | | | | | | | 4,5,8 | 6,7,10,11,13 | 8,9,12,13 |
| 12 | | | | | | | | | | | | 1,2,3 ² ,6,7, | 4,5,6,7,8,9, |
| | | | | | | | | | | | | 8,9,12 ² ,13 | 10,11,12,13 |
| 13 | | | | | | | | | | | | | 1,2,3 ² ,4,5 |
| | | | | | | | | | | | | | 10,11,12,13 ² |

TABLE IV. Conjugacy classes of $\Delta(600)$.

| \mathcal{C} | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 |
|-----------------|----|----|----|----|----|-----|----|----|----|----|----|----|----|----|----|----|----|
| Order | 1 | 5 | 5 | 5 | 5 | 5 | 5 | 2 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 |
| $ \mathcal{C} $ | 1 | 6 | 6 | 3 | 3 | 3 | 3 | 3 | 6 | 6 | 3 | 6 | 6 | 3 | 6 | 6 | 3 |
| \mathcal{C} | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | |
| Order | 10 | 10 | 10 | 10 | 10 | 3 | 2 | 10 | 10 | 10 | 10 | 4 | 20 | 20 | 20 | 20 | |
| $ \mathcal{C} $ | 6 | 6 | 3 | 6 | 6 | 200 | 30 | 30 | 30 | 30 | 30 | 30 | 30 | 30 | 30 | 30 | |

TABLE V. Irreducible representations of $\Delta(600)$.

| IR | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 |
|------|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| c.c. | 1 | 2 | 3 | 4 | 5 | 21 | 22 | 23 | 24 | 25 | 16 | 17 | 18 | 19 | 20 | 11 | 12 |
| dim | 1 | 1 | 2 | 3 | 3 | 3 | 3 | 3 | 3 | 6 | 3 | 3 | 3 | 3 | 6 | 3 | 3 |
| IR | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | |
| c.c. | 13 | 14 | 15 | 6 | 7 | 8 | 9 | 10 | 26 | 27 | 29 | 28 | 30 | 33 | 32 | 31 | |
| dim | 3 | 3 | 6 | 3 | 3 | 3 | 3 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | |

Substituting (4) and (5) into (8), we get the nontrivial commutation relations with the $Z_{10} \times Z_{10}$ generators to be

$$\begin{aligned} f_2 f_4 f_2^{-1} &= f_4^6 f_3, & f_2 f_3 f_2^{-1} &= f_4^7 f_3^3, \\ f_1 f_4 f_1^{-1} &= f_4^9 f_3, & f_1 f_3 f_1^{-1} &= f_3. \end{aligned} \quad (8)$$

The group $\Delta(600)$ has 33 classes and 33 IRs, as listed in Tables IV and V. We have no need for the character table or the Clebsch-Gordan series in this article, so they will not be given here.

The classes containing order-2 elements are \mathcal{C}_8 and \mathcal{C}_{24} . Unlike $\Delta(150)$, it is possible to find two mutually commuting order-2 elements in this group. The significance of this remark will be explained in Sec. VI.

III. LEFT-HANDED SYMMETRY FOR $\Delta(150)$

In this section we first review the general theory of mixing based on a symmetry of effective left-handed mass matrices (to be abbreviated as left-handed symmetry) [10]. We will also provide much more detailed information on the specific case of $\Delta(150)$.

Every neutrino mixing matrix U has a natural $Z_2 \times Z_2$ symmetry generated by

$$G_i = \pm u_i u_i^\dagger \mp u_j u_j^\dagger \pm u_k u_k^\dagger, \quad (i = 1, 2, 3) \quad (9)$$

where u_1, u_2, u_3 are the three columns of U , referred to as *mixing vectors* below, and (i, j, k) is a permutation of $(1, 2, 3)$. It is easy to see that G_i is unitary, $G_i^2 = 1$, and $G_i G_j = \pm G_k$. G_i is a symmetry of the Majorana neutrino mass matrix $\bar{M}_\nu = \bar{M}_\nu^T$ in the basis where the left-handed effective charged-lepton matrix $\bar{M}_e := M_e^\dagger M_e$ is diagonal. This is so because

$$G_i^T \bar{M}_\nu G_i = \bar{M}_\nu \quad (10)$$

follows from $\bar{M}_\nu = \sum_i m_i u_i u_i^T$, with m_i being the masses of the active neutrinos with Majorana phases.

This symmetry in the neutrino sector is accompanied by a Z_n symmetry in the charged-lepton sector. Since the \bar{M}_e is diagonal, every unitary diagonal 3×3 matrix F satisfying $F^\dagger \bar{M}_e F = \bar{M}_e$ is a symmetry. If $F^n = 1$, then F generates the said Z_n symmetry. F cannot be identical to any G_i , otherwise the mixing column u_i would have two zeros, contrary to experiments.

\bar{M}_ν and \bar{M}_e are the (effective) left-handed mass matrices mentioned earlier.

So far everything is general. The group theory of mixing is based on the assumption that F and at least one G_i (hereafter referred to simply as G) originate from an unbroken non-Abelian symmetry \mathcal{G} and remain a symmetry after the spontaneous symmetry breaking, although the other members of \mathcal{G} may not. From now on we shall assume \mathcal{G} to be a finite group.

The minimal symmetry group \mathcal{G} consistent with this assumption is generated by F and G . Conversely, given any \mathcal{G} , we can construct mixing vectors v (one of the u_i above) in the following way.

Collect from \mathcal{G} all possible pairs of elements (F, G) belonging to all possible three-dimensional irreducible representations in which F and G are unitary. In order to ensure that \bar{M}_e is diagonal when F is, we require the three eigenvalues of F to be distinct; hence, its order n must be at least 3. G should be of order 2 because $G^2 = 1$. Let $v = (v_1, v_2, v_3)^T$ be the normalized eigenvector of G with eigenvalue $-\text{Tr}(G) (= \pm 1)$ in the F -diagonal basis; then v is a possible mixing vector and is uniquely determined from G . In particular, if it is the mixing vector for the third column of the neutrino mixing matrix U , then $|v_1| = |\sin \theta_{13}|$, $|v_2| = |\cos \theta_{13} \sin \theta_{23}|$, and $|v_3| = |\cos \theta_{13} \cos \theta_{23}|$. Since the CP and Majorana phases are unknown, many distinct v 's can give rise to the same $|v_i|$,

which is all that can be measured. Moreover, by rearranging the entries of the diagonal F , the components of v_i are similarly rearranged. We shall refer to these distinct v 's that give rise to the same $|v_i|$ with some rearrangement as *equivalent mixing vectors*. For experimental comparisons they need not be distinguished.

In addition to the G whose mixing vector v fits the third column of U , if we can find another G' commuting with G whose mixing vector v' fits the first or the second column, then the unitarity of U ensures that the remaining column agrees with experiment. In this way we could get a purely symmetric explanation of all the mixing parameters, without invoking any free parameter. In other words, we have then a full $Z_2 \times Z_2$ rather than a single Z_2 symmetry in the neutrino sector. A necessary condition for this full symmetry to happen is to have two mutually commuting order-2 elements G, G' in the group. As remarked in the last section, this can never happen in $\Delta(150)$, but it does happen in $\Delta(600)$.

There are clearly many (F, G) pairs for each group. The choice in G is somewhat limited because it must belong to a conjugacy class of order 2, but F can come from essentially everything else except the identity. Fortunately, it is sufficient to take only one F per conjugacy class of order ≥ 3 , provided we choose G to be all possible elements of order 2. This is so because the pair (F, G) and the pair (gFg^{-1}, gGg^{-1}) , for any $g \in \mathcal{G}$, give equivalent mixing vectors v , so other F 's in the same conjugacy cannot yield an inequivalent mixing vector.

Let us now specialize to $\mathcal{G} = \Delta(150)$. As shown in Table I, it has 13 classes C_i and 13 irreducible representations. G must be taken from C_2 , with $|C_2| = 15$, but F can be taken from C_3 to C_{13} ; hence, there are 11 possible F 's. There are eight three-dimensional irreducible representations (IR4 to IR11), so altogether there are $15 \times 11 \times 8 = 1320$ distinct (F, G) pairs. Most of these pairs do not yield an acceptable mixing vector for the third column, but many pairs do, all giving equivalent mixing vectors $|v| = (.170, .607, .777)^T$, corresponding to $\sin^2 2\theta_{13} = 0.11$ and $\sin^2 2\theta_{23} = 0.94$. These pairs occur in IR_n , with F in C_i , and G being the k th element of C_2 . Table VI consists of all these possible triplets (n, i, k) .

