

Discrete symmetries and model-independent patterns of lepton mixing

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In the context of discrete flavor symmetries, we elaborate a method that allows one to obtain relations between the mixing parameters in a model-independent way. Under very general conditions, we show that flavor groups of the von Dyck type, that are not necessarily finite, determine the absolute values of the entries of one column of the mixing matrix. We apply our formalism to finite subgroups of the infinite von Dyck groups, such as the modular groups, and find cases that yield an excellent agreement with the best fit values for the mixing angles. We explore the Klein group as the residual symmetry of the neutrino sector and explain the permutation property that appears between the elements of the mixing matrix in this case.

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I. INTRODUCTION

During the last ten years, the dominant approach for understanding lepton mixing has been based on the idea that it originates from a flavor symmetry G_f , broken in such a way that the residual symmetries of the neutrino and charged lepton sector, G_ν and G_l , respectively, are different [1]. To a large extent, this approach was motivated by the very peculiar tribimaximal (TBM) mixing which was in agreement with experimental data [2]. TBM implied the form invariance of the mass matrix—a situation in which the mixing matrix is determined by symmetry alone and, in particular, does not depend on the masses of neutrinos and charged leptons.

Given the flavor group G_f , this generic program was realized in several ways depending on which symmetries remained in the neutrino and charged lepton sectors. The different residual symmetries in which G_f could be broken were related to the Majorana nature of neutrinos and to different assignments of G_l charges for the right-handed components of the charged leptons. Taking G_f as a non-Abelian discrete group with a small number of elements, most possibilities were explored and shown to successfully produce interesting mixing patterns including the TBM [3–6].

Yet, despite its appeal, this approach has certain generic problems. Since G_f is broken in two different ways, there must be two distinct Higgs sectors, one for the neutrinos and another for the charged leptons, with two different vacuum alignments. Usually some auxiliary symmetry is introduced to keep these sectors separated. Moreover, the necessary vacuum expectation values are achieved by introducing a number of *ad hoc* assumptions and new free parameters. In fact, no fully convincing model has been constructed. In some specific realizations, the mixing is determined by the additional assumptions and auxiliary symmetries rather than by the original flavor symmetry.

The now established relatively large 1–3 mixing and indications of significant deviations of 2–3 mixing from maximal [7] imply substantial deviations from TBM mixing. This casts even more doubt on the validity of the whole approach. At the same time, it was shown that the residual symmetries can still be consistent with nonzero 1–3 mixing and nonmaximal θ_{23} [8].

In our previous paper, we began a study whose ultimate goal is to clarify up to which extent the flavor symmetry G_f can fix the mixing in a model-independent way [9]. We developed an algorithm for “symmetry building” and used it to obtain relations between the mixing parameters without explicit reference to any particular model. Since G_f is completely broken in the whole theory, there are corrections to the mixing patterns derived that do depend on the details of the specific model and cannot be obtained in our approach. Our results constitute a good approximation to those models in which the corrections are small.

More specifically, we showed under general assumptions that the flavor symmetry group, reconstructed from G_ν and G_l , is of the von Dyck type. We derived general relations between the elements of the mixing matrix which follow directly from the group structure. It was shown that if a \mathbf{Z}_2 symmetry is taken as the residual symmetry of the neutrino sector, one column of the mixing matrix turns out to be determined. This allows in particular for nonzero 1–3 mixing and a deviation of 2–3 mixing from maximal. Fixing one column of the mixing matrix produces two relations between the mixing angles and CP phase, and we explored how those relations compare with the experimental values when G_f is finite.

A number of assumptions were made in Ref. [9] that simplified our considerations but reduced generality. In this paper, we generalize the results of Ref. [9] in the following directions:

- (i) We allow the charged leptons to carry arbitrary charges under G_l , only subject to the constraint that G_f is a subgroup of $SU(3)$. This makes the results applicable to a wider class of groups.

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We obtain the general conditions imposed on the mixing matrix in this case.

- (ii) We apply the formalism to finite groups that are not of the von Dyck type but that have been used nonetheless for building models of flavor. In this paper we consider the groups $PSL(2, \mathbf{Z}_7)$, $\Delta(96)$ and $\Delta(384)$ that derive from restricting the finite modular groups Γ_N (with $N > 5$) to their $SU(3)$ representations. The formalism is applicable because these groups are subgroups of infinite von Dyck groups and they have a presentation in which one generator gives rise to a \mathbf{Z}_2 subgroup. $PSL(2, \mathbf{Z}_7)$, $\Delta(96)$, and $\Delta(384)$ were first studied systematically in Ref. [10], where it was shown that they can lead to interesting patterns of mixing that are in agreement with recent experimental results. We show that the pattern of mixing derived from the corresponding von Dyck group, applies unchanged to the mixing patterns of these groups.
- (iii) Finally, we consider the possibility that G_ν is the Klein group ($\mathbf{Z}_2 \otimes \mathbf{Z}_2$). We use the formalism to explain a permutation property that appears in the mixing matrices in this case.

In Ref. [9] we assumed that neutrinos are Majorana particles. We restrict to that assumption in this work as well. Nonetheless, the method developed in these papers can be generalized to the case in which fermion masses are Dirac that serves in particular for analyzing how quarks fit inside this scheme. This is outside the scope of this work and will be analyzed elsewhere.

The paper is organized as follows. In Sec. II we present the ‘‘symmetry building’’ approach, which is a generalization of the approach in our previous paper [9]. We proceed to obtain the model-independent symmetry relations between the mixing matrix elements, and we show the application of this general formalism to the finite von Dyck groups. In Sec. III we generalize our formalism to finite subgroups of the infinite von Dyck groups. Section IV is devoted to the mixing patterns that are obtained when the complete Klein group is taken as the residual symmetry in the neutrino sector. Section V contains conclusions.

II. SYMMETRY BUILDING

A. Symmetries of the mass matrices and the flavor group

We take a bottom-up approach: starting from the residual symmetries of the neutrino and charged lepton mass matrices, G_ν and G_l with $\{G_\nu, G_l\} \subset G$, we reconstruct the flavor symmetry G_f of the whole theory. We assume for the rest of the paper that G_f is a subgroup of $SU(3)$ and that neutrinos are Majorana particles.

In the flavor basis, the part of the leptonic Lagrangian which describes charged current interactions and masses reads

$$\mathcal{L} = \frac{g}{\sqrt{2}} \bar{\ell}_L \gamma^\mu \nu_L W_\mu^+ + \bar{E}_R m_\ell \ell_L + \frac{1}{2} \bar{\nu}_L^c M_{\nu U} \nu_L + \text{H.c.} \quad (1)$$

Here $\nu_L \equiv (\nu_e, \nu_\mu, \nu_\tau)_L^T$, $\ell_L \equiv (e, \mu, \tau)_L^T$, $E_R \equiv (e, \mu, \tau)_R^T$, and $m_\ell \equiv \text{diag}\{m_e, m_\mu, m_\tau\}$. The neutrino mass matrix in the flavor basis, $M_{\nu U}$, can be expressed via the mixing matrix U_{PMNS} as

$$M_{\nu U} = U_{PMNS}^* m_\nu U_{PMNS}^\dagger, \quad (2)$$

where $m_\nu \equiv \text{diag}\{m_1, m_2, m_3\}$ is the matrix of the neutrino mass eigenvalues.

