Correct light deflection in Weyl conformal gravity

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The conformal gravity fit to observed galactic rotation curves requires $\gamma > 0$. On the other hand, the conventional method for light deflection by galaxies gives a negative contribution to the Schwarzschild value for $\gamma > 0$, which is contrary to observation. Thus, it is very important that the contribution to bending should in principle be *positive*, no matter how small its magnitude is. Here we show that the Rindler-Ishak method gives a positive contribution to Schwarzschild deflection for $\gamma > 0$, as desired. We also obtain the exact local coupling term derived earlier by Sereno. These results indicate that conformal gravity can potentially test well against all astrophysical observations to date.

The metric exterior to a static spherically symmetric distribution in Weyl conformal gravity has been obtained by Mannheim and Kazanas [1]. Recently, the solution has been used to predict rotation curves of many galaxy samples [2] and that the model can provide a good idea of the possible size of individual galaxies [3]. The metric reads ($G = c = 1$):

$$d\tau^2 = -B(r)dt^2 + \frac{1}{B(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2),$$
$$B(r) = \alpha - \frac{2M}{r} + \gamma r - kr^2, \tag{1}$$

where $\alpha = (1 - 6M\gamma)^{1/2}$, M is the luminous mass, and k and γ are arbitrary constants that could be appropriately fixed by using the fit to rotation curves. For distances neither too small nor too large, one may take $\alpha = 1$ but in what follows we shall not make any such approximation. Now, conventional calculations for light deflection show that the constant k does not appear in the relevant equations, leading finally to the two way deflection as [4]

$$2\epsilon = \frac{4M}{r_0} - \gamma r_0, \tag{2}$$

where r_0 is the distance of closest approach. The difficulty is that the fit to observed rotation curve requires $\gamma > 0$, and for consistency, all other astrophysical observations should respect this sign. Evidently, for $\gamma > 0$ in Eq. (2), the light deflection by a galaxy falls short of the Schwarzschild value $\frac{4M}{r_0}$, while observations tell us that $2\epsilon > \frac{4M}{r_0}$. The purpose of this brief paper is to show that conformal

gravity does give a positive contribution to Schwarzschild deflection, removing the above impasse.

The resolution is based on the realization that conventional methods do not apply to asymptotically nonflat spacetimes as the limit $r \rightarrow \infty$ makes no sense in it [5]. The Rindler-Ishak method of invariant angle is most appropriate in such situations, and we show that it gives a positive contribution to light bending proportional to $+\gamma$, as required. The bending angle in general is defined by $\epsilon = \psi - \varphi$. Rindler and Ishak considered the case $\varphi = 0$ so that the deflection angle is $\epsilon = \psi$ given by [5]

$$\tan\psi = \frac{B^{1/2}r}{|A|}, \tag{3}$$

where $A(r, \varphi) = \frac{dr}{d\varphi}$. With $u = \frac{1}{r}$, the photon trajectory from (1) is given by

$$\frac{d^2u}{d\varphi^2} = -\alpha u + 3Mu^2 - \frac{\gamma}{2}. \tag{4}$$

As evident, k has disappeared from the above equation. This is a nonlinear differential equation that has to be solved perturbatively in powers of M. Following Bodenner and Will [6], we linearize the equation by expanding u in orders of M. To first order, we have, for small perturbation u_1

$$\frac{1}{r} = u = u_0 + u_1. \tag{5}$$

Then the zeroth and first order linearized equations respectively become

$$\frac{d^2u_0}{d\varphi^2} + \alpha u_0 = -\frac{\gamma}{2}, \tag{6}$$

$$\frac{d^2u_1}{d\varphi^2} + \alpha u_1 = 3Mu_0^2. \tag{7}$$

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In Eq. (6), redefine $\alpha u_0 = \tilde{u}_0$, $\bar{u}_0 = \tilde{u}_0 + \frac{\gamma}{2}$, $\sqrt{\alpha}\varphi = \bar{\varphi}$; then it transforms into

$$\frac{d^2 \bar{u}_0}{d\bar{\varphi}^2} + \bar{u}_0 = 0, \tag{8}$$

which yields

$$\bar{u}_0 = \frac{1}{R}\cos(\bar{\varphi}). \tag{9}$$

Reverting to original variables, we get

$$u_0 = \frac{1}{\alpha}\left(-\frac{\gamma}{2} + \frac{1}{R}\cos\{\sqrt{\alpha}\varphi\}\right). \tag{10}$$