We see from Table VI that the successful F comes from C_3 , whose GAP-basis expression turns out to be

$$F = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (11)$$

for all the IRs in Table VI. It can be diagonalized by V , yielding $V^\dagger F V = \text{diag}(\omega^2, \omega, 1)$, with $\omega := \exp(2\pi i/3)$, where

$$V = \frac{1}{\sqrt{3}} \begin{pmatrix} \omega & \omega^2 & 1 \\ \omega^2 & \omega & 1 \\ 1 & 1 & 1 \end{pmatrix}. \quad (12)$$

TABLE VI. Origin of equivalent mixing vectors.

| n | i | k | n | i | k | n | i | k | n | i | k |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 4 | 3 | 5 | 5 | 3 | 5 | 6 | 3 | 3 | 7 | 3 | 3 |
| 8 | 3 | 9 | 9 | 3 | 9 | 10 | 3 | 6 | 11 | 3 | 6 |
| 4 | 3 | 6 | 5 | 3 | 6 | 6 | 3 | 9 | 7 | 3 | 9 |
| 8 | 3 | 3 | 9 | 3 | 3 | 10 | 3 | 5 | 11 | 3 | 5 |
| 4 | 3 | 10 | 5 | 3 | 10 | 6 | 3 | 14 | 7 | 3 | 14 |
| 8 | 3 | 8 | 9 | 3 | 8 | 10 | 3 | 12 | 11 | 3 | 12 |
| 4 | 3 | 11 | 5 | 3 | 11 | 6 | 3 | 7 | 7 | 3 | 7 |
| 8 | 3 | 15 | 9 | 3 | 15 | 10 | 3 | 13 | 11 | 3 | 13 |
| 4 | 3 | 12 | 5 | 3 | 12 | 6 | 3 | 8 | 7 | 3 | 8 |
| 8 | 3 | 14 | 9 | 3 | 14 | 10 | 3 | 10 | 11 | 3 | 10 |
| 4 | 3 | 13 | 5 | 3 | 13 | 6 | 3 | 15 | 7 | 3 | 15 |
| 8 | 3 | 7 | 9 | 3 | 7 | 10 | 3 | 11 | 11 | 3 | 11 |

If we permute the columns of V , then the entries of the diagonal form of F are similarly permuted.

The mixing vector \tilde{v} in the GAP basis is the eigenvector of G in that basis, with eigenvalue $-\text{Tr}(G)$. The mixing vector v in the F -diagonal basis which yields a possible column of U is related to it by $v = V^\dagger \tilde{v}$.

It turns out that there are only six different G and \tilde{v} pairs, to be labeled by a, b, c, d, e, f below. a comes from the first two rows of Table VI, b the next rows, and so on, and f comes from the last two rows. They are

$$\begin{aligned} G_a &= \pm \begin{pmatrix} 0 & \eta^3 & 0 \\ \eta^2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & G_b &= \pm \begin{pmatrix} 0 & \eta^2 & 0 \\ \eta^3 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ G_c &= \pm \begin{pmatrix} 0 & 0 & \eta^2 \\ 0 & 1 & 0 \\ \eta^3 & 0 & 0 \end{pmatrix}, & G_d &= \pm \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \eta^3 \\ 0 & \eta^2 & 0 \end{pmatrix}, \\ G_e &= \pm \begin{pmatrix} 0 & 0 & \eta^3 \\ 0 & 1 & 0 \\ \eta^2 & 0 & 0 \end{pmatrix}, & G_f &= \pm \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \eta^2 \\ 0 & \eta^3 & 0 \end{pmatrix}, \end{aligned} \quad (13)$$

where $\eta := \exp(2\pi i/5)$. There are four big columns in Table VI. The G 's coming from columns 1 and 3 have $\det(G) = -1$, and those coming from columns 2 and 4 have $\det(G) = +1$.

The corresponding unnormalized \tilde{v} are

$$\begin{aligned} \tilde{v}_a &= \begin{pmatrix} -\eta^3 \\ 1 \\ 0 \end{pmatrix}, & \tilde{v}_b &= \begin{pmatrix} -\eta^2 \\ 1 \\ 0 \end{pmatrix}, & \tilde{v}_c &= \begin{pmatrix} -\eta^2 \\ 0 \\ 1 \end{pmatrix}, \\ \tilde{v}_d &= \begin{pmatrix} 0 \\ -\eta^3 \\ 1 \end{pmatrix}, & \tilde{v}_e &= \begin{pmatrix} -\eta^3 \\ 0 \\ 1 \end{pmatrix}, & \tilde{v}_f &= \begin{pmatrix} 0 \\ -\eta^2 \\ 1 \end{pmatrix}. \end{aligned} \quad (14)$$

The corresponding normalized mixing vectors $v = V^\dagger \tilde{v}$ are

$$\begin{aligned}
v_a &= \frac{1}{\sqrt{6}} \begin{pmatrix} \omega(1 - \omega\eta^3) \\ \omega^2(1 - \omega^2\eta^3) \\ 1 - \eta^3 \end{pmatrix}, & v_b &= \frac{1}{\sqrt{6}} \begin{pmatrix} \omega(1 - \omega\eta^2) \\ \omega^2(1 - \omega^2\eta^2) \\ 1 - \eta^2 \end{pmatrix}, \\
v_c &= \frac{1}{\sqrt{6}} \begin{pmatrix} 1 - \omega^2\eta^2 \\ 1 - \omega\eta^2 \\ 1 - \eta^2 \end{pmatrix}, & v_d &= \frac{1}{\sqrt{6}} \begin{pmatrix} 1 - \omega\eta^3 \\ 1 - \omega^2\eta^3 \\ 1 - \eta^3 \end{pmatrix}, \\
v_e &= \frac{1}{\sqrt{6}} \begin{pmatrix} 1 - \omega^2\eta^3 \\ 1 - \omega\eta^3 \\ 1 - \eta^3 \end{pmatrix}, & v_f &= \frac{1}{\sqrt{6}} \begin{pmatrix} 1 - \omega\eta^2 \\ 1 - \omega^2\eta^2 \\ 1 - \eta^2 \end{pmatrix}.
\end{aligned} \tag{15}$$

Since complex conjugation interchanges ω with ω^2 and η^2 with η^3 , and since $|v| = |v^*|$, we conclude that $|v_a| = |v_c| = |v_d|$ and $|v_b| = |v_e| = |v_f|$. Moreover, $|v_b|$ is obtained from $|v_a|$ by interchanging the last two rows. Hence, all six cases give rise to the same equivalent mixing vector whose normalized magnitude is $|v| = (.170, .607, .777)^T$.

The mass matrix $\bar{M}_e = \bar{M}_e^\dagger$ can be obtained from the symmetry condition $F^\dagger \bar{M}_e F$ to be

$$\bar{M}_e = \begin{pmatrix} \alpha & \beta & \beta^* \\ \beta^* & \alpha & \beta \\ \beta & \beta^* & \alpha \end{pmatrix}, \tag{16}$$

where α is a real parameter and β a complex parameter. The masses m_e^2 , m_μ^2 , and m_τ^2 can be obtained from α , $\beta_R := \text{Re}(\beta)$ and $\beta_I := \text{Im}(\beta)$ to be $\alpha - \beta_R + \sqrt{3}\beta_I$, $\alpha - \beta_R - \sqrt{3}\beta_I$, $\alpha + 2\beta_R$.

Similarly, the neutrino mass $\bar{M}_\nu = \bar{M}_\nu^T$ can be obtained from the symmetry condition $G^T \bar{M}_\nu G = \bar{M}_\nu$. Corresponding to the six G_a , we get, respectively,

$$\begin{aligned}
\bar{M}_{\nu a} &= \begin{pmatrix} a & b & c \\ b & a\eta & c\eta^3 \\ c & c\eta^3 & f \end{pmatrix}, & \bar{M}_{\nu b} &= \begin{pmatrix} a & b & c \\ b & a\eta^4 & c\eta^2 \\ c & c\eta^2 & f \end{pmatrix}, \\
\bar{M}_{\nu c} &= \begin{pmatrix} a & b & c \\ b & f & b\eta^2 \\ c & b\eta^2 & a\eta^4 \end{pmatrix}, & \bar{M}_{\nu d} &= \begin{pmatrix} a & b & b\eta^3 \\ b & f\eta^4 & c \\ b\eta^3 & c & f \end{pmatrix}, \\
\bar{M}_{\nu e} &= \begin{pmatrix} a & b & c \\ b & f & b\eta^3 \\ c & b\eta^3 & a\eta \end{pmatrix}, & \bar{M}_{\nu f} &= \begin{pmatrix} a & b & b\eta^2 \\ b & f\eta & c \\ b\eta^2 & c & f \end{pmatrix},
\end{aligned} \tag{17}$$

where a, b, c, f are complex parameters which can be used to fit the solar angle, the CP phase, the three neutrino masses and the three Majorana phases (one of them is an overall phase).

A. A simple model

From now on we will study case a in more detail and will write $\bar{M}_{\nu a}$ simply as \bar{M}_ν . As has already been remarked, the four complex parameters a, b, c, f are just

enough to fit the neutrino masses and Majorana phases, the solar angle, and the CP phase. In this subsection, we consider a simple model with $c = 0$, which turns out to give the right solar angle as well. The remaining parameters a, b, f can be used to determine the neutrino masses m_i and the Majorana phases.

With $c = 0$, the mass matrix becomes

$$(\bar{M}_\nu)_0 = \begin{pmatrix} a & b & 0 \\ b & a\eta & 0 \\ 0 & 0 & f \end{pmatrix}. \tag{18}$$

It can be diagonalized by the matrix

$$W_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} \eta^3 & 0 & -\eta^3 \\ 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \end{pmatrix}, \tag{19}$$

resulting in

$$W_0^T (\bar{M}_\nu)_0 W_0 = \begin{pmatrix} a\eta - b\eta^3 & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & a\eta + b\eta^3 \end{pmatrix}. \tag{20}$$

The neutrino mixing matrix in this case is

$$U_0 = V^\dagger W_0 = \frac{1}{\sqrt{6}} \begin{pmatrix} \omega + \omega^2\eta^3 & \sqrt{2} & \omega - \omega^2\eta^3 \\ \omega^2 + \omega\eta^3 & \sqrt{2} & \omega^2 - \omega\eta^3 \\ 1 + \eta^3 & \sqrt{2} & 1 - \eta^3 \end{pmatrix}, \tag{21}$$

where V is the matrix in (12) used to diagonalize F .