Let us identify the symmetries G_ν and G_l of the neutrino and charged lepton mass terms. The generic neutrino mass matrix in Eq. (2) is invariant under the transformations

$$\nu \rightarrow S_{iU} \nu, \quad i = 1, 2, 3, \quad (3)$$

where

$$S_{iU} = U_{PMNS} S_i U_{PMNS}^\dagger \quad (4)$$

and

$$S_1 = \text{diag}\{1, -1, -1\}, \quad S_2 = \text{diag}\{-1, 1, -1\}, \quad (5)$$

$$S_3 = \text{diag}\{-1, -1, 1\}.$$

Notice that $S_3 = S_1 S_2$, which implies

$$S_{3U} = S_{1U} S_{2U}. \quad (6)$$

S_{1U} and S_{2U} commute¹ and, consequently, the transformations S_{1U} and S_{2U} generate a Klein group, $\mathbf{Z}_2 \otimes \mathbf{Z}_2$. Let $\mathbf{Z}_2^{(i)}$ for $i = 1, 2$, and 3 be the \mathbf{Z}_2 subgroup of the Klein group generated by S_{iU} . Then, G_ν can be identified either with the Klein group or with one of the \mathbf{Z}_{2i} . The matrices S_{iU} satisfy $\text{Det}[S_{iU}] = 1$, consistent with the assumption that $G \subset SU(3)$.

The charged lepton mass term has a full $U(1)^3$ symmetry. We assume that the residual symmetry in this sector is $G_l = \mathbf{Z}_m$, which is a subgroup of $U(1)^3$. A representation of G_l is given by the matrix T such that ℓ_L and E_R transform as [9],

$$\ell \rightarrow T \ell_L, \quad \ell_R \rightarrow T \ell_R, \quad (7)$$

where

$$T \equiv \text{diag}\{e^{i\phi_e}, e^{i\phi_\mu}, e^{i\phi_\tau}\} \quad (8)$$

and

$$\phi_e \equiv 2\pi \frac{k_e}{m}, \quad \phi_\mu \equiv 2\pi \frac{k_\mu}{m}, \quad (9)$$

$$\phi_\tau \equiv 2\pi \frac{k_\tau}{m} = -2\pi \frac{k_e + k_\mu}{m}.$$

¹Notice that S_1 and S_2 are the symmetry transformations of the neutrino mass matrix in the mass basis.

T satisfies the condition $T^m = \mathbb{1}$. The equality

$$\phi_e + \phi_\mu + \phi_\tau = 0 \quad (10)$$

ensures that $\text{Det}[T] = 1$, according to the assumption that $G \subset SU(3)$.

Let us now reconstruct the full flavor group G_f by taking the residual symmetries of the neutrino and charged lepton sectors as its generating elements. We consider here $G_\nu = \mathbf{Z}_2^{(i)}$ and postpone to Sec. IV the analysis of the case in which G_ν is the complete Klein group. For the charged leptons we take $G_\ell = \mathbf{Z}_m$. G_f is then defined as the group formed by all $SU(3)$ matrices that can be written as a product of the generating elements of G_ν and G_ℓ , S_{iU} and T , respectively. In particular, the matrix W_{iU} defined as

$$W_{iU} \equiv S_{iU}T, \quad (11)$$

satisfies $W_{iU} \in G$. If G_f is a finite group, there must exist an integer p such that

$$W_{iU}^p = (S_{iU}T)^p = \mathbb{1}. \quad (12)$$

Thus, a minimal set of relations that can serve as a presentation of G_f is given by

$$S_{iU}^2 = T^m = W_{iU}^p = \mathbb{1}. \quad (13)$$

The three relations in Eq. (13) define the von Dyck group $D(2, m, p)$.

In our construction it was assumed that G_f is finite. The necessary and sufficient condition for the von Dyck group $D(n, m, p)$ to be finite is

$$\frac{1}{n} + \frac{1}{m} + \frac{1}{p} > 1. \quad (14)$$

According to Eq. (14), the complete list of the finite von Dyck groups is given by²

$$\begin{aligned} D(2, 2, n) &= \mathbf{D}_n, & D(2, 3, 3) &= \mathbf{A}_4, \\ D(2, 3, 4) &= \mathbf{S}_4, & D(2, 3, 5) &= \mathbf{A}_5, \end{aligned} \quad (15)$$

which contains the most popular groups used in literature for model building.

For large p and m , the condition in Eq. (14) is not satisfied and the von Dyck group is infinite. Finite subgroups of the infinite von Dyck groups can be generated if one imposes new relations in addition to those in Eqs. (13). These new relations should be consistent with Eq. (12). We shall discuss those cases in Sec. III.

²In this paper we do not study the dihedral family $\mathbf{D}_n = D(2, 2, n)$. They have been less used for model building because they do not have irreducible representations of dimension three.

B. Symmetry and relations between mixing matrix elements

Using Eq. (4), the condition in Eq. (12) can be written explicitly as

$$W_{iU}^p = (U_{PMNS} S_i U_{PMNS}^\dagger T)^p = \mathbb{1}. \quad (16)$$

For S_i and T fixed, this equation can be considered as a condition on U_{PMNS} . It connects the group properties, determined by S_i and T , with the mixing matrix.

The explicit relations between the elements of U_{PMNS} imposed by Eq. (16) can be obtained in the following way. The equation $W_{iU}^p = \mathbb{1}$ implies that the three eigenvalues of W_{iU} , λ_j ($j = 1, 2, 3$) must satisfy³

$$\lambda_j^p = 1. \quad (17)$$

Consider then the characteristic equation for W_{iU} [9],

$$\text{Det}[W_{iU} - \lambda \mathbb{1}] = \lambda^3 - a\lambda^2 + a^*\lambda - 1 = 0, \quad (18)$$

where

$$\text{Tr}[W_{iU}] = a. \quad (19)$$

Notice that if a is known, the eigenvalues of W_{iU} are completely determined. On the other hand, according to Eq. (17), a must be a sum of three p th roots of unity,

$$a = \sum_{j=1}^3 \lambda_j^{(p)}, \quad [\lambda_j^{(p)}]^p = 1. \quad (20)$$

In general, Eq. (20) does not determine a uniquely for any given p . However, it implies that a takes a discrete set of values which can be scanned systematically. For instance, we can write the possible values of a for $p = 3, 4$,

$$p = 3 \quad a = 1 + \omega + \omega^2 = 0; \quad (21)$$

$$p = 4 \quad a = \begin{cases} 1, \\ -1 + 2i, \\ -1 - 2i, \end{cases} \quad (22)$$

where we have taken into account that $\prod_{j=1}^3 \lambda_j^{(p)} = 1$. With a determined as above, Eq. (19) is a condition on W_{iU} and, consequently, on U_{PMNS} .