Note that a $\sqrt{\alpha}$ factor has sneaked into the argument of the trigonometric function and also appears at other places. Their contributions must also be included in the deflection angle. The integration of the linear Eq. (7) can be straightforwardly performed by using the standard method.[1] The solution is

$$u_1 = \frac{M}{4R^2\alpha^3}[6 + 3R^2\gamma^2 - 6R\gamma\cos\{\sqrt{\alpha}\varphi\} - 2\cos\{2\sqrt{\alpha}\varphi\} - 6R\sqrt{\alpha}\gamma\varphi\sin\{\sqrt{\alpha}\varphi\}]. \tag{11}$$

Then the perturbative orbit equation, after changing $\varphi \to \pi/2 - \varphi$ on the right-hand sides of Eqs. (10) and (11), is given by

$$
\begin{aligned}
u = u_0 + u_1 \\
= \frac{1}{\alpha}\left[-\frac{\gamma}{2} + \frac{1}{R}\cos\left\{\frac{\sqrt{\alpha}}{2}(\pi - 2\varphi)\right\}\right] + \frac{M}{4R^2\alpha^3}\left[6 + 3R^2\gamma^2 - 6R\gamma\cos\left\{\frac{\sqrt{\alpha}}{2}(\pi - 2\varphi)\right\}\right. \\
\left. - 2\cos\{\sqrt{\alpha}(\pi - 2\varphi)\} - 3\pi R\sqrt{\alpha}\gamma\sin\left\{\frac{\sqrt{\alpha}}{2}(\pi - 2\varphi)\right\} + 6R\sqrt{\alpha}\gamma\varphi\sin\left\{\frac{\sqrt{\alpha}}{2}(\pi - 2\varphi)\right\}\right].
\end{aligned}
\tag{12}
$$

Note that the usual Schwarzschild orbit equation $u = \frac{1}{R}\sin\varphi + \frac{M}{2R^2}(3 + \cos 2\varphi)$ is recovered at $\gamma = 0$ and $\alpha = 1$. From Eq. (12), we can find r at $\varphi = 0$ as

$$r = \frac{4\alpha^3 R^2}{X}, \tag{13}$$

where

$$X \equiv 6M - 2\alpha^2 R^2\gamma + 3MR^2\gamma^2 - R(6M\gamma - 4\alpha^2)\cos\left\{\frac{\pi\sqrt{\alpha}}{2}\right\} - 2M\cos\{\pi\sqrt{\alpha}\} - 3MR\pi\sqrt{\alpha}\gamma\sin\left\{\frac{\pi\sqrt{\alpha}}{2}\right\}. \tag{14}$$

Also, at $\varphi = 0$, we find that

$$|A| = \frac{4R^2\alpha^{7/2}[3MR\pi\sqrt{\alpha}\gamma\cos\{\frac{\pi\sqrt{\alpha}}{2}\} + 4R\alpha^2\sin\{\frac{\pi\sqrt{\alpha}}{2}\} - 4M\sin\{\pi\sqrt{\alpha}\}]}{X^2}. \tag{15}$$

It can be seen again that, at $\gamma = 0$, we recover the Schwarzschild values $r = \frac{R^2}{2M}$ and $|A| = \frac{R^3}{4M^2}$. Using the value of r from Eq. (13) and $|A|$ from Eq. (15), we get from Eq. (3) the required deflection angle

$$\tan\psi = \frac{X\sqrt{B(r)}}{3M\pi R\gamma\alpha\cos\{\frac{\pi\sqrt{\alpha}}{2}\} + 4\sqrt{\alpha}[R\alpha^2\sin\{\frac{\pi\sqrt{\alpha}}{2}\} - M\sin\{\pi\sqrt{\alpha}\}]}. \tag{16}$$