The third column of U_0 is just v_a of (15), which gives the desired reactor and atmospheric angles. The second column gives trimaximal mixing, yielding a solar angle θ_{12} and satisfying

$$\sin^2 2\theta_{12} = 0.90, \tag{22}$$

consistent with the PDG value [11] $\sin^2 2\theta_{12} = 0.95 \pm 0.10 \pm 0.01$. The first column is determined from the other two by the unitarity of U_0 .

The CP phase can also be read off from the (21), (22), and (31), or (32) element of $|(U_0)_{ij}|$. The result is $\delta_{CP} = 0$.

The neutrino masses are the absolute values of the diagonal elements in (20):

$$m_1 = |a\eta - b\eta^3|, \quad m_2 = |f|, \quad m_3 = |a\eta + b\eta^3|. \tag{23}$$

There are enough parameters to accommodate the measured mass gaps, either with a normal hierarchy or an inverted hierarchy. In fact, it can fit any Majorana phase as well. For example, if it should turn out that the Majorana phases are all zero, then we would take $a\eta \mp b\eta^3$ and f to be positive. In a normal hierarchy with $m_1 = 0$, $m_2^2 = f^2$ is the solar gap and $m_3^2 = 4|b|^2$ is the atmospheric gap. We can do so similarly for an inverted hierarchy.

The charged-lepton mass squares are given below (16) to be $m_e^2 = \alpha - \beta_R + \sqrt{3}\beta_I$, $m_\mu^2 = \alpha - \beta_R - \sqrt{3}\beta_I$, and

$m_\tau^2 = \alpha + 2\beta_R$. Thus, $m_\mu^2 - m_e^2 = -2\sqrt{3}\beta_I$ tells us that β_I is negative, $\alpha = (m_e^2 + m_\mu^2 + m_\tau^2)/3$ is positive, and $2\beta_R = m_\tau^2 - \alpha$ is also positive.

IV. BOTH-HANDED SYMMETRY FOR $\Delta(150)$

The left-handed mass matrices $\bar{M}_e = M_e^\dagger M_e$ and $\bar{M}_\nu = M_\nu^T M_N^{-1} M_\nu$ given in (16) and (17) are the most general that yield the correct reactor and atmospheric angles from $\Delta(150)$. They have just enough free parameters to fit the unknown quantities: charged-lepton and neutrino masses, the solar angle, and the CP and the Majorana phases. Left-handed symmetry alone cannot determine these parameters, nor can it tell us what the Dirac mass matrices M_e, M_ν and the Majorana mass matrix M_N are. For the latter we need to impose additional assumptions regarding how the right-handed leptons transform.

Suppose the right-handed charged leptons transform according to a (reducible or irreducible) representation C' , and the right-handed Majorana neutrinos N according to a representation C . Let $F_{C'}, G_C$ be the unitary matrices for the residual symmetries in these representations. With these symmetries, the Dirac and Majorana mass matrices should obey the constraint

$$M_e = F_{C'}^\dagger M_e F, \quad M_\nu = G_C^T M_\nu G, \quad M_N = G_C^T M_N G_C, \quad (24)$$

where F is given in (11) and G is one of the six equivalent forms in (17). In what follows we shall confine ourselves to case a , namely,

$$F = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad G = - \begin{pmatrix} 0 & \eta^3 & 0 \\ \eta^2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\tilde{\nu} = \begin{pmatrix} -\eta^3 \\ 1 \\ 0 \end{pmatrix} \quad (25)$$

and

$$\bar{M}_e = \begin{pmatrix} \alpha & \beta & \beta^* \\ \beta^* & \alpha & \beta \\ \beta & \beta^* & \alpha \end{pmatrix}, \quad \bar{M}_\nu = \begin{pmatrix} a & b & c \\ b & a\eta & c\eta^3 \\ c & c\eta^3 & f \end{pmatrix}. \quad (26)$$

These matrices are taken from IR5, with F coming from class C_3 and G from C_2 . More specifically, in terms of the generators f_i discussed in Sec. II,

$$F = f_2, \quad G = f_1 f_3^2 f_4^4. \quad (27)$$

In order for the left-handed matrices $\bar{M}_e = M_e^\dagger M_e$ and $\bar{M}_\nu = M_\nu^T M_N^{-1} M_\nu$ to be given by (26), G_C must be real as well as unitary, a requirement which can also be seen from

the Majorana nature of N . If N is described by a Majorana field ψ , then $\psi_c = C\bar{\psi} = \psi$, where C is the charge conjugation operator. To preserve the Majorana nature under an internal symmetry transformation $\psi \rightarrow R\psi$, it is necessary to have $R = R^*$ because $\psi_c \rightarrow R^* \psi_c$. As a result, $G = G^T$ because G is unitary and of order 2, and M_N^{-1} transforms like M_N :

$$M_N^{-1} = G^T M_N^{-1} G. \quad (28)$$

With the introduction of right-handed symmetry, we expected more parameters to appear in M_e, M_ν , and M_N than in \bar{M}_e and \bar{M}_ν . Nevertheless, when we use the former three to calculate the latter two, we must still get back to (26) and the parametrizations there. On the one hand, since there are more parameters in the former than in the latter, we expect the resulting parameters in the latter to remain independent. On the other hand, the composite nature of the latter may produce occasional surprises arising from special composite features. A case in point which we will discuss later occurs in the model in Sec. III A, where $c = 0$. There are four different right-handed assignments giving the same \bar{M}_ν , and in three of the four cases, m_3 is forced to be zero as well.

If future experiment should reveal that $m_3 \neq 0$, then this is a strong support for the symmetry assignment of N to be the fourth. This conclusion is what we gain by imposing the right-handed symmetry.

We will assume fermions of different generations to be distinguished by different family quantum numbers; hence, if C' and C are reducible, the irreducible representations they contain must not repeat themselves. Also, included in the latitude of the choice of C is the number ρ of N 's, which is unknown. We reject $\rho = 1$ to retain the possibility of leptogenesis, and we will confine ourselves here to $\rho = 2$ and 3.

A. Charged-lepton sector

There are three right-handed charged leptons, so the dimension of C' must be 3. If it is irreducible, from Table I we see that it must belong to one of IR4 to IR11. If it is reducible, it could either be IR3 + IR1 or IR3 + IR2.

It turns out that $F_{C'} = F$ if C' is one of IR4 to IR11. Even so, the solution of (24) for M_e is not the same as solution \bar{M}_e in (26) because \bar{M}_e has to be Hermitian, but M_e may not be. The general solution of (24) for M_e turns out to be

$$M_e = \begin{pmatrix} \zeta & \sigma & \tau \\ \tau & \zeta & \sigma \\ \sigma & \tau & \zeta \end{pmatrix}. \quad (29)$$

If it is Hermitian, then $\tau = \sigma^*$ and we get back to the form of \bar{M}_e in (26). In any case, we can compute $\bar{M}_e = M_e^\dagger M_e$ from (29) and obtain (26) with $\alpha = |\zeta|^2 + |\sigma|^2 + |\tau|^2$ and $\beta = \zeta^* \sigma + \sigma^* \tau + \tau^* \zeta$.

Both \bar{M}_e and M_e give the same physics, so why then is the former described by three real parameters, α, β_R, β_I , but the latter is expressed in terms of three complex parameters ζ, σ, τ ? This puzzle can be solved by computing $V^\dagger M_e V$ to diagonalize M_e . One finds its three eigenvalues to be complex. Their magnitudes are the masses and their phases are unphysical and can be absorbed into the right-handed charged-lepton fields.

Next, suppose C' is IR3 + IR1 or IR3 + IR2. Then using the real representations given in (A8), we obtain

$$F_{C'} = \frac{1}{2} \begin{pmatrix} -1 & -\sqrt{3} & 0 \\ \sqrt{3} & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad (30)$$

for both cases. The symmetry constraint (24) then yields

$$M_e = \begin{pmatrix} \zeta & \sigma & -\zeta - \sigma \\ -(2\sigma + \zeta)/\sqrt{3} & (\sigma + 2\zeta)/\sqrt{3} & (\sigma - \zeta)\sqrt{3} \\ \tau & \tau & \tau \end{pmatrix}. \quad (31)$$

If we compute $\bar{M}_e = M_e^\dagger M_e$ from (31), we get back to (26) with $\alpha = 2(|\zeta|^2 + |\sigma|^2 + |\zeta + \sigma|^2)/3 + |\tau|^2$ and $\beta = 2(-|\zeta + \sigma|^2 - \sigma^* \zeta)/3 + |\tau|^2$.

B. Neutrino sector

As mentioned before, the representation of a Majorana neutrino must be real. Since IR5 is complex, it is not

possible to realize \bar{M}_ν in a type-II seesaw mechanism. There is no problem to realize it with a type-I seesaw, as implicitly assumed before, as long as the representation C of N is real. The only real representations of $\Delta(150)$ (see Table I and the Appendix) are IR1, IR2, IR3, IR12, and IR13. The dimensions of IR12 and IR13 are six, so we can ignore them if $\rho < 6$. This leaves only two possibilities each for $\rho = 2$ and 3. For $\rho = 2$, $C = \text{IR3}$ or $\text{IR1} + \text{IR2}$. For $\rho = 3$, C is either $\text{IR3} + \text{IR1}$ or $\text{IR3} + \text{IR2}$. In fact, if $\rho < 6$, the only other possibility is $\rho = 4$ with $C = \text{IR3} + \text{IR2} + \text{IR1}$, but that contains many parameters and sheds no light on the existing physics so we will consider it further.