Thus, the condition in Eq. (16) is reduced to Eqs. (19) and (20). Computing the trace of W_{iU} and inserting it in Eq. (19), we have

$$\sum_{\alpha} e^{i\phi_{\alpha}} \xi_{\alpha i} = a, \quad (23)$$

where

$$\xi_{\alpha i} \equiv 2|U_{\alpha i}|^2 - 1. \quad (24)$$

³This can be seen immediately since W_{iU} can be diagonalized by unitary matrix: $W_{iU} = V^\dagger \hat{\lambda} V$, where $\hat{\lambda} \equiv \text{diag}(\lambda_1, \lambda_2, \lambda_3)$. Then the condition $W_{iU} = \mathbb{1}$ is reduced to $\hat{\lambda}^p = \mathbb{1}$.

Equation (23) gives two constraints on the parameters $|U_{\alpha i}|^2$, which correspond to the real and imaginary part of a . Let us introduce the notations

$$a_R \equiv \text{Re}[a], \quad a_I \equiv \text{Im}[a] \quad (25)$$

and

$$\phi_{\alpha\beta} \equiv \phi_\alpha - \phi_\beta, \quad \alpha, \beta = e, \mu, \tau. \quad (26)$$

Then, using the unitarity condition, $\sum_\alpha |U_{\alpha i}|^2 = 1$, it is straightforward to solve Eq. (23) for the absolute values of the entries of the mixing matrix,

$$|U_{ei}|^2 = \frac{a_R \cos \frac{\phi_e}{2} + \cos \frac{3\phi_e}{2} - a_I \sin \frac{\phi_e}{2}}{4 \sin \frac{\phi_{e\mu}}{2} \sin \frac{\phi_{e\tau}}{2}}, \quad (27)$$

$$|U_{\mu i}|^2 = \frac{a_R \cos \frac{\phi_\mu}{2} + \cos \frac{3\phi_\mu}{2} - a_I \sin \frac{\phi_\mu}{2}}{4 \sin \frac{\phi_{e\mu}}{2} \sin \frac{\phi_{\mu\tau}}{2}}, \quad (28)$$

$$|U_{\tau i}|^2 = \frac{a_R \cos \frac{\phi_\tau}{2} + \cos \frac{3\phi_\tau}{2} - a_I \sin \frac{\phi_\tau}{2}}{4 \sin \frac{\phi_{e\tau}}{2} \sin \frac{\phi_{\mu\tau}}{2}}. \quad (29)$$

These equations are the central result of this paper. For a specific S_i and a selected set of parameters $\{m, k_e, k_\mu, p, a\}$, they give the absolute values of the i th column of U_{PMNS} ,

$$\mathcal{V}_i^T(m, k_e, k_\mu, p, a) \equiv (|U_{ei}|^2, |U_{\mu i}|^2, |U_{\tau i}|^2). \quad (30)$$

We call the five parameters $\{m, k_e, k_\mu, p, a\}$ the ‘‘symmetry assignment.’’ The first parameter is the order of the residual symmetry G_I , while the second and third ones are the lepton charges under this subgroup. The fourth and fifth parameters are properties of W_{iU} . They link G_p and G_I and give the restrictions on the mixing matrix. The von Dyck group is defined by m and p , while k_e, k_μ , and a specify its three-dimensional representation.

It can be easily shown that a permutation of m and p corresponds to the same von Dyck group. Indeed, taking into account that $S_{iU}^2 = \mathbb{1}$, one has

$$S_{iU} W_{iU} = T. \quad (31)$$

Substituting T from this equation in Eq. (13), we obtain

$$S_{iU}^2 = W_{iU}^p = (S_{iU} W_{iU})^m = \mathbb{1}. \quad (32)$$

These relations define the von Dyck group $D(2, p, m)$, which proves that $D(2, m, p) = D(2, p, m)$.

The presentation in Eq. (32) has the same form of Eq. (13) with the exchanges $W_{iU} \leftrightarrow T$ and $m \leftrightarrow p$. This implies that W_{iU} can be taken as the generator of G_I instead of T . In that case, the last equality in Eq. (32) imposes relations between the mixing matrix elements in the same way as the equality $(S_{iU} T)^p = \mathbb{1}$ did before.

However, since W_{iU} is not diagonal, the presentation in Eq. (32) is in a basis that is not the flavor basis. To get the mixing matrix in the flavor basis, we apply the unitary

transformation V that diagonalizes W_{iU} on the elements of G_f . Hence, we define

$$T' \equiv V^\dagger W_{iU} V, \quad S'_{iU} \equiv V^\dagger S_{iU} V, \quad (33)$$

where T' is diagonal. The new generator of G_I can be written as

$$T' = \text{diag}\{e^{i\phi'_e}, e^{i\phi'_\mu}, e^{i\phi'_\tau}\} \quad (34)$$

with

$$\begin{aligned} \phi'_e &\equiv 2\pi \frac{k'_e}{p}, & \phi'_\mu &\equiv 2\pi \frac{k'_\mu}{p}, \\ \phi'_\tau &\equiv 2\pi \frac{k'_\tau}{p} = -2\pi \frac{k'_e + k'_\mu}{p}. \end{aligned} \quad (35)$$

The group presentation does not change when written in terms of S' and T' , so we have

$$S'^2_{iU} = T'^p = (W'_{iU})^m = \mathbb{1}, \quad (36)$$

where

$$W'_{iU} \equiv S'_{iU} T' = V^\dagger T V. \quad (37)$$

Following our derivation of Eq. (19), we obtain that the last equality in Eq. (36) leads to a condition for the elements of the new mixing matrix U'_{PMNS} that we can write as

$$\text{Tr}[W'_{iU}] = a' = \text{Tr}[T]. \quad (38)$$

Here the last equality follows from Eq. (37). Thus, a' can be obtained immediately by computing the trace of the original matrix T .

We call the symmetry assignment $\{p, k'_e, k'_\mu, m, a'\}$ ‘‘ mp -permuted’’ with respect to $\{m, k_e, k_\mu, p, a\}$. The new mixing matrix U'_{PMNS} can be obtained from the matrix of the original assignment, U_{PMNS} , by noticing that

$$S'_{iU} = V^\dagger S_{iU} V = V^\dagger U'_{PMNS} S_i U'_{PMNS}{}^\dagger V, \quad (39)$$

and therefore

$$U'_{PMNS} = V^\dagger U_{PMNS}. \quad (40)$$

C. Applications. Finite von Dyck groups

As it was established in the previous section, the procedure of finding relations between the mixing matrix elements, given a specific symmetry assignment, consists of the following steps:

- (1) Using Eq. (20), find $a = a(\lambda_i^{(p)})$.
- (2) From m, k_e, k_μ , determine the phases $\phi_\alpha = \phi_\alpha(m, k_e, k_\mu)$ and $\phi_{\alpha\beta} = \phi_{\alpha\beta}(m, k_e, k_\mu)$, according to Eq. (9).
- (3) Using a_R, a_I, ϕ_α , and $\phi_{\alpha\beta}$, determine $|U_{\alpha i}|^2$ from Eqs. (27)–(29).