This is the result we get considering the exact metric without any *a priori* approximation on u or α. Restoring the value of α, expanding in the first power of γ and then in

first power in R, we obtain for small ψ, after converting to r_0 via $\frac{1}{r_0} = \frac{1}{R} + \frac{M}{R^2} \Rightarrow R \simeq r_0$, the leading order terms

$$2\psi = \frac{4M}{r_0} - \frac{kr_0^3}{2M} + \frac{15M^2\gamma}{r_0}. \tag{17}$$

[1]Define the operator $D \equiv \frac{d}{d\varphi}$ and write the particular integral of Eq. (7) as $u_1 = \frac{1}{(D^2+\alpha)}[\frac{C}{2} + A + B\cos\{\sqrt{\alpha}\varphi\} + \frac{C}{2}\cos\{2\sqrt{\alpha}\varphi\}]$ where the constants are $A = \frac{3M\gamma^2}{4\alpha^2}$, $B = -\frac{3M\gamma}{R\alpha^2}$, $C = \frac{3M}{R^2\alpha^2}$. Note that $(D^2+\alpha)(\frac{C}{2\alpha}) = \frac{C}{2} \Rightarrow \frac{C/2}{(D^2+\alpha)} = \frac{C}{2\alpha} = (\frac{M}{4R^2\alpha^3}) \times 6$ etc. Also use $(D^2+\alpha)\varphi\sin(\sqrt{\alpha}\varphi) = 2\sqrt{\alpha}\cos(\sqrt{\alpha}\varphi) \Rightarrow \frac{\cos(\sqrt{\alpha}\varphi)}{(D^2+\alpha)} = \frac{\varphi\sin(\sqrt{\alpha}\varphi)}{2\sqrt{\alpha}}$. Similarly, $\frac{1}{(D^2+\alpha)} \times \cos\{2\sqrt{\alpha}\varphi\} = -\frac{1}{3\alpha}\cos\{2\sqrt{\alpha}\varphi\}$. Adding the characteristic function from $(D^2+\alpha)u_1 = 0$, we arrive at Eq. (11).

The second term is the same as the one obtained by Rindler and Ishak [4] for the deflection in the Schwarzschild—de Sitter spacetime. Using $k = \Lambda/3$ for comparison with literature and expressing r_0 in terms of the impact parameter b as $\frac{1}{r_0} \simeq \frac{1}{b} + \frac{\Lambda b}{6}$, we have the relevant terms

$$2\psi = \frac{4M}{r_0} + \frac{15M^2\gamma}{r_0} \simeq \frac{4M}{b} + \frac{2Mb\Lambda}{3} + \frac{15M^2\gamma}{b} > \frac{4M}{r_0}. \tag{18}$$

Note that we have also obtained the local coupling term $\frac{2Mb\Lambda}{3}$ derived earlier by Sereno [5] by a completely different method, namely, by integrating the first order differential equation of light orbit. Our main result is that we have obtained a positive contribution $+\frac{15M^2\gamma}{b}$ instead of a negative contribution. This positivity is important *as a principle* since it lends physical consistency to conformal gravity predictions.

Here we wish to point out that Sultana and Kazanas [7] have first tackled the present problem of light deflection. To make contact with their calculation, we should redefine our M as

$$M = \frac{\beta}{2}(2 - 3\beta\gamma) \Rightarrow \alpha = (1 - 6M\gamma)^{1/2} = 1 - 3\beta\gamma. \tag{19}$$