We consider these four cases separately below.

1. $\rho = 2, C = \text{IR3}$

From (27), Table I and (A8), we get

$$G_C = \bar{f}_1(3)\bar{f}_3(3)\bar{f}_4(3)^4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (32)$$

The solution of (24) is

$$M_N^{-1} = \begin{pmatrix} P & Q \\ Q & P \end{pmatrix}, \quad M_\nu := \begin{pmatrix} x & y & z \\ -y\eta^2 & -x\eta^3 & -z \end{pmatrix}. \quad (33)$$

Computing from these $\bar{M}_\nu = M_\nu^T M_N^{-1} M_\nu$, we obtain (26) with

$$\begin{aligned} a &= (x^2 + y^2\eta^4)P - 2xy\eta^2Q, & b &= 2xyP - (x^2 + y^2\eta)\eta^2Q, \\ c &= z(x + y\eta^2)(P - Q), & f &= 2z^2(P - Q), & \Rightarrow \\ a + b\eta^2 &= (x + y\eta^2)^2(P - Q). \end{aligned} \quad (34)$$

2. $\rho = 2, C = \text{IR1} + \text{IR2}$

From Table I, we get

$$G_C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (35)$$

The solution of (24) is

$$M_N^{-1} = \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix}, \quad M_\nu := \begin{pmatrix} x & -x\eta^3 & 0 \\ y & y\eta^3 & z \end{pmatrix}. \quad (36)$$

Computing from these $\bar{M}_\nu = M_\nu^T M_N^{-1} M_\nu$, we obtain (26) with

$$\begin{aligned} a &= x^2P + y^2Q, & b &= \eta^3(-x^2P + y^2Q), \\ c &= yzQ, & f &= z^2Q, & \Rightarrow \\ a + b\eta^2 &= 2y^2Q. \end{aligned} \quad (37)$$

3. $\rho = 3, C = \text{IR3} + \text{IR1}$

In this case,

$$G_C = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_N^{-1} = \begin{pmatrix} P & Q & R \\ Q & P & R \\ R & R & S \end{pmatrix}, \quad (38)$$

$$M_\nu := \begin{pmatrix} x & y & z \\ -y\eta^2 & -x\eta^3 & -z \\ w & -w\eta^3 & 0 \end{pmatrix}.$$

Note that the 3×3 Dirac mass matrix M_ν here is a composite, with its first two rows taken from the M_ν in (33), and the last row taken from the first row (with a change of symbol) of the M_ν in (36).

Computing from these $\bar{M}_\nu = M_\nu^T M_N^{-1} M_\nu$, we obtain (26) with

$$\begin{aligned}
 a &= (x^2 + y^2\eta^4)P - 2xy\eta^2Q + 2(x - y\eta^2)wR + w^2S, \\
 b &= 2xyP - (x^2\eta + y^2)\eta^2Q + 2(-x\eta^3 + y)wR - w^2\eta^3S, \\
 c &= z(x + y\eta^2)(P - Q), \quad f = 2z^2(P - Q), \quad \Rightarrow \\
 a + b\eta^2 &= (x + y\eta^2)^2(P - Q).
 \end{aligned} \tag{39}$$

If we set $w = R = S = 0$, we return to (34) as it should.

4. $\rho = 2, C = \mathbf{IR3} + \mathbf{IR2}$

In this case,

$$G_C = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad M_N^{-1} = \begin{pmatrix} P & Q & R \\ Q & P & -R \\ R & -R & S \end{pmatrix},$$

$$M_\nu := \begin{pmatrix} x & y & z \\ -y\eta^2 & -x\eta^3 & -z \\ w & w\eta^3 & u \end{pmatrix}. \tag{40}$$

The 3×3 Dirac mass matrix M_ν here is also a composite, with its first two rows taken from the M_ν in (33), and the last row taken from the second row (with a change of symbol) of the M_ν in (36).

Computing from these $\bar{M}_\nu = M_\nu^T M_N^{-1} M_\nu$, we obtain (26) with

$$\begin{aligned}
 a &= (x^2 + y^2\eta^4)P - 2xy\eta^2Q + 2(x + y\eta^2)wR \\
 &\quad + w^2S, \\
 b &= 2xyP - (x^2\eta + y^2)\eta^2Q + 2(x\eta^3 + y)wR \\
 &\quad + w^2\eta^3S, \\
 c &= z(x + y\eta^2)(P - Q) + (2zw + u(x + y\eta^2))R \\
 &\quad + uwS, \\
 f &= 2z^2(P - Q) + 4zuR + u^2S,
 \end{aligned}$$

$$a + b\eta^2 = (x + y\eta^2)[(x + y\eta^2)(P - Q)] + 4wR + 2w^2S. \tag{41}$$

If we set $u = w = R = S = 0$, we return to (34) as we should.

C. General remarks

In all the cases considered above, both in the charged-lepton and in the neutrino sectors, there are more parameters than necessary to fix \bar{M}_e and \bar{M}_ν in (26), from which all the low-energy leptonic physical quantities can be determined. This is especially so in case 4 above. As a result, the situation is very complicated and there are many degenerate parameters that cannot be resolved by the measured quantities. In the rest of this section, we will concentrate on studying the model $c = 0$ in more detail, where things become a bit simpler and more transparent.

D. The model $c = 0$

Consider \bar{M}_N in (26) with $c = 0$, a case already considered in Sec. III A. Recall that all the experimental mixing angles can be obtained with this choice, together with $\delta_{CP} = 0$. We would like to know what happens if we also impose a right-handed symmetry in the model. For that purpose, let us concentrate on Eqs. (34), (37), (39), (41), and (20).

It can be seen from these equations that $f = 0$ often follows from $c = 0$. This is undesirable because phenomenologically $m_2 = |f|$ can never be zero, although either m_1 or m_3 may be. In the first three cases, the only way to render $c = 0$ and $f \neq 0$ is to set

$$\begin{aligned}
 x + y\eta^2 &= 0 \text{ in (35),} & y &= 0 \text{ in (39),} \\
 x + y\eta^2 &= 0 \text{ in (42).}
 \end{aligned} \tag{42}$$

In all three cases, this automatically implies $m_3 = |a + b\eta^2| = 0$. This unexpected prediction of $m_3 = 0$, and hence an inverted hierarchy, stems from the right-handed symmetries imposed on these three cases; so at least in these cases, the imposition of right-handed symmetry does produce additional predictions.

Beyond that, there are enough free parameters left in each case to fit any m_1 and m_2 . Specifically, for case 1 in (34),

$$\begin{aligned}
 m_1 &= |a - b\eta^2| = |4x^2(P + Q)|, \\
 m_2 &= |f| = |2z^2(P - Q)|.
 \end{aligned} \tag{43}$$

For case 2 in (37),

$$m_1 = |a - b\eta^2| = |x^2P|, \quad m_2 = |f| = |z^2Q|. \tag{44}$$

For case 3 in (39),

$$\begin{aligned}
 m_1 &= |a - b\eta^2| = |4x^2(P + Q) + 8xwR + 2w^2S|, \\
 m_2 &= |f| = |2z^2(P - Q)|.
 \end{aligned} \tag{45}$$

The last case is more complicated because it involves by far the most numbers of parameters. There seem to be many ways to impose $c = 0$ and $f \neq 0$, and it seems quite possible to have $m_3 \neq 0$ with $c = 0$. For now let us just concentrate on a simple choice of parameters to render $c = 0$, by letting $x + y\eta^2 = 0$ and $w = 0$. With this choice, $m_3 = 0$ as well, and as before there are enough parameters left to fix

$$\begin{aligned} m_1 &= |a - b\eta^2| = 2|a| = 4|x^2(P + Q)|, \\ m_2 &= |f| = |2z^2(P - Q) + 4zuR + u^2S| \end{aligned} \quad (46)$$

in many ways.

V. DYNAMICAL MODELS FOR $\Delta(150)$

It was assumed in the last two sections that the family symmetry $\mathcal{G} = \Delta(150)$ broke down to the residual symmetries generated by F and G , leaving unspecified how to achieve that dynamically. In this section we make a further assumption that this is caused by the vacuum expectation values of scalar fields coupled to fermions in \mathcal{G} -invariant Yukawa interactions. We will assume these scalar fields ϕ , called valons, to carry only \mathcal{G} -quantum numbers but no Standard Model quantum numbers. The Standard Model quantum numbers are carried by the usual Higgs field H , so the compound field $H\phi$ appearing together in the Yukawa terms carries both Standard Model and family quantum numbers. Since we are only interested in the family structure here, we will omit all spacetime and Standard Model details, and replace the Higgs field by its expectation value $\langle v \rangle$. In the presence of the compound scalar field, the Yukawa terms have dimension 5, so the coupling constant is inversely proportional to some heavy scale Λ . The Yukawa terms can then be written in the form

$$\sum_{B,a,b,c} h_B \bar{\psi}_c^C \chi_a^A \phi_b^B \langle Cc|Bb, Aa \rangle + \text{H.c.}, \quad (47)$$

in which the factor $\langle v \rangle/\Lambda$ has been absorbed into the Yukawa coupling constant h_B . The fermion fields ψ and χ are assumed to transform according to representations C and A , and the valon field ϕ according to representation B . The indices a, b, c are the components of A, B, C , and $\langle Cc|Bb, Aa \rangle$ are the Clebsch-Gordan coefficients needed to render (47) \mathcal{G} invariant.