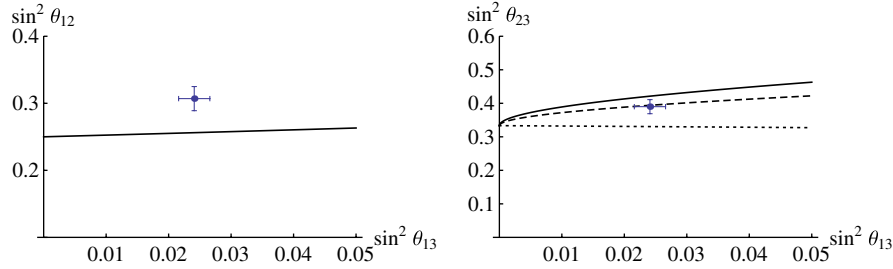


FIG. 1 (color online). $\sin^2\theta_{12}$ (left panel) and $\sin^2\theta_{23}$ (right panel) as functions of $\sin^2\theta_{13}$ for the assignment $\{m, k_e, k_\mu, p, a\} = \{4, 3, 0, 3, 0\}$. The curves in the right panel correspond to $\delta = \pi$ (thick), $3\pi/4$ (dashed), $\pi/2$ (dotted).

- (4) Substituting the standard parametrization of U_{PMNS} in Eqs. (27)–(29), obtain the two conditions for the three mixing angles and CP phase.

We apply this procedure in what follows.

Equations (27)–(29) generalize the results obtained in Ref. [9], where a vanishing value for one of k_e , k_μ , or k_τ was taken. This corresponds to $a_I = 0$ and real values of both $\text{Tr}[W]$ and $\text{Tr}[T]$. In this case $T = T_\alpha$, where $\alpha = e, \mu$ or τ , and it has the diagonal elements:

$$T_{\alpha\alpha} = 1, \quad T_{\beta\beta} = T_{\gamma\gamma}^* = e^{2\pi ik/m}, \quad \beta \neq \gamma \neq \alpha. \quad (41)$$

For instance, $T_e = \text{diag}\{1, e^{2\pi ik/m}, e^{-2\pi ik/m}\}$, etc. For each of these cases, Eqs. (27)–(29) reduce to

$$|U_{\alpha i}|^2 = \eta, \quad |U_{\beta i}|^2 = |U_{\gamma i}|^2 \quad \beta, \gamma \neq \alpha, \quad (42)$$

where

$$\eta \equiv \frac{1+a}{4\sin^2(\frac{\pi k}{m})}. \quad (43)$$

In Ref. [9], several cases were analyzed in which Eqs. (42) and (43) led to values for the mixing angles that are compatible with the experimental results. For instance, taking the symmetry assignment $\{m, k_e, k_\mu, p, a\} = \{3, 0, 1, 4, 1\}$, which corresponds to taking T_e in Eq. (41), we obtain the column of mixing parameters,

$$\mathcal{V}_i^T = \left(\frac{2}{3}, \frac{1}{6}, \frac{1}{6}\right). \quad (44)$$

For $\{m, k_e, k_\mu, p, a\} = \{3, 0, 1, 3, 0\}$ the column equals

$$\mathcal{V}_i^T = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right). \quad (45)$$

Both predictions in Eqs. (44) and (45) are in agreement with nonzero θ_{13} . The case of Eq. (44) applied to $i = 1$ is known in the literature as the trimaximal 1 (TM1) mixing pattern, while the second case, Eq. (45) for $i = 2$, is the trimaximal 2 (TM2) pattern [11]. The derivation presented here shows that TM1 and TM2 are specific cases of the general formulas in Eqs. (27)–(29), which, in turn, can be derived directly from the group structure.

Let us consider the choice of parameters $\{m, k_e, k_\mu, p, a\} = \{4, 3, 0, 3, 0\}$ that has not been explored

in previous works. T_μ is given by $\text{diag}\{-i, 1, i\}$ and the corresponding group is \mathbf{S}_4 . In this case $|U_{\mu 2}|^2 = \frac{1}{2}$ and the second column fixed at

$$\mathcal{V}_2^T = \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right). \quad (46)$$

Using the standard parametrization, Eq. (46) yields the following relations between the mixing angles:

$$\sin^2\theta_{12}\cos^2\theta_{13} = \frac{1}{4}, \quad (47)$$

$$\cos^2\theta_{12}\cos^2\theta_{23} + \sin^2\theta_{12}\sin^2\theta_{23}\sin^2\theta_{13} - 2\cos\theta_{12}\sin\theta_{12}\cos\theta_{23}\sin\theta_{23}\sin\theta_{13}\cos\delta = \frac{1}{2}. \quad (48)$$

In the limit of $\theta_{13} \rightarrow 0$ these equalities give $\sin^2\theta_{12} = 1/4$ and $\sin^2\theta_{23} = 1/3$. In Fig. 1 we plot the values of $\sin^2\theta_{12}$ and $\sin^2\theta_{23}$ that result from Eqs. (47) and (48). The prediction for $\sin^2\theta_{12}$ is more than 3σ away from the best-fit value, so that this case could correspond to a first-order approximation, and large corrections are required.

For the case of finite von Dyck groups, we performed a scan over all possible assignments in Eqs. (27)–(29). We find that the only assignments compatible with the requirement that $0 \leq |U_{\alpha i}|^2 \leq 1$ are precisely the ones that correspond to real $\text{Tr}[W]$ and $\text{Tr}[T]$. In other words, the relations found in Ref. [9] and in this section describe all the possible mixing patterns that can be derived from the finite von Dyck groups in this framework.

III. FINITE SUBGROUPS OF THE INFINITE VON DYCK GROUPS

A. General considerations

In this section we extend the analysis of the previous sections to finite subgroups of infinite von Dyck groups. As examples, we will consider the groups $PSL(2, \mathbf{Z}_7)$, $\Delta(96)$ and $\Delta(384)$ explored in Ref. [10]. The results of this section are also consistent with the scan of mixing patterns that can appear from subgroups of $SU(3)$ of order less than 511 that was presented in Ref. [12].

$PSL(2, \mathbf{Z}_7)$, $\Delta(96)$, and $\Delta(384)$ can be generated by the matrices S_{IU} and T that satisfy the von Dyck relations,

Eq. (13), with $\{p, m\} = \{7, 3\}$ or $\{7, 4\}$ for $PSL(2, \mathbf{Z}_7)$, $\{p, m\} = \{8, 3\}$ for $\Delta(96)$, and $\{p, m\} = \{16, 3\}$ for $\Delta(384)$. In order to render the group finite, a relation of the form

$$X^q = \mathbb{1}, \quad X \equiv S_{iU} T^{-1} S_{iU} T, \quad (49)$$

where q is an integer, is added to Eq. (13). For the case of the finite von Dyck groups discussed in the previous section, Eq. (49) is redundant after Eq. (13). In the case of infinite groups, the equality in Eq. (49) is in general invalid as we will see below. Hence, it imposes additional restriction on the parameters of the group.

