

**Torque anomaly in quantum field theory**

S. A. Fulling\*

*Departments of Mathematics and Physics, Texas A&M University, College Station, Texas 77843-3368, USA*

F. D. Mera

*Department of Mathematics, Texas A&M University, College Station, Texas 77843-3368, USA*

C. S. Trendafilova†

*Department of Physics, University of Texas, Austin, Texas 78712-0264, USA*

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The expectation values of energy density and pressure of a quantum field inside a wedge-shaped region appear to violate the expected relationship between torque and total energy as a function of angle. In particular, this is true of the well-known Deutsch-Candelas stress tensor for the electromagnetic field, whose definition requires no regularization except possibly at the vertex. Unlike a similar anomaly in the pressure exerted by a reflecting boundary against a perpendicular wall, this problem cannot be dismissed as an artifact of an *ad hoc* regularization.

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**I. INTRODUCTION**

The law of conservation of energy requires that the change in energy of a system, such as a fluid in a box, in response to an infinitesimal motion of an element of its boundary be equal to the negative of the force on that boundary times its perpendicular displacement. Thus, for perpendicular motion of a flat boundary at  $x = \text{const}$ , one has

$$-\frac{\partial E}{\partial x} = F = \iint p dy dz, \quad (1)$$

where  $E$  is the energy,  $p$  is the pressure, and the integral is over the moving element of boundary. Similarly, for a wedge of opening angle  $\alpha$ , one expects the change in energy with respect to angle to be related to the torque on the moving side:

$$-\frac{\partial E}{\partial \alpha} = \tau = \iint r p dr dz, \quad (2)$$

in cylindrical coordinates. Such relations (sometimes called instances of the “principle of virtual work”) do not follow automatically from the local energy-momentum conservation law,  $\frac{\partial T^{\mu\nu}}{\partial x^\mu} = 0$ ; the equation of state of the matter (dependence of pressure on energy density) must be consistent with the dependence of the energy density on the parameter concerned.

Recent work on quantum vacuum energy has displayed violations of Eq. (1) that have been traced to cutoffs introduced, without adequate physical basis, to remove divergences in the total energy near an idealized boundary. (See Ref. [1] and references therein.) An *ad hoc* remedy

was achieved (reviewed below). In our more recent work on wedges, it was expected that a similar problem and remedy would arise in connection with Eq. (2). As reported below, the problem arose, but the remedy did not work. More importantly, we point out here that Eq. (2) is violated already for conformally invariant fields [2], where there is no boundary divergence in a wedge (except at the axis), and hence the result cannot be blamed on a bad regularization method.

**II. CLASSICAL FLUID**

We begin by reviewing how Eq. (2) manifests itself for a classical fluid or a more general radially layered system. Assume an equation of state  $p = \beta\rho$ , where, at first,  $\rho$  and  $p$  are homogeneous (but depend on  $\alpha$ ). Consider a wedge region ( $0 < \theta < \alpha$ ) with large outer radius  $R$ , and assume that there is no shear stress  $T^{r\theta}$  on that outer boundary. All quantities are considered *per unit length* in the  $z$  direction (e.g., the “volume”  $V$  has units of area). Then,  $E = \rho V = \frac{1}{2}R^2\alpha\rho$ , so

$$\frac{\partial E}{\partial \alpha} = \frac{1}{2}R^2\left(\rho + \alpha\frac{\partial \rho}{\partial \alpha}\right). \quad (3)$$

Also,

$$\tau = \int_0^R p r dr = \frac{1}{2}R^2\beta\rho. \quad (4)$$

So, we expect that

$$\rho + \alpha\frac{\partial \rho}{\partial \alpha} = -\beta\rho, \quad (5)$$

to satisfy Eq. (2). Equation (5) implies that  $\rho \propto \alpha^{-(\beta+1)}$ , and hence  $E$  has the form

$$E = c_1\alpha^{-\beta} = c_2V^{-\beta}. \quad (6)$$

\*fulling@math.tamu.edu; <http://www.math.tamu.edu/~fulling>  
†cytrendafilova@gmail.com

The final formula is shape-independent and *equivalent* to the equation of state.

Since the torque balance holds locally for each  $r$ , this discussion generalizes to  $\rho$ ,  $p$ , and  $\beta$  dependent on  $r$  (for example, to surface tension in a cylindrical membrane).