Let g_A, g_B, g_C be elements of \mathcal{G} in representations A, B, C , respectively. A basis is chosen so that these matrices are unitary, and real if the representation is real (see the Appendix). That is also the basis in which the Clebsch-Gordan coefficients $\langle Cc|Bb, Aa \rangle$ are computed. From the covariance relation of Clebsch-Gordan coefficients [12]

$$\sum_{a',b',c'} (g_C^\dagger)_{cc'} (g_B)_{b'b} (g_A)_{a'a} \langle Cc'|Bb', Aa' \rangle = \langle Cc|Bb, Aa \rangle, \quad (48)$$

we see that the Yukawa term (47) is invariant under any \mathcal{G} transformation

$$\psi \rightarrow g_C \psi, \quad \chi \rightarrow g_A \chi, \quad \phi \rightarrow g_B \phi. \quad (49)$$

To break the symmetry from \mathcal{G} down to the residual symmetry, we impose vacuum expectation values $\langle \phi^B \rangle$, determined by a \mathcal{G} -invariant valon potential $\mathcal{V}(\phi)$ via the equation of motion $\partial \mathcal{V} / \partial \phi^{D^*} = 0$, where D^* is the complex conjugated representation of D . In order to

preserve the residual symmetry, the invariance (49) must remain true for $g = F$ in the charged-lepton sector, and $g = G$ in the neutrino sector. This requires

$$F_B \langle \phi^B \rangle_e = \langle \phi^B \rangle_e, \quad G_B \langle \phi^B \rangle_\nu = \langle \phi^B \rangle_\nu \quad (50)$$

for every B , where the subscripts e and ν indicate in which sector the vacuum expectation values apply. Incidentally, since the normalization of $\langle \phi^B \rangle$ cannot be determined from (50), it is conventional to take out the energy unit with an arbitrary numerical constant and absorb it into the Yukawa constant h_B . What remains is usually called the vacuum alignment.

In other words, the alignment in the charged-lepton sector must be an invariant eigenvector of F , and the alignment in the neutrino sector must be an invariant eigenvector of G . By invariant eigenvector, we mean an eigenvector with eigenvalue +1.

The mass matrix after symmetry breaking can be read off from (47) to be

$$\begin{aligned} (M_e)_{ca} &= \sum_{B,b} h_B \langle \phi_b^B \rangle_e \langle C'c|Bb, Aa \rangle, \\ (M_\nu)_{ca} &= \sum_{B,b} h_B \langle \phi_b^B \rangle_\nu \langle Cc|Bb, Aa \rangle, \\ (M_N)_{ca} &= \sum_{B,b} h_B \langle \phi_b^B \rangle_\nu \langle Cc|Bb, Ca \rangle, \end{aligned} \quad (51)$$

where the range of B, b and the value of h_B may be different for the three cases. The \mathcal{G} representation of the left-handed isodoublets is A , that of the right-handed charged leptons is C' , and that of the right-handed neutrinos N is C , with C being a real representation. It follows from (48) and (50) that the constraint on the mass matrices so obtained,

$$M_e = F_C^\dagger M_e F_A, \quad M_\nu = G_C^T M_\nu G_A, \quad M_N = G_C^T M_N G_C, \quad (52)$$

is the same as (24), except that in (24) F_A was abbreviated as F and G_A was abbreviated as G . In other words, dynamical models are consistent with both-handed symmetries and may be considered as a refinement of the latter.

Now that we have decided on the desired alignment to preserve residual symmetries, the question is whether a potential \mathcal{V} , invariant under \mathcal{G} , can be devised to provide such alignments. It turns out that every invariant potential does provide alignment solutions which are invariant eigenvectors of F, G , or any other element g of \mathcal{G} . The reason for this will be explained below.

Let us assume $\mathcal{V} = \mathcal{V}^{(34)} + \mathcal{V}^{(2)}$ to be a polynomial consisting of two parts: $\mathcal{V}^{(34)}$ of degrees 3 and 4, and $\mathcal{V}^{(2)}$ of degree 2. The quadratic part can be written as

$$\mathcal{V}^{(2)} = - \sum_B \mu_B^2 \phi^{B^*} \phi^B, \quad (53)$$

where $\phi^{B^*} = (\phi^B)^*$ is ϕ in the representation B^* . Accordingly, the equation of motion can be written as

$$\mu_D^2 \phi^D = \partial \mathcal{V}^{(34)} / \partial \phi^{D*} := \mathcal{Q}_D(\phi^B). \quad (54)$$

The unspecified index B in the argument of \mathcal{Q}_D simply means that this may be a function of ϕ^B for several B 's. To obtain $\langle \phi^D \rangle$ in the lowest order, we may simply consider the fields ϕ^D in (54) to be classical and identify them with $\langle \phi^D \rangle$.

Let g_B be an element of \mathcal{G} in representation B . We will now show that if $\hat{\phi}^B$ is a normalized invariant eigenvector of g_B , with $g_B \langle \hat{\phi}^B \rangle = \langle \hat{\phi}^B \rangle$, then $\phi^D = \kappa \hat{\phi}^D$ is a solution of (54) for every D , provided $\hat{\phi}^D$ is the *unique* invariant eigenvector of g_D . The normalization factor κ is a constant to be determined from (54) in a way to be explained.

To show this assertion, first note that

$$g_D \mathcal{Q}_D(\phi^B) = \mathcal{Q}_D(g_B \phi^B) = \mathcal{Q}_D(\phi^B), \quad (55)$$

so \mathcal{Q}_D is an invariant eigenvector of g_D , whatever \mathcal{Q}_D and the normalization of ϕ^B are. Suppose \mathcal{Q}_D consists of a number of monomials q_D^i of degree 2 and a number of monomials c_D^j of degree 3, so that $\mathcal{Q}_D = \sum_i q_D^i + \sum_j c_D^j$. Since every $q_D^i(\hat{\phi}^B)$ and every $c_D^j(\hat{\phi}^B)$ is an invariant eigenvector of g_B , and since this eigenvector is unique, we must have $q_D^i(\hat{\phi}^B) = \alpha_i \hat{\phi}^D$ and $c_j(\hat{\phi}^B) = \beta_j \hat{\phi}^D$ for some constants α_i and β_j that are determined by the structure of q^i and c^j . Since q^i is quadratic and c^j cubic, it follows that $q_D^i(\kappa \hat{\phi}^B) = \alpha_i \kappa^2 \hat{\phi}^D$ and $c_j(\kappa \hat{\phi}^B) = \beta_j \kappa^3 \hat{\phi}^D$; hence, $\mathcal{Q}_D(\kappa \hat{\phi}^B) = (\sum_i \alpha_i \kappa^2 + \sum_j \beta_j \kappa^3) \hat{\phi}^D$. Consequently, if $\kappa \neq 0$ is chosen to be the solution of the quadratic equation

$$\mu_D^2 = \sum_i \alpha_i \kappa + \sum_j \beta_j \kappa^2, \quad (56)$$

then $\phi^D = \kappa \hat{\phi}^D$ is a solution of (54).

Let us now denote the ϕ that couples to fermions in the charged-lepton sector of (47) by ϕ_e , and the one that couples in the neutrino sector by ϕ_ν . What we want then is a solution with $\langle \phi_e^B \rangle = \langle \phi^B \rangle_e$ and $\langle \phi_\nu^B \rangle = \langle \phi^B \rangle_\nu$. This can be accomplished if we choose the valon potential to be $\mathcal{V} = \mathcal{V}_e(\phi_e) + \mathcal{V}_\nu(\phi_\nu)$, with \mathcal{V}_e and \mathcal{V}_ν being any two \mathcal{G} -invariant potentials. However, if \mathcal{V} includes an interacting potential $\mathcal{V}_{e\nu}(\phi_e, \phi_\nu)$, then like spin-spin interaction, the resulting alignments of $\langle \phi_e^B \rangle$ and $\langle \phi_\nu^B \rangle$ would be shifted. To be consistent with everything up to now, we must assume either additional ‘‘shaping symmetry’’ can be found [13] to effectively forbid $\mathcal{V}_{e\nu}$, or that nature provides us with a small $\mathcal{V}_{e\nu}$ so that everything said up to now would be approximately true.

The dynamical model reviewed above is the most popular method to implement family symmetry, but it requires the presence of valon fields which may or may not exist in nature. Since the Yukawa terms are of dimension 5, the theory is at best a lower-energy effective theory used to explain leptonic masses and mixing. The heaviest leptonic mass is $m_\tau < 2$ GeV, so one might expect the valon

degrees of freedom to show up around there, but certainly there is no sign of any of them up to the present. Moreover, if there are really new degrees of freedom present, one might expect the Higgs coupling to fermion pairs to deviate from the Standard Model [14], but hitherto there is no sign of that either in the 125-GeV Higgs candidate.

In contrast, either the left-handed or both-handed symmetry approach to the mass matrix does not require the presence of valons, though they do leave open the question of how the family symmetry is broken to the desired residual symmetry. In this sense they are similar to the texture-zero approach to mass matrices, which, for example, yields the celebrated relation of the Cabibbo angle in terms of a ratio of quark masses [15], but the origin and the location of the zeros generally remain somewhat obscured.