Notice that $S_{iU} X^\dagger S_{iU} = X$ and, taking into account that $S = S^{-1}$, we find that X and X^\dagger are similar matrices with the same eigenvalues. Therefore, if λ is an eigenvalue of X , λ^* must also be. Since X is a 3×3 matrix, we conclude that X must have one eigenvalue equal to 1 and that $\text{Tr}[X]$ must be a real number.

We will use the same method as in in Secs. II and III to analyze the relation in Eq. (49). In particular, from the equality in Eq. (49), we obtain

$$\text{Tr}[X] = x, \quad (50)$$

where x is the sum of three q th roots of unity,

$$x = \sum_{j=1}^3 \lambda_j^{(q)}, \quad [\lambda_j^{(q)}]^q = 1, \quad (51)$$

and it takes a discrete set of values. The trace in Eq. (50) equals

$$\begin{aligned} \text{Tr}[X] &= \xi_{ei}^2 + \xi_{\mu i}^2 + \xi_{\tau i}^2 + 8[\cos \phi_{e\mu} |U_{ei}|^2 |U_{\mu i}|^2 \\ &\quad + \cos \phi_{e\tau} |U_{ei}|^2 |U_{\tau i}|^2 + \cos \phi_{\mu\tau} |U_{\mu i}|^2 |U_{\tau i}|^2] \\ &= |\text{Tr}[W]|^2 - 2[\cos \phi_{e\mu} \xi_{\tau i} + \cos \phi_{\mu\tau} \xi_{ei} \\ &\quad + \cos \phi_{\tau e} \xi_{\mu i}]. \end{aligned} \quad (52)$$

Defining

$$A = 2 + e^{i\phi_{e\mu}} + e^{-i\phi_{e\mu}} + e^{i\phi_{\mu\tau}} + e^{-i\phi_{\mu\tau}} + e^{i\phi_{\tau e}} + e^{-i\phi_{\tau e}}, \quad (53)$$

we can rewrite Eq. (52) as

$$\text{Tr}[X] = |\text{Tr}[W_{iU}]|^2 + \text{Tr}[W_{iU}]^\dagger \text{Tr}[T] + \text{Tr}[W_{iU}] \text{Tr}[T]^\dagger + A. \quad (54)$$

Note that the rhs in Eq. (54) is invariant under the permutation $T \leftrightarrow T^\dagger$. That is, taking $\{S_{iU}, T\}$ or $\{S_{iU}, T^\dagger\}$ as generators leads to the same $\text{Tr}[X]$.

Substituting Eq. (19) in Eq. (54), we obtain

$$x = |a|^2 + a \text{Tr}[T]^\dagger + a^\dagger \text{Tr}[T] + A. \quad (55)$$

This equality does not depend on U_{PMNS} and, therefore, the additional relation in Eq. (49)—or Eq. (50)—does not add any new constraint to the mixing parameters. Instead,

Eq. (55) places a condition that the eigenvalues of the matrix T must satisfy.

For some particular values of p and a , Eq. (55) acquires a particularly simple form. For $p = 3$ and $a = 0$, Eq. (54) reduces to

$$x = A. \quad (56)$$

For $p = 4$ and $a = 1$, Eq. (54) reads

$$x = 1 + \text{Tr}[T] + \text{Tr}[T]^\dagger + A. \quad (57)$$

Notice that Eq. (57) is satisfied, in particular, when $x = A$ and $\text{Re}(\text{Tr}[T]) = -1/2$. If A is found from Eqs. (56) or (57), or in general from (55), then the Eq. (53) with known A can be used to obtain k_e and k_μ .

As in the case of finite von Dyck groups, the mp -permuted solution exists. Indeed, since $T = S_{iU} W_{iU}$, the matrix X can be rewritten in terms of S_{iU} and W_{iU} as

$$X = S_{iU} (T^{-1} S_{iU}) S_{iU} (S_{iU} T) = S_{iU} W_{iU}^{-1} S_{iU} W_{iU}. \quad (58)$$

That is, the element X has the same form when written in terms of S_{iU} and T and in terms of S_{iU} and W_{iU} so that the elements W_{iU} and T can exchange roles. Using S'_{iU} and T' as given in Eq. (33), we obtain

$$S'^2_{iU} = T'^p = (W'_{iU})^m = (S'_{iU} T'^{-1} S'_{iU} T')^q = \mathbb{1}. \quad (59)$$

Thus, S'_{iU} and T' generate the same group G_f . Imposing that T' is the residual symmetry of the lepton sector leads to a different mixing matrix U'_{PMNS} given by Eq. (40). Similarly to Eq. (38), the values of k_e and k_μ which determine T also determine the eigenvalues of W'_{iU} .

B. Symmetry relations from $PSL(2, \mathbf{Z}_7)$

The group $PSL(2, \mathbf{Z}_7)$ is a subgroup of the infinite von Dyck group with $p = 3$ and $m = 7$, so that the group presentation is

$$S^2_{iU} = T^7 = (S_{iU} T)^3 = \mathbb{1}. \quad (60)$$

The additional relation, Eq. (49), reads

$$X^4 = \mathbb{1}. \quad (61)$$

Since $q = 4$, the eigenvalues of X are fourth roots of unity and can be taken as $\{1, i, -i\}$, for which we have $x = 1$. For $p = 3$ we have the equality in Eq. (56) which gives $A = 1$. Substituting this value into Eq. (53) we find that it is satisfied for $k_e = 5$, $k_\mu = 3$ and $k_\tau = 6$. Thus, the symmetry assignment is

$$\{m, k_e, k_\mu, p, a\} = \{7, 5, 3, 3, 0\}. \quad (62)$$

We can now plug in the values of k_e and k_μ in Eqs. (27)–(29) to obtain the values of the mixing elements,

$$|U_{\mu i}|^2 = \frac{1}{4[1 + \sin \frac{\pi}{14}]}, \quad |U_{\tau i}|^2 = \frac{1}{4[1 + \cos \frac{\pi}{7}]}, \quad (63)$$

and $|U_{ei}|^2 = 1 - |U_{\mu i}|^2 - |U_{\tau i}|^2$.

For $i = 2$, the values of mixing parameters in Eq. (63) are in a good agreement with experiment. We call this mixing pattern, the *PSL7A* pattern; see upper panels of Fig. 2. The predicted *CP* phase has a central value of $\delta = 87.5^\circ$ and for the 1σ allowed regions of mixing angles, δ in the interval $\delta = 82^\circ - 93^\circ$.

Note that T is fixed up to permutations of its eigenvalues. The predicted entries of U_{PMNS} depend not only on i but also on the particular ordering of the k_α in T . For *PSL7A*, we chose the order that gives the best agreement with experimental data.