### III. RECTILINEAR PRESSURE ANOMALY AND ITS RESOLUTION

Next, we review Ref. [1]. Consider a scalar field with the simplest curvature coupling,  $\xi = \frac{1}{4}$ . (Other choices do not affect the situation significantly.) At distance  $x$  from a perfectly reflecting plane boundary, the expectation value of the energy density is  $\rho = (32\pi^2 x^4)^{-1}$ , and that of the pressure parallel to the plane is the negative of that; therefore, on a test surface perpendicular to the plane, the density and pressure satisfy Eq. (1) pointwise, but the total energy and force (integrated over  $x$ ) are divergent. Of course, a real boundary cannot be perfectly reflecting at arbitrarily high frequencies, and an arbitrary, but physically plausible, response is to insert an exponential ultraviolet cutoff. The resulting energy and force violate Eq. (1) by a factor of  $-2$ . [Please note that the rightmost member of Eq. (20) in Ref. [1] has the wrong sign.] The ultraviolet cutoff is related to point splitting in the time direction, so it is natural to consider splittings (by distance  $\epsilon$ ) in the various space directions; choosing the remaining orthogonal space direction yields functions satisfying Eq. (1) (and, incidentally, agreeing with the ultraviolet regularization for the pressure, not for the energy):

$$\rho = \frac{1}{2\pi^2(\epsilon^2 + 4x^2)^2} = -p. \quad (7)$$

$$f(\alpha) = \begin{cases} \frac{\pi^2}{2\alpha^2} \left( \frac{\pi^2}{\alpha^2} - \frac{\alpha^2}{\pi^2} \right) & \text{for the conformally coupled scalar field,} \\ \left( \frac{\pi^2}{\alpha^2} + 11 \right) \left( \frac{\pi^2}{\alpha^2} - 1 \right) & \text{for the electromagnetic field.} \end{cases} \quad (9)$$

Confining attention to a finite interval  $r_{\min} < r < r_{\max}$ , consider the torque on the side of the wedge at  $\theta = \alpha$  (the other side remaining at  $\theta = 0$ ). One may consider  $r_{\max}/r_{\min}$  to be large, so that the region approximates a complete wedge, or to be small, to fix attention on a cylindrical layer of vacuum energy at one particular  $r$ ; the result is completely uniform in this parameter. From the pressure  $T_\theta^\theta$ , the torque is (per unit  $z$ )

$$\tau = \int_{r_{\min}}^{r_{\max}} r dr T_\theta^\theta(r) = - \int_{r_{\min}}^{r_{\max}} r dr \frac{1}{720\pi^2 r^4} 3f(\alpha). \quad (10)$$

But we should also be able to calculate it from the  $\alpha$  derivative of the energy (per unit  $z$ )

### IV. SCALAR QUANTUM FIELD THEORY IN A WEDGE

In Refs. [3,4], the cutoff technologies of Ref. [1] and predecessor papers were applied in cones and wedges. The obvious analog of the cutoff successful in the rectilinear case is point splitting in the axial ( $z$ ) direction. There were two surprises: (1) The cutoff did not completely remove the divergence at the axis of the wedge, and (2) the expected equality (2) was not satisfied, even with the axial cutoff. We intend to improve and extend our MATHEMATICA calculations to this effect before publishing any details. Our attempts to resolve the conundrum about the torque were interrupted by the observation reported in the next section.

### V. THE DEUTSCH-CANDELAS STRESS TENSORS FOR CONFORMALLY INVARIANT FIELDS

In a classic paper [2], whose results have been confirmed by independent calculations (e.g., Refs. [5–7]), Deutsch and Candelas calculated the energy density and pressure in a wedge for the conformally coupled ( $\xi = \frac{1}{6}$ ) scalar field and the electromagnetic field. For these fields, there is no divergence against a flat boundary (though the divergence at the axis remains, and weaker divergences emerge for curved, smooth boundaries). Therefore, for strictly positive values of  $r$ , one can meaningfully study the unregularized quantities, and the issue of a trustworthy cutoff does not arise.