With the assumption of the existence of valons, the dynamical model does allow a better control of symmetry, because it provides a physical explanation for the parameters appearing in M_e , M_ν , and M_N as Yukawa couplings, which in principle can be directly measured if valons do exist. Moreover, one might imagine the possibility of turning off certain Yukawa couplings on the grounds that the corresponding valons do not exist, thereby obtaining relations between the parameters of mass matrices, which may provide relations between physical quantities.

We do not expect this last possibility to materialize in the charged-lepton sector. As discussed in the last two sections, there are three positive parameters present to determine the three charged-lepton masses. If a relation among these three parameters exists, it would imply a relation among the charged-lepton masses. The only known relation among these masses is the Koide relation [16], but that involves the square root of masses that cannot be obtained by these kinds of Yukawa terms. For that reason we will concentrate on the neutrino sector in what follows. The four cases considered below correspond to the four cases studied in the last section. In these cases, the left-handed neutrinos ν always transform in representation $A = \text{IR}5$, but the right-handed neutrinos N transform according to a different representation C in different cases. With ρ being the number of N 's, the first two cases have $\rho = 2$ and the last two have $\rho = 3$.

Before going into the details of these four cases, it might be useful to provide a brief summary. First of all, there are always enough Yukawa coupling constants h_B in every case to provide the necessary parameters $x, y, z, u, w, P, Q, R, S$ in the last section. Since some of these parameters are redundant in determining the parameters a, b, c, f of \vec{M}_ν needed to fix the measured quantities, some of the Yukawa coupling constants can indeed be set to zero without giving rise to any new physical predictions. Next, one might ask whether the model studied in the last two sections with $c = 0$ can be obtained by the absence of some valons, or equivalently, by setting some Yukawa couplings to zero.

The answer is no, though $c = 0$ can be obtained by specific relations between two nonzero Yukawa parameters.

The Clebsch-Gordan coefficients needed to calculate the mass matrix in (51) can be obtained with the help of GAP using (48), which states that $\langle Cc|Bc, Aa \rangle$ is the invariant eigenvector of $g_C^\dagger \otimes g_B^T \otimes g_A^T$. With that the computation of the Clebsch-Gordan coefficients is reduced to an algebraic problem of finding the invariant eigenvectors from the representations of group elements given by GAP.

If $\langle Aa|$, $\langle Bb|$ are the a , b components of two vectors in the representation spaces A and B , respectively, then

$$\langle Cc| = \sum_{a,b} \langle Cc|Bb, Aa \rangle \langle Bb| \langle Aa| \quad (57)$$

is the c component of a vector in the representation space C . In what follows we will list a few Clebsch-Gordan coefficients using (57) and the following notation. a is a singlet in IR1, b is a singlet in IR2, (c, d) is a doublet in IR3, and (e, f, g) , (e', f', g') , (e'', f'', g'') are triplets in IR5, IR10, IR11, respectively. Subscripts may be used to distinguish two multiplets transforming the same way. The product of a multiplet with a always reproduces the multiplet, so we will skip those relations. Some other useful ones are

$$\begin{aligned} \begin{pmatrix} c \\ d \end{pmatrix} &= \frac{1}{2\sqrt{3}} \begin{pmatrix} (1 - \sqrt{3})ee' + (1 + \sqrt{3})ff' - 2gg' \\ -(1 + \sqrt{3})ee' + (-1 + \sqrt{3})ff' + 2gg' \end{pmatrix}, \\ \begin{pmatrix} c \\ d \end{pmatrix} &= \frac{1}{2\sqrt{3}} \begin{pmatrix} (-1 - \sqrt{3})ee'' + (-1 + \sqrt{3})ff'' + 2gg'' \\ (-1 + \sqrt{3})ee'' + (-1 - \sqrt{3})ff'' + 2gg'' \end{pmatrix}, \\ \begin{pmatrix} c \\ d \end{pmatrix} &= \frac{1}{2} \begin{pmatrix} c_1c_2 - c_1d_2 - d_1c_2 - d_1d_2 \\ -c_1c_2 - c_1d_2 - d_1c_2 + d_1d_2 \end{pmatrix}, \\ a &= \frac{1}{\sqrt{3}}(ee'' + ff'' + gg''), \quad b = \frac{1}{\sqrt{3}}(ee' + ff' + gg'), \\ a &= \frac{1}{\sqrt{2}}(c_1c_2 + d_1d_2), \quad b = \frac{1}{\sqrt{2}}(c_1d_2 - c_2d_1). \end{aligned} \quad (58)$$

We will now proceed to study the four cases separately. Since there is really nothing new we can get that way, all that we can do is to relate the parameters $x, y, z, u, w, P, Q, R, S$ of the last section to the Yukawa coupling constants h_B , and to determine the relations between the h_B 's that can render $c = 0$.

A. $\rho = 2, C = \text{IR3}$

1. Dirac mass matrix M_ν

From Table I, we know that $\text{IR5} \times \text{IR10}$ and $\text{IR5} \times \text{IR11}$ both contain IR3, so B in (51) consists of IR10 and IR11. The corresponding representation G_B for the residual symmetry is

$$G_{10} = -G_{11} = \begin{pmatrix} 0 & \eta^2 & 0 \\ \eta^3 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (59)$$

The alignments obtained from (50) are $\langle \phi^{11} \rangle = (-\eta^2, 1, 0)^T$ in one case, and an arbitrary linear combination of $\langle \phi^{10a} \rangle = (\eta^2, 1, 0)^T$ and $\langle \phi^{10b} \rangle = (0, 0, 1)^T$ in the other case. The reason why IR11 has one invariant eigenvector and IR10 has 2 is because $\det(G_{11}) = +1$ but $\det(G_{10}) = -1$. There are two Yukawa coupling constants h_{10} and h_{11} , but since the combination of the two invariant eigenvectors of IR10 is arbitrary, there are effectively three unknown coefficients, which we will refer to as h_{10a} , h_{10b} , and h_{11} . Using (51) and (58), we find that M_ν is of the form given in (33), with

$$\begin{aligned} x &= \frac{1}{2\sqrt{3}}(h_{10a}x^{10a} + h_{11}x^{11}), \quad y = \frac{1}{2\sqrt{3}}(h_{10a}y^{10a} + h_{11}y^{11}), \\ z &= -\frac{1}{\sqrt{3}}h_{10b}, \end{aligned} \quad (60)$$

where

$$\begin{aligned} x^{10a} &= \eta^2(1 - \sqrt{3}), & y^{10a} &= 1 + \sqrt{3}, \\ x^{11} &= \eta^2(1 + \sqrt{3}), & y^{11} &= -1 + \sqrt{3}. \end{aligned} \quad (61)$$

Using them to compute $(x + y\eta^2)$, which is proportional to c , we get

$$\begin{aligned} (x + y\eta^2)^{10a} &= 2\eta^2, & (x + y\eta^2)^{10b} &= 0; \\ (x + y\eta^2)^{11} &= 2\eta^2\sqrt{3}, \end{aligned} \quad (62)$$

hence,

$$\sqrt{3}(x + y\eta^2)^{10a} - (x + y\eta^2)^{11} = 0. \quad (63)$$

Thus, if $h_{11} = -\sqrt{3}h_{10a}$, then $x + y\eta$ is zero, giving $c = 0$ and the model considered in the last two sections. As discussed in the last section, $m_3 = 0$ follows automatically in this case.

2. Majorana mass matrix M_N

There is no difficulty in computing the Majorana mass matrix M_N , as we shall see, but M_N^{-1} is needed for \vec{M}_ν , so a matrix inversion has to be performed. Since M_N and M_N^{-1} satisfy the same symmetry constraints, (24) and (28), their solutions must have the same form. Henceforth we will use the form of M_N^{-1} and the same letters *with a prime* to parametrize M_N . In the case of (33), the relations between these two sets of parameters are

$$P = P'/D_1, \quad Q = -Q'/D_1, \quad D_1 := P'^2 - Q'^2. \quad (64)$$

From Table I, we see that $\text{IR3} \times \text{IRB} = \text{IR3}$ if $B = 1, 2$, or 3. The residual symmetry G_B for $B = 3$ is given in (32), and the other two, $G_1 = -G_2 = +1$, are given in Table I. The alignments are $\langle \phi^3 \rangle = (1, 1)^T$, $\langle \phi^1 \rangle = 1$,

and $\langle \phi^2 \rangle = 0$, so only $B = \text{IR1}$ and IR3 contribute. The Majorana mass matrix M_N^{-1} of the form given in (33), with $P' = h_1$ and $Q' = -h_3$. The parameters P, Q for M_N^{-1} can be obtained from (64).

B. $\rho = 2, C = \text{IR1} + \text{IR2}$

1. Dirac mass matrix M_ν

From Table I, we know that $\text{IR5} \times \text{IR11}$ contains IR1 , and $\text{IR5} \times \text{IR10}$ contains IR2 , so once again B in (51) consists of IR10 and IR11 . The corresponding representation G_B and the alignments are the same as those given in Sec. VA. The Dirac mass matrix M_ν is the sum of M_ν^a and M_ν^b , obtained using, respectively, alignments $\langle \phi^{10a} \rangle$ and $\langle \phi^{10b} \rangle$ for IR2 , and alignment $\langle \phi^{11} \rangle$ for IR1 . M_ν can be calculated from (51) and (58). It is of the form in (36), with

$$x = -h_{11} \eta^2 / \sqrt{3}, \quad y = h_{10a} \eta^2 / \sqrt{3}, \quad z = h_{10b} / \sqrt{3}. \quad (65)$$

From (37), we see that in order to reproduce the model with $c = 0$ and $f \neq 0$, we need to have $y = 0$ and hence $h_{10a} = 0$.