Consider now the mp -permuted assignment

$$\{m, k'_e, k'_\mu, p, a\} = \{3, 1, 0, 7, -(1 + i\sqrt{7})/2\}. \quad (64)$$

Now the mixing pattern can be found directly from the mixing obtained in the previous case as described in Sec. II. We get

$$\mathcal{V}_i^T = \left[\frac{1}{12}(5 + \sqrt{21}), \frac{1}{6}, \frac{1}{12}(5 - \sqrt{21}) \right]. \quad (65)$$

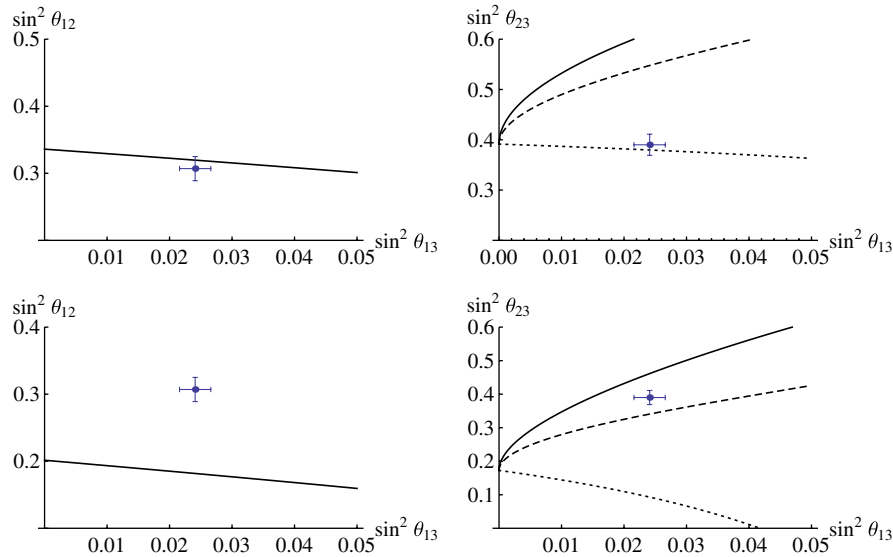


FIG. 2 (color online). $\sin^2 \theta_{12}$ (left panels) and $\sin^2 \theta_{23}$ (right panels) as functions of $\sin^2 \theta_{13}$ for the patterns *PSL7A* (top panels) and *PSL7B* (bottom panels). The curves in the right panel correspond to $\delta = 0$ (thick), $\pi/4$ (dashed), $\pi/2$ (dotted).

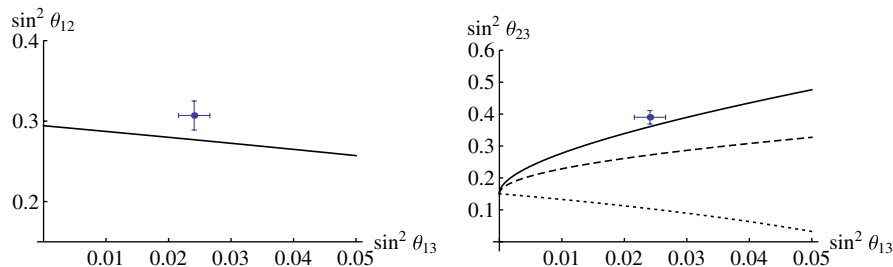


FIG. 3 (color online). $\sin^2 \theta_{12}$ (left panel) and $\sin^2 \theta_{23}$ (right panel) as functions of $\sin^2 \theta_{13}$ for the pattern *PSL7C*. The curves in the right panel correspond to $\delta = 0$ (thick), $\pi/4$ (dashed), $\pi/2$ (dotted).

For $i = 1$, we call this pattern *PSL7B*. The values of $\sin^2 \theta_{12}$ and $\sin^2 \theta_{23}$ as functions of $\sin^2 \theta_{13}$ are shown in the bottom panels of Fig. 2. The predicted value of θ_{12} is substantially smaller than the observed one in this case. Thus, large corrections would be required in order to make the pattern viable.

Besides Eq. (60) the group $PSL(2, \mathbf{Z}_7)$ has the presentation

$$S_{iU}^2 = T^7 = (S_{iU}T)^4 = \mathbb{1}, \quad X^4 = \mathbb{1}. \quad (66)$$

It leads to the same values for the mixing parameters as in Eq. (63) if $\{k_e, k_\mu, k_\tau\} = \{6, 5, 3\}$, i.e., if the T charges are permuted with respect to those in Eq. (62).

A new mixing pattern appears in the case of the mp -permuted assignment,

$$\{m, k'_e, k'_\mu, p, a\} = \{4, 3, 0, 7, -(1 + i\sqrt{7})/2\}. \quad (67)$$

In this case we find

$$\mathcal{V}_i^T = \left(\frac{1}{8}(3 + \sqrt{7}), \frac{1}{4}, \frac{1}{8}(3 - \sqrt{7}) \right). \quad (68)$$

For $i = 1$, the column in Eq. (68), which we refer to as pattern *PSL7C*, leads to the mixing angles shown in Fig. 3. Although this case still requires some corrections, the disagreement with data is smaller than in the pattern *PSL7B*.

C. Symmetry relations from $\Delta(96)$

The group $\Delta(96)$ is a finite subgroup of the infinite von Dyck group with presentation

$$S_{iU}^2 = T^8 = (S_{iU}T)^3 = \mathbb{1} \quad (69)$$

to which the additional relation

$$X^3 = \mathbb{1} \quad (70)$$

is added in order to render it finite. From Eq. (70) one obtains $x = 0$ and, consequently, we must have $A = 0$ according to Eq. (56), which holds for $p = 3$. For vanishing A , Eq. (53) is satisfied for $k_e = 5$ and $k_\mu = 2$. Substituting these values in Eqs. (27)–(29), we find the column $\mathcal{V}_i^T = \{1/4, 1/4, 1/2\}$ and the possibilities that derive from all possible permutations. These cases were studied in Sec. II.

Turning to the mp -permuted assignment, we have

$$\{m, k'_e, k'_\mu, p, a'\} = \{3, 1, 0, 8, i\}, \quad (71)$$

which leads to the mixing pattern

$$\mathcal{V}_i^T = \left(\frac{2 + \sqrt{3}}{6}, \frac{1}{3}, \frac{2 - \sqrt{3}}{6} \right). \quad (72)$$

The mixing angles which follow from this column setting $i = 1$, mixing pattern *D96*, are shown in Fig. 4. There is no good agreement with observations and large corrections are required for this pattern to be viable.

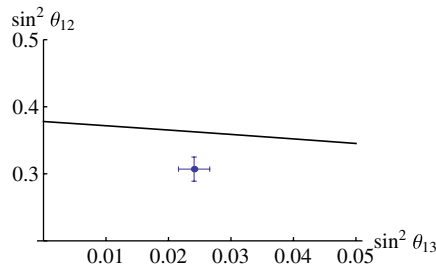
D. Symmetry relations from $\Delta(384)$

The group $\Delta(384)$ is the subgroup of the infinite von Dyck group with presentation

$$S_{iU}^2 = T^{16} = (S_{iU}T)^3 = \mathbb{1}, \quad (73)$$

where S_{iU} and T satisfy the additional relation,

$$X^3 = \mathbb{1}. \quad (74)$$



Now, $a = x = 0$, and therefore $A = 0$. Because of large $m = 16$, there are many *a priori* possibilities for the matrix T . For zero A and $m = 16$, we find that Eq. (53) is satisfied for the two sets of values of $\{k_e, k_\mu\}$, $\{-7, 1\}$ and $\{-5, 3\}$. No new mixing pattern appears when T is the residual symmetry of the charged lepton sector. However, the mp -permuted assignment leads to interesting mixing patterns, which correspond to two possible choices of a' . In the first case, for $\{k_e, k_\mu\} = \{-7, 1\}$, one finds the assignment

$$\{m, k'_e, k'_\mu, p, a'\} = \left\{ 3, 1, 1, 16, \frac{-1+i}{\sqrt{2}} \right\} \quad (75)$$

and then

$$\mathcal{V}_i^T = \left(\frac{4 + \sqrt{2} + \sqrt{6}}{12}, \frac{4 + \sqrt{2} - \sqrt{6}}{12}, \frac{2 - \sqrt{2}}{6} \right). \quad (76)$$

For $i = 1$ we call this pattern *D384A*, and the predicted mixing angles are plotted in Fig. 5. This pattern is able to fit all mixing angles for $\delta = 55^\circ$ with the 1σ range for the mixing angles giving a range for δ , $49^\circ < \delta < 61^\circ$.