According to Ref. [2], the vacuum stress tensor inside a wedge, in coordinates  $(t, r, \theta, z)$  and metric signature  $g_{00} < 0$ , is

$$T_\mu^\nu = \frac{f(\alpha)}{720\pi^2 r^4} \text{diag}(1, 1, -3, 1), \quad (8)$$

with

$$E = - \int_{r_{\min}}^{r_{\max}} r dr \int_0^\alpha d\theta T_{00} = - \int_{r_{\min}}^{r_{\max}} r dr \frac{1}{720\pi^2 r^4} \alpha f(\alpha). \quad (11)$$

Thus,

$$\tau = - \frac{\partial E}{\partial \alpha} = \int_{r_{\min}}^{r_{\max}} r dr \frac{1}{720\pi^2 r^4} \frac{d}{d\alpha} [\alpha f(\alpha)]. \quad (12)$$

Consistency will, therefore, be achieved if

$$g(\alpha) \equiv \frac{d}{d\alpha} [\alpha f(\alpha)] + 3f(\alpha) = 0. \quad (13)$$

We have

$$\alpha f(\alpha) = \begin{cases} \frac{1}{2}\pi^4\alpha^{-3} - \frac{1}{2}\alpha & \text{scalar,} \\ \pi^4\alpha^{-3} + 10\pi^2\alpha^{-1} - 11\alpha & \text{electromagnetic,} \end{cases} \quad (14)$$

and thus

$$g(\alpha) = \begin{cases} -2 & \text{scalar,} \\ 20\pi^2\alpha^{-2} - 44 & \text{electromagnetic.} \end{cases} \quad (15)$$

Therefore, there is a discrepancy that has nothing to do with a bad cutoff but is inherent in either the quantum field theory or the basic physics of torque. Note that the anomaly has a constant sign as a function of  $r$ , so it cannot disappear when the radial integral is evaluated.

In the framework of Eqs. (3)–(6), the relevant component of pressure satisfies  $p = \beta\rho$  with  $\beta = 3$ , just as for parallel plates [to which Eq. (8) formally reduces in the limit of small  $\alpha$  and large  $r$ ]. But the associated energy density does not satisfy Eq. (6) (except in that small- $\alpha$  limit); it is inconsistent with the equation of state.

## VI. CONCLUSION

Unless some elementary blunder is being made, the violation of the torque balance equation (2) by the Deutsch-Candelas stress tensor indicates some fundamental problem in our understanding of vacuum energy. It is not an artifact of regularization, because no cutoff has been introduced in the analysis. It apparently has nothing to do with divergences or boundary terms in the energy, but rather with the true Casimir energy of the bulk region.

If the wedge is made of thin plates, one might argue that the force from outside the wedge, and the corresponding variation in energy, must be taken into account. These quantities are obtained by replacing  $\alpha$  by  $2\pi - \alpha$  and reversing the sign. In the scalar case, the anomaly  $g$  is independent of  $\alpha$ , so the total anomaly does vanish in that case. In the electromagnetic case, however, it does not cancel (unless  $\alpha = \pi$ ). Our MATHEMATICA calculations indicate the same conclusion for the scalar field with

$\xi = \frac{1}{4}$  and the axial cutoff: The anomaly for  $\alpha = \frac{3\pi}{4}$  is not the same as that for  $\alpha = \frac{\pi}{4}$ .

It has been suggested that the persistent divergence at  $r \rightarrow 0$  spoils the argument: The conclusion is not convincing unless all quantities are finite, or at least all infinities are cleanly cancelled by considering the exterior of the region along with the interior. Therefore, we are presently investigating the Deutsch-Candelas stress tensors in an “annular sector” and its exteriors. That is, we consider conducting boundaries at  $r = r_{\min} > 0$ ,  $r = r_{\max} < \infty$ ,  $\theta = 0$  with  $r_{\min} < r < r_{\max}$ , and  $\theta = \alpha$  with  $r_{\min} < r < r_{\max}$ , and allow the last boundary to move. This model has been studied in various ways in Refs. [8,9], but those works do not answer all the questions we need to ask. The divergence at  $r = 0$  is now removed (and would be independent of  $\alpha$  anyway). However, new divergences are now introduced by the curved boundaries. The usual leading-order surface divergences will cancel (in the force) between the inside and outside of the wedge surfaces (and between electric and magnetic terms in the electromagnetic case); we expect them to be nonanomalous anyway, on the basis of Ref. [1]. There are also higher-order divergences associated with the curvature of the boundary. The leading such term will cancel, the exterior of the annulus against the interior. Also, corner energies are independent of  $\alpha$ . The crux of the problem is how the residual bulk Casimir term depends on  $\alpha$ . One remaining complication is that in the presence of wedge boundaries, stress tensors, in general, are not diagonal; the possibility of a nonzero shear force on the curved sides must be considered.

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