2. Majorana mass matrix M_N

Since $\text{IR1} \times \text{IR1} = \text{IR1}$ and $\text{IR2} \times \text{IR1} = \text{IR2}$, if we let the two Yukawa coupling constants be h_1 and h'_1 , respectively, then the Majorana matrix M_N^{-1} is given by (36), with $P = h_1^{-1}$ and $Q = h_1'^{-1}$.

C. $\rho = 3, C = \text{IR3} + \text{IR1}$

1. Dirac mass matrix M_ν

As remarked in Sec. IV B 3, the M_ν here is a composite, with the first two rows given by the M_ν in VA, and the last row given by the first row of the M_ν in Sec. VB. The parameters x, y, z, w in (38) can now be copied from (60) and (65) to be

$$x = \frac{1}{2\sqrt{3}}(h_{10a}x^{10a} + h_{11}x^{11}), \quad y = \frac{1}{2\sqrt{3}}(h_{10a}y^{10a} + h_{11}y^{11}), \\ z = -\frac{1}{\sqrt{3}}h_{10b}, \quad w = -\frac{1}{\sqrt{3}}h_{11}'\eta^2, \quad (66)$$

where $x^{10a}, y^{10a}, x^{11}, y^{11}$ are given in (61), h_B is the Yukawa coupling to IR3 , and h'_B is the Yukawa coupling to IR1 .

Again, as in Sec. VA, because of (63), to render $c = 0$ we need to have $h_{11} = -\sqrt{3}h_{10a}$.

2. Majorana mass matrix M_N

Both M_N and M_N^{-1} are of the form (38), with their parameters related by

$$P = (P'S' - R^2)/D_{3a}, \quad Q = -(Q'S' - R^2)/D_{3a}, \\ D_{3a} := (P' - Q')D_{3b}, \quad R = -R'/D_{3b}, \\ S = (P' + Q')/D_{3b}, \quad D_{3b} := (P' + Q')S' - 2R'^2. \quad (67)$$

The parameters for M_N can be partially copied from Secs. VA and VB:

$$P' = h_1, \quad Q' = -h_3, \quad R' = h'_3/\sqrt{2}, \quad S' = h'_1, \quad (68)$$

where the couplings are, respectively, for $\text{IR3} \times \text{IR1} \rightarrow \text{IR3}$, $\text{IR3} \times \text{IR3} \rightarrow \text{IR3}$, $\text{IR3} \times \text{IR3} \rightarrow \text{IR1}$, and $\text{IR1} \times \text{IR1} \rightarrow \text{IR1}$.

D. $\rho = 3, C = \text{IR3} + \text{IR2}$

1. Dirac mass matrix M_ν

As remarked in Sec. IV B 4, the M_ν here is a composite, with the first two rows given by the M_ν in Sec. VA, and the last row given by the second row of the M_ν in Sec. VB. The parameters x, y, z, w in (38) can now be copied from (60) and (65) to be

$$x = \frac{1}{2\sqrt{3}}(h_{10a}x^{10a} + h_{11}x^{11}), \quad y = \frac{1}{2\sqrt{3}}(h_{10a}y^{10a} + h_{11}y^{11}), \\ z = -\frac{1}{\sqrt{3}}h_{10b}, \quad w = -\frac{1}{\sqrt{3}}h_{11}'\eta^2, \quad (69)$$

where $x^{10a}, y^{10a}, x^{11}, y^{11}$ are given in (61), h_B is the Yukawa coupling to IR3 , and h'_B is the Yukawa coupling to IR1 .

Again, as in Sec. VA, because of (63), to render $c = 0$ we need to have $h_{11} = -\sqrt{3}h_{10a}$.

2. Majorana mass matrix M_N

Both M_N and M_N^{-1} are of the form (40), with their parameters related by

$$P = (P'S' - R^2)/D_{4a}, \quad Q = -(Q'S' + R^2)/D_{4a}, \\ D_{4a} := (P' + Q')D_{4b}, \quad R = -R'/D_{4b}, \\ S = (P' - Q')/D_{4b}, \quad D_{4b} := (P' - Q')S' - 2R'^2. \quad (70)$$

The parameters for M_N can be partially copied from Secs. VA and VB:

$$P' = h_1, \quad Q' = -h_3, \quad R' = h'_3/\sqrt{2}, \quad S' = h'_1, \quad (71)$$

where the couplings are, respectively, for $\text{IR3} \times \text{IR1} \rightarrow \text{IR3}$, $\text{IR3} \times \text{IR3} \rightarrow \text{IR3}$, $\text{IR3} \times \text{IR3} \rightarrow \text{IR2}$, and $\text{IR2} \times \text{IR1} \rightarrow \text{IR2}$.

VI. NEUTRINO MIXING OF $\Delta(600)$

A parallel was mentioned in the Introduction between symmetry considerations of the old and the new mixing data. The zero reactor angle and the maximal atmospheric angle of the old data can be explained by a S_3 symmetry, a group generated by the matrices $\tilde{F} = \text{diag}(1, \omega, \omega^2)$, and an order-2 unitary matrix \tilde{G} with an invariant eigenvector $(0, 1, -1)^T$. To explain the solar angle by symmetry as well, we need another order-2 unitary matrix \tilde{G}' which commutes with \tilde{G} . Such a matrix with an invariant eigenvector $(1, 1, 1)^T$ does exist and is contained in the group $S_4 \supset S_3$. These two invariant eigenvectors together give rise to tribimaximal mixing and the correct solar angle.

With the new data of a nonzero reactor angle and possibly nonmaximal atmospheric mixing, we saw in the last three sections that they can now be explained by the symmetry group $\Delta(150)$, a group generated by F and G of (25) and (27). Again, to explain the solar angle by symmetry, we need to find another order-2 unitary operator G' which commutes with G . The smallest group containing such an operator, as well as the subgroup $\Delta(150)$, is $\Delta(600)$.

This motivation for the full-symmetry group $\Delta(600)$ is based completely on symmetry considerations. There is another motivation based on dynamics which also has a parallel with the old data. The most popular dynamical model for the old data is based on the symmetry group A_4 [17], generated by \tilde{F} and \tilde{G}' above. If its neutrino coupling to the $\mathbf{1}'$ and $\mathbf{1}''$ valons are dynamically suppressed, then the solar angle comes out correct, the matrix \tilde{G} also becomes a symmetry, and the group A_4 is promoted to S_4 .

The analogy of A_4 with the new data is $\Delta(150)$. By setting the parameter c dynamically to zero, we saw in the last three sections that the solar angle can also be explained. In the GAP basis where the generators of $\Delta(150)$ are the F, G of (25) and (27), the mixing vector for the model with $c = 0$ is $(0, 0, 1)^T$. It turns out this is just the invariant eigenvector of the operator G' in the group $\Delta(600)$.

To see that, let us first find out what G' is. It must be an order-2 element of $\Delta(600)$ that commutes with G of (27). In terms of the generators of $\Delta(600)$ given in Sec. II B, $G = f_1 c_3^2 c_4^4$. According to (7), f_1, c_3, c_4 all commute with the order-2 element d_3 , so a natural candidate for G' is a three-dimensional representation of d_3 . From Table V, there are 18 three-dimensional representations. We must choose one for f and g which reproduces F and G in (25). This turns out to be IR14. The representation of d_3 in IR14 is $G' = \text{diag}(-1, -1, 1)$, which does possess the desired invariant eigenvector $(0, 0, 1)^T$.

We know from Sec. II B that F and G' generate an A_4 subgroup of $\Delta(600)$, and we already know that F and G generate the subgroup $\Delta(150)$. In this way, $\Delta(600)$ for

the new data is like S_4 for the old data, which contain and are generated by the matrices \tilde{F}, \tilde{G} , and \tilde{G}' . In fact, if we use V of (12) to convert the F diagonal to the GAP representation, \tilde{F} is the same as F , and \tilde{G}' is the same as G' , though \tilde{G} and G are different. \tilde{F}, \tilde{G} generate an S_3 subgroup of S_4 , and F, G generate a $\Delta(150)$ subgroup of $\Delta(600)$.

A. Left-handed mass matrices for $\Delta(600)$

Applying the symmetry constraints

$$F^\dagger \bar{M}_e F = \bar{M}_e, \quad G^T \bar{M}_\nu G = \bar{M}_\nu, \quad G'^T \bar{M}_\nu G' = \bar{M}_\nu, \quad (72)$$

to the Hermitian \bar{M}_e and the symmetric \bar{M}_ν , we get the same solution \bar{M}_e , as shown in (26). For \bar{M}_ν , the extra symmetry constraint provided by G' reduces the form in (26) to the form in (18) for the special model. Hence, the $c = 0$ model of $\Delta(150)$ can be realized by the increased symmetry of $\Delta(600)$, just like the zero $\mathbf{1}', \mathbf{1}''$ model of A_4 can be realized by the increased symmetry of S_4 .

VII. SUMMARY

The symmetry group $\Delta(150)$ is applied to neutrino mixing to obtain the correct reactor and atmospheric angles. Three methods are used to implement this symmetry: by imposing it on the left-handed fermions alone, by imposing it on both the left-handed and right-handed fermions, and by constructing dynamical models based on the symmetry. The relative merits of these three methods are discussed. Generally speaking, as far as mixing parameters are concerned, it is sufficient to use the left-handed symmetry. A simple model with the parameter $c = 0$ has been discussed in some detail. This model gives the correct solar angle and can be reproduced from the enlarged group $\Delta(600)$ by symmetry alone.