In the second case, $\{k_e, k_\mu\} = \{-5, 3\}$, the mp -permuted assignment is

$$(m, k'_e, k'_\mu, p, a') = \left(3, -1, 1, 16, \frac{1+i}{\sqrt{2}} \right). \quad (77)$$

It leads to

$$\mathcal{V}_i^T = \left(\frac{4 + \sqrt{2} + \sqrt{6}}{12}, \frac{4 + \sqrt{2} - \sqrt{6}}{12}, \frac{2 - \sqrt{2}}{6} \right). \quad (78)$$

This column comes closer to the experimental values for $i = 3$. In that case, we call this pattern *D384B*. It leaves δ and θ_{12} undetermined but gives predictions for θ_{13} and θ_{23} ,

$$\sin^2 \theta_{12} = \frac{4 - \sqrt{2} - \sqrt{6}}{12} \approx 0.011, \quad (79)$$

$$\sin^2 \theta_{23} = \frac{4 - \sqrt{2} + \sqrt{6}}{8 + \sqrt{2} + \sqrt{6}} \approx 0.424$$

in very good agreement with experiment.

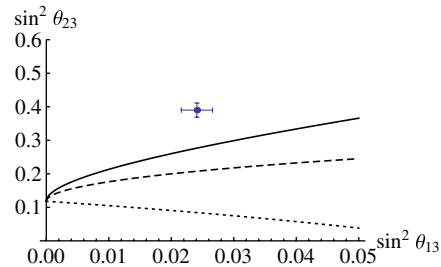


FIG. 4 (color online). $\sin^2 \theta_{12}$ (left panel) and $\sin^2 \theta_{23}$ (right panel) as functions of $\sin^2 \theta_{13}$ for the pattern *D96*. The curves in the right panel correspond to $\delta = 0$ (thick), $\pi/4$ (dashed), $\pi/2$ (dotted).

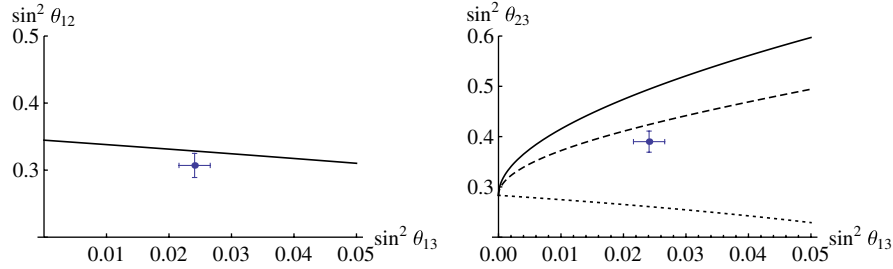


FIG. 5 (color online). $\sin^2 \theta_{12}$ (left panel) and $\sin^2 \theta_{23}$ (right panel) as functions of $\sin^2 \theta_{13}$ for the pattern $D384A$. The curves in the right panel correspond to $\delta = 0$ (thick), $\pi/4$ (dashed), $\pi/2$ (dotted).

IV. THE RESIDUAL KLEIN GROUP FOR NEUTRINOS

Let us take the full Klein group $\mathbf{Z}_2 \otimes \mathbf{Z}_2$ as the residual symmetry of the neutrino sector. Such a possibility has been realized in most of the specific models [3–5]. For definiteness, we take S_{1U} and S_{2U} as the generators of the group, so that its presentation is given by⁴

$$S_{1U}^2 = S_{2U}^2 = T^m = W_{1U}^{p_1} = W_{2U}^{p_2} = \mathbb{1}, \quad [S_{1U}, S_{2U}] = 0. \quad (80)$$

In Eq. (80) we have two sets of relations, one for S_{1U} and another for S_{2U} , of the type considered in the previous sections. These relations impose two sets of conditions on U_{PMNS} which can be written as

$$\sum_{\alpha} e^{i\phi_{\alpha}} \xi_{\alpha i} = a_i, \quad \text{for } i = 1, 2, \quad (81)$$

where $a_i \equiv \sum_j \lambda_j^{(p_i)}$ is a sum of three p_i th roots of unity. Equation (81) gives four relations, that determine two columns of the mixing matrix, which are enough to determine U_{PMNS} completely. An example of this situation appears in Eqs. (44) and (45) with $\{p_1, p_2, m\} = \{4, 3, 3\}$. The only mixing matrix compatible with both Eqs. (44) and (45) is the TBM.

In Ref. [10] it was found that, in some cases, the mixing matrix derived from Eq. (81) has the property that the absolute values of the entries of two column were equal up to a permutation. In what follows we provide a general explanation of this fact.

Consider the elements g_k of G_f that can be written as

$$g_k = S_{1U} T^k, \quad (82)$$

where k is an integer. If G_f is finite then there exists an integer r , with $r < m$, such that

$$g_k^r = (S_{1U} T^k)^r = \mathbb{1}. \quad (83)$$

Notice that g_k generates a \mathbf{Z}_r subgroup of G_f .

⁴Which S_{iU} are chosen in Eq. (80) is irrelevant due to Eq. (6).

Equation (83) implies that the entries of U_{PMNS} satisfy another relation, similar to Eq. (81) for $i = 1$, but extracted by substituting $T \rightarrow T^k$. This new relation can be obtained from Eq. (81) by replacing a and the phases correspondingly. We obtain

$$\sum_{\alpha} e^{ik\phi_{\alpha}} \xi_{\alpha 1} = \tilde{a}_k(r), \quad (84)$$

where $\tilde{a}_k(r) \equiv \sum \lambda_j^{(r)}$. Suppose now that there exists a value of k such that

$$\{k\phi_e, k\phi_{\mu}, k\phi_{\tau}\} = \text{permutation of } \{\phi_e, \phi_{\mu}, \phi_{\tau}\} \pmod{2\pi} \\ \equiv \{\phi_{h(e)}, \phi_{h(\mu)}, \phi_{h(\tau)}\}, \quad (85)$$

where h represents a permutation of e , μ , and τ . In that case, Eq. (84) can be expressed as

$$\sum_{\alpha} e^{i\phi_{h(\alpha)}} \xi_{\alpha 1} = \tilde{a}_k(r) \quad (86)$$

and, according to Eq. (85), this can be rewritten as

$$\sum_{\alpha} e^{i\phi_{\alpha}} \xi_{f(\alpha)1} = \tilde{a}_k(r), \quad (87)$$

where $f \equiv h^{-1}$ is the inverse permutation of h . If, moreover,

$$\tilde{a}_k(r) = a_2, \quad (88)$$

then Eq. (87) becomes

$$\sum_{\alpha} e^{i\phi_{\alpha}} \xi_{f(\alpha)1} = a_2. \quad (89)$$

Confronting this equation with Eq. (81) for $i = 2$, we conclude that

$$\xi_{\alpha 2} = \xi_{f(\alpha)1}. \quad (90)$$

That is, the elements of column 2 are equal to the permuted elements of column 1.