They used the path equation to first order in γ as

$$u_{\mathrm{SK}} = \left(\frac{\sin\varphi}{b} - \frac{\gamma}{2}\right) + \left[\frac{3\beta(2 - 3\beta\gamma)}{4b^2} \right. $$
$$\left. + \frac{\beta(2 - 3\beta\gamma)}{4b^2}\cos 2\varphi\right] - \frac{3\beta\gamma}{2b}\varphi\cos\varphi \tag{20}$$

that yielded a negative contribution $-\frac{4\beta^2\gamma}{b}$ to two way deflection. However, note that in searching for the first power effect of γ, it is only logical that one must retain all the first power terms in γ in relevant expansions, which in turn implies that one must retain $\alpha \neq 1$ in the trigonometric arguments and elsewhere. Then the expression for u from Eq. (12), to first order in γ, reads

$$u = \left(\frac{\sin\varphi}{b} - \frac{\gamma}{2}\right) + \frac{3\beta\gamma}{2b}\sin\varphi + \left[\frac{3\beta(2 + 15\beta\gamma)}{4b^2}\right.$$
$$\left. + \frac{\beta(2 + 15\beta\gamma)}{4b^2}\cos 2\varphi + \frac{3\beta^2\gamma}{4b^2}(2\varphi - \pi)\sin 2\varphi\right], \tag{21}$$

which is widely different from u_{SK}. Thus the negative contribution seems ruled out. To see the actual contribution, it is enough to convert $2\psi = \frac{4M}{r_0} + \frac{15M^2\gamma}{r_0}$ in terms of the notation β used in Ref. [7], which would then yield, to first order in γ, the result $2\psi = \frac{4\beta}{r_0} + \frac{9\beta^2\gamma}{r_0}$. Yet again, the positive γ contribution is quite evident.

We can incorporate the light bending Eq. (18) in the lensing equation, ignoring the local coupling term, which is numerically much smaller than the γ term by several orders of magnitude for typical galaxies. The lens equation is given by

$$\theta D_{\mathrm{os}} = \beta D_{\mathrm{os}} + (2\psi)D_{\mathrm{ls}}. \tag{22}$$

When the observer, lens and source are aligned in one direction, we have $\beta = 0$, which yields, in the small angle approximation $b = \theta D_{\mathrm{ol}}$, the "Weyl angle" as

$$\theta_{\mathrm{Weyl}} = \left(\frac{4M + 15M^2\gamma}{D}\right)^{1/2}, \tag{23}$$

where $D \equiv \frac{D_{\mathrm{ol}}D_{\mathrm{os}}}{D_{\mathrm{ls}}}$. The Einstein angle is of course $\theta_{\mathrm{Einstein}} = (\frac{4M}{D})^{1/2}$, which means that the Schwarzschild mass is only to be redefined as $\bar{M} = M + \frac{15}{4}M^2\gamma$ to obtain the Weyl angle. Of course, for galactic lenses, these masses do not differ enormously. This is expected as the luminous matter obeys $M_{\mathrm{L}}(r) \propto r^3$, while flat rotation curves demand $M_{\mathrm{DM}}(r) \propto r$ in the halo region that increases with radius more slowly with distance than $M_{\mathrm{L}}(r)$ and thus is comparatively rarer.[2] Customarily, the observed light deflection is explained by a total mass distribution that includes also the hypothetical "dark matter" in *excess* of the luminous component [4]. On the other hand, in conformal gravity, this hypothesis is not required [1–3]. Our result here supports this central aspect of conformal gravity in that the mass gets *automatically* enhanced to $\bar{M} > M$ due necessarily to the positive contribution $+\frac{15}{4}M^2\gamma$, as we promised to show.

It has been pointed out to us that the γr term can be absorbed into a conformal factor [8] and that the sign of the γ contribution to bending can alter under different choices of conformal factors [9].[3] While we agree with these facts, we still worked only in the conformal frame as exactly fixed by Eq. (1), i.e., in the metric used by Sultana and Kazanas [7], for the single reason that it has remarkably explained observations for appropriate choices of γ and a quadratic potential [2]. It is true that the bending effect is exceedingly small and, as it stands, incompatible with the observations but our aim was to argue that the sign must be positive in the first place for qualitative validity of conformal gravity theory.[4] Thus, it remains for us to explore if the theory can show also quantitative validity with respect to bending observations. Work is under way.

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[4]Under the practical assumption that $R \gg M$, $\frac{1}{R} \simeq \frac{1}{r_0} \simeq \frac{1}{b} + \frac{\Lambda b}{6}$, the γ bending is always positive though exceedingly small.

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