APPENDIX: REAL REPRESENTATIONS OF $\Delta(150)$

We see from Table I in Sec. II that the characters of IR1, 2, 3, 12, 13 are real. As noted in Sec. II A, these are actually real rather than quaternionic representations because their Frobenius-Schur indices are +1. Let $f_i(n)$ denote the representation of the generator f_i in $\text{IR}n$. For IR1 and IR2, the irreducible representation is the same as the character, which is explicitly real. For IR3, the representation given in GAP

$$\begin{aligned} f_1(3) &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & f_2(3) &= \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}, \\ f_3(3) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & f_4(3) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned} \quad (\text{A1})$$

is not all real, and neither is the case for IR12 and IR13:

$$\begin{aligned}
 f_1(12) &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, & f_2(12) &= \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \\
 f_3(12) &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \eta^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & \eta^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \eta^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta^3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, & f_4(12) &= \begin{pmatrix} \eta & 0 & 0 & 0 & 0 & 0 \\ 0 & \eta^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & \eta & 0 & 0 & 0 \\ 0 & 0 & 0 & \eta^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta^4 & 0 \\ 0 & 0 & 0 & 0 & 0 & \eta^4 \end{pmatrix};
 \end{aligned} \tag{A2}$$

$$\begin{aligned}
 f_1(13) &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, & f_2(13) &= \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \\
 f_3(13) &= \begin{pmatrix} \eta & 0 & 0 & 0 & 0 & 0 \\ 0 & \eta^4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta^4 & 0 \\ 0 & 0 & 0 & 0 & 0 & \eta \end{pmatrix}, & f_4(13) &= \begin{pmatrix} \eta & 0 & 0 & 0 & 0 & 0 \\ 0 & \eta^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \eta^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \eta^3 & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta^4 & 0 \\ 0 & 0 & 0 & 0 & 0 & \eta^3 \end{pmatrix}.
 \end{aligned} \tag{A3}$$

Although these matrices may not be real, they are all unitary. In that case [12], there is a unitary and symmetric matrix $B(n)$ which renders $B(n)f_i(n)B(n)^{-1} = \bar{f}_i(n)$ explicitly real. Moreover, $S(n)f_i(n)S(n)^{-1} = f_i(n)^*$ when $S(n) = B(n)^2$. Our task is to find $S(n)$ and $B(n)$, and here is how.

To simplify the writing, let us drop the dependence on n and i . If v is an eigenvector of \bar{f} with eigenvalue λ , then v^* is an eigenvector of f with eigenvalue λ^* . From $BfB^{-1} = \bar{f} = \bar{f}^*$, we conclude that $u := B^{-1}v$ is an eigenvector of f with eigenvalue λ , and $w := B^{-1}v^*$ is an eigenvector of f with eigenvalue λ^* . If λ is complex and nondegenerate, then $v = Bu = (Bw)^*$; hence,

$$Su = w^*, \tag{A4}$$

because B is unitary and symmetric. Even if λ is degenerate or real, $(Su)^*$ is still an eigenvector of f with eigenvalue λ . Together with (A4), these relations can be used to obtain $S(n)$ from $u_i(n)$ and $w_i(n)$.

Once S is known, $B = \sqrt{S}$ can be computed as follows. Suppose e_a is the normalized eigenvector of S with eigenvalue σ_a ; then

$$S = \sum_a \sigma_a e_a e_a^\dagger, \quad \Rightarrow \quad B = \sum_a \sqrt{\sigma_a} e_a e_a^\dagger. \tag{A5}$$

Since each $\sqrt{\sigma_a}$ has two values, there can be many B 's, but any one of them would do.

1. $n = 3$

Only $f_2(3)$ in (A1) is complex. $u = (1, 0)^T$ is its eigenvector with eigenvalue $\lambda = \omega$, and $w = (0, 1)^T$ is its eigenvector with eigenvalue $\lambda^* = \omega^2$. Thus, $Su = w^*$ implies

$$S(3) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = e_1 e_1^\dagger - e_2 e_2^\dagger, \tag{A6}$$

with $e_1 = (1, 1)^T/\sqrt{2}$ and $e_2 = (1, -1)^T/\sqrt{2}$. Hence,

$$B(3) = e_1 e_1^\dagger + i e_2 e_2^\dagger = \begin{pmatrix} x & x^* \\ x^* & x \end{pmatrix}, \quad (\text{A7})$$

where $x = (1 + i)/2 = \exp(\pi i/4)/\sqrt{2}$. The real representations $B(3)f_i(3)B(3)^{-1} = \bar{f}_i(3)$ are given by

$$\begin{aligned} \bar{f}_1(3) &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & \bar{f}_2(3) &= -\frac{1}{2} \begin{pmatrix} 1 & 2c_{7/12} \\ -2c_{7/12} & 1 \end{pmatrix}, \\ \bar{f}_3(3) &= \bar{f}_4(3) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \end{aligned} \quad (\text{A8})$$

where

$$c_r := \cos(2\pi ir), \quad s_r := \sin(2\pi ir). \quad (\text{A9})$$

We computed $S(3)$ using (A4) to illustrate the general technique. In the present case, it can be obtained without this heavy machinery. Since $f_2(3)^*$ is obtained from $f_2(3)$ by interchanging the (11) and (22) entries, it is obvious that $S(3)$ must be given by (A6).

2. $n = 12$

Since the complex matrices $f_3(12)$ and $f_4(12)$ are diagonal, it is again easy to obtain $S(12)$ directly as follows. $f_3(12)^*$ is obtained from $f_3(12)$ by interchanging the (11) entry either with the (22) or the (51) entry, but comparing $f_4(12)^*$ with $f_4(12)$ tells us that it must be the (51) and not the (22) entry. From f_3 we then know that (22) must swap with (62) to get f_3^* ; then from f_4 we know that (33) must swap with, thereby obtaining

$$S(12) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (\text{A10})$$

Taking the square root of $S(12)$, we get

$$B(12) = \begin{pmatrix} x & 0 & 0 & 0 & 0 & x^* \\ 0 & x & 0 & x^* & 0 & 0 \\ 0 & 0 & x & 0 & x^* & 0 \\ 0 & x^* & 0 & x & 0 & 0 \\ 0 & 0 & x^* & 0 & x & 0 \\ x^* & 0 & 0 & 0 & 0 & x \end{pmatrix}. \quad (\text{A11})$$

The real representations $B(12)f_i(12)B(12)^{-1} = \bar{f}_i(12)$ are then given by

$$\begin{aligned} \bar{f}_1(12) &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, & \bar{f}_2(12) &= \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \\ \bar{f}_3(12) &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & c_{2/5} & 0 & c_{3/20} & 0 & 0 \\ 0 & 0 & c_{2/5} & 0 & c_{7/20} & 0 \\ 0 & c_{7/20} & 0 & c_{2/5} & 0 & 0 \\ 0 & 0 & c_{3/20} & 0 & c_{2/5} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, & \bar{f}_4(12) &= \begin{pmatrix} c_{1/5} & 0 & 0 & 0 & 0 & c_{9/20} \\ 0 & c_{2/5} & 0 & c_{3/20} & 0 & 0 \\ 0 & 0 & c_{1/5} & 0 & c_{9/20} & 0 \\ 0 & c_{7/20} & 0 & c_{2/5} & 0 & 0 \\ 0 & 0 & c_{1/20} & 0 & c_{1/5} & 0 \\ c_{1/20} & 0 & 0 & 0 & 0 & c_{1/5} \end{pmatrix}. \end{aligned} \quad (\text{A12})$$

a. $n = 13$ The computation is similar to the $n = 12$ case. The results are

$$\begin{aligned}
S(13) &= \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, & B(13) &= \begin{pmatrix} x & 0 & 0 & 0 & x^* & 0 \\ 0 & x & 0 & 0 & 0 & x^* \\ 0 & 0 & x & x^* & 0 & 0 \\ 0 & 0 & x^* & x & 0 & 0 \\ x^* & 0 & 0 & 0 & x & 0 \\ 0 & x^* & 0 & 0 & 0 & x \end{pmatrix}, \\
\bar{f}_1(13) &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, & \bar{f}_2(13) &= \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \\
\bar{f}_3(13) &= \begin{pmatrix} c_{1/5} & 0 & 0 & 0 & c_{9/20} & 0 \\ 0 & c_{1/5} & 0 & 0 & 0 & -c_{9/20} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -c_{9/20} & 0 & 0 & 0 & c_{1/5} & 0 \\ 0 & c_{9/20} & 0 & 0 & 0 & c_{1/5} \end{pmatrix}, \\
\bar{f}_4(13) &= \begin{pmatrix} c_{1/5} & 0 & 0 & 0 & c_{9/20} & 0 \\ 0 & c_{2/5} & 0 & 0 & 0 & c_{7/20} \\ 0 & 0 & c_{2/5} & c_{7/20} & 0 & 0 \\ 0 & 0 & c_{3/20} & c_{2/5} & 0 & 0 \\ -c_{9/20} & 0 & 0 & 0 & c_{1/5} & 0 \\ 0 & c_{3/20} & 0 & 0 & 0 & c_{2/5} \end{pmatrix}. \tag{A13}
\end{aligned}$$

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