Equations (85) and (88) are the necessary and sufficient conditions for the second column of U_{PMNS} to be a permutation of the first one. The analysis does not depend on which i and j are chosen in Eq. (80), so that the same arguments apply to other pairs of columns of U_{PMNS} .

We illustrate this permutation property with the groups studied in Sec. V. The pattern $PSL7A$ corresponds to T with phases

$$\{\phi_\alpha\} \equiv \{\phi_e, \phi_\mu, \phi_\tau\} = \frac{2\pi}{7}\{5, 3, 6\}. \quad (91)$$

Considering the group element $g_2 = S_{1U}T^2$, so that $k = 2$ in Eq. (82), we find that

$$2\{\phi_\alpha\} \pmod{2\pi} = \frac{2\pi}{7}\{3, 6, 5\}. \quad (92)$$

Therefore, according to Eq. (90), we have

$$\xi_{ej} = \xi_{\tau 1}, \quad \xi_{\mu j} = \xi_{e 1}, \quad \xi_{\tau j} = \xi_{\mu 1}, \quad (93)$$

provided that

$$a_j = \tilde{a}_2. \quad (94)$$

The elements $|U_{\alpha j}|^2$ for $j \neq 1$ are a cyclic permutation of $|U_{\alpha 1}|^2$. According to Eq. (93) for $j = 2$, the full matrix of mixing parameters equals

$$|||U_{\alpha i}|^2|| = \begin{pmatrix} c_e & c_\tau & c_\mu \\ c_\mu & c_e & c_\tau \\ c_\tau & c_\mu & c_e \end{pmatrix}, \quad (95)$$

where $c_e \equiv |U_{e1}|^2$, $c_\mu \equiv |U_{\mu 1}|^2$ and $c_\tau \equiv |U_{\tau 1}|^2$ are given in Eq. (63).

Once U_{PMNS} is found, the validity of the assumption in Eq. (94) can be checked. Knowing the explicit expression for $|||U_{\alpha i}|^2||$, Eq. (95), we can calculate the traces of the matrices $S_{1U}T^2$ and $S_{2U}T$: $\text{Tr}[S_{1U}T^2] = \text{Tr}[g_2] = \tilde{a}_2$, $\text{Tr}[S_{2U}T] = \text{Tr}[W_{U2}] = a_2$ and check that they are equal.

Note that in the case of pattern $PSL7A$, we can also choose $k = 4$ in Eq. (85) which leads to

$$a_j = \tilde{a}_4(r), \quad \{\xi_{\tau j}, \xi_{ej}, \xi_{\mu j}\} = \{\xi_{e1}, \xi_{\mu 1}, \xi_{\tau 1}\}. \quad (96)$$

This is consistent with the permutation of $|U_{\alpha 1}|^2$ that appears in the third column, $|U_{\alpha 3}|^2$, in Eq. (95). These columns can, of course, be interchanged. Among the various possibilities in the context of $PSL7A$, the matrix in Eq. (95) is the closest one to the experimentally determined matrix. Hence, the choice of group parameters which leads to Eq. (95) is the best choice from the experimental point of view.

The same method can be also applied to the patterns $PSL7B$, $PSL7C$, and $D96$. They all have $m < p$ and, in

particular, the charged-lepton generator is T^l , with $T^{l3} = \mathbb{1}$. Thus, the eigenvalues of the generator T^l are cubic roots of unity which implies that one can only have $k = 2$ in Eq. (84). Following the same steps as above, one can find the following matrices of the moduli of mixing elements squared, $|||U_{\alpha i}|^2||$:

$$\frac{1}{12} \begin{pmatrix} 5 + \sqrt{21} & 2 & 5 - \sqrt{21} \\ 2 & 8 & 2 \\ 5 - \sqrt{21} & 2 & 5 + \sqrt{21} \end{pmatrix}, \quad (97)$$

$$\frac{1}{8} \begin{pmatrix} 3 + \sqrt{7} & 2 & 3 - \sqrt{7} \\ 2 & 4 & 2 \\ 3 - \sqrt{7} & 2 & 3 + \sqrt{7} \end{pmatrix}$$

for the patterns $PSL7B$, $PSL7C$, correspondingly, and

$$\frac{1}{6} \begin{pmatrix} 2 + \sqrt{3} & 2 & 2 - \sqrt{3} \\ 2 & 2 & 2 \\ 2 - \sqrt{3} & 2 & 2 + \sqrt{3} \end{pmatrix} \quad (98)$$

for the pattern $D96$. In contrast to the cyclic permutations of pattern $PSL7A$, in all these cases we find permutations of two elements in two different columns. This is related to the fact that T^l for these models has an eigenvalue that is equal to 1.

V. CONCLUSIONS

In this paper we extended the formalism presented in Ref. [9] in several directions.

- (i) A closed formula for the column elements of the mixing matrix was obtained without any of the assumptions in Ref. [9]. This formula allowed us to explore new flavor groups that were not considered in our previous work.
- (ii) We systematically analyzed all possible mixing patterns that can be obtained with the finite non-dihedral von Dyck groups. We conclude that relations obtained in Ref. [9] and in Sec. II C of this paper cover all physically relevant relations that can be obtained within this framework. In Sec. II C we present a mixing pattern so far overlooked in the literature in which the elements of the second column are predicted by the symmetry to be $\{1/4, 1/2, 1/4\}$. It corresponds to a transposition of the BM mixing pattern. This mixing pattern does not provide a good fit but could be used as a zero order pattern for which significant corrections are required.
- (iii) The groups $PSL(2, \mathbf{Z}_7)$, $\Delta(96)$, and $\Delta(384)$ are subgroups of infinite von Dyck groups and can be analyzed within our formalism. We obtained the constraints on U_{PMNS} for these cases and showed

that the symmetry assignments corresponding to the patterns *PSL7A* and *D384A* provide an excellent agreement with experiment. For these patterns we have the following predictions for the *CP* phase: $\delta \sim 87^\circ$ for *PSL7A* and $\delta \sim 55^\circ$ for *D384A*.

- (iv) Imposing the Klein symmetry in the neutrino sector fixes completely the mixing matrix. Using our approach, we were able to explain the appearance of mixing patterns in which two or three columns

of U_{PMNS} are composed of the same but permuted elements.

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