Probability of the standard model appearance from a matrix model

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The standard model of particle physics lies in an enormous number of string vacua. In a nonperturbative formulation of string theory, various string vacua can, in principle, be compared dynamically, and the probability distribution over the vacuum space could be calculated. In this paper, we consider situations where the IIB matrix model is compactified on a six-dimensional torus with various gauge groups and various magnetic fluxes, find matrix configurations that provide the standard model matter content, and estimate semiclassically the probability of their appearance.

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I. INTRODUCTION

Matrix models (MM) are a promising candidate to formulate the superstring theory nonperturbatively [1–3], and they indeed include quantum gravity and gauge theory. One of the important subjects in those studies is to connect these models to phenomenology. Spacetime structures can be analyzed dynamically and four-dimensionality seems to be preferred in the IIB matrix model [4–6]. Assuming that our spacetime is obtained, we next want to show the standard model (SM) of particle physics on it.

Here, we give two comments regarding the importance of these studies. First, a path connecting the MM and the SM would give us a guide for bringing them close to each other: from the SM side, when one tries to go beyond the SM, there are too many phenomenological models, but this path may give us a hint about which way to go; from the MM side, there also remain important problems, for instance, interpretations of spacetime and matter in matrices, how to take a large-N limit, and so on. In order to justify or modify the formulation of MM, whether or not one can obtain the SM at low energies gives us a criterion. Secondly, since the MM has a definite measure and action, we can, in principle, calculate everything, such as spacetime dimensions, gauge groups, and matter contents. We could dynamically compare various string vacua, and obtain a probability distribution[1] over the string landscape [9]. This is an advantage that MM has over the perturbative formulations of superstring theories.

An important ingredient of the SM is the chirality of fermions. Chiral symmetry also ensures the existence of massless fermions, since otherwise quantum corrections would induce a mass of the order of the Planck scale or of the Kaluza-Klein scale in general. (Gauge fields are protected to be massless by gauge symmetry.) We usually obtain a chiral spectrum on our spacetime by introducing nontrivial topologies—which then give chiral zero modes—in the extra dimensions: Euler characteristics of compactified manifolds, special boundary conditions at orbifold singularities, the intersection numbers of D-branes, etc., give nontrivial topologies. Also from the MM, chiral fermions and the SM matter content were obtained by considering toroidal compactifications with magnetic fluxes [10] and intersecting D-branes [11].[2]

In this paper, we will study the case of toroidal compactifications in more detail. We first study matrix configurations that provide the SM matter content. Within the configurations that provide the SM gauge group plus an extra U(1) and the SM fermion species with three generations, the minimal number of extra U(1)'s turns out to be three. Even within this case, there still can be a large number of matrix configurations with various fluxes, but actually they are determined almost uniquely. We then calculate their classical actions, argue how to take the large-N limit, and estimate semiclassically the probability of their appearance.

In Sec. II, we briefly review a formulation of topological configurations on a torus. We then find matrix configurations that provide the SM matter content in Sec. III. In Sec. IV, we study semiclassical analyses of MM dynamics. Section V is devoted to conclusions and a discussion. In the Appendix, detailed calculations for determining q_l^{ab} are shown.

II. TOPOLOGICAL CONFIGURATIONS ON A TORUS

Let us begin with a review of the IIB MM [2]. Its action is written as

$$S_{\mathrm{IIBMM}} = -\frac{1}{g_{\mathrm{IIBMM}}^2}\,\mathrm{tr}\Big(\frac{1}{4}[A_M, A_N][A^M, A^N] + \frac{1}{2}\bar{\Psi}\Gamma^M[A_M, \Psi]\Big), \tag{2.1}$$

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[1] Studies based on number countings of the flux vacua [7] and cosmological evolutions on the landscape [8] were given. However, an underlying theory of the entire landscape with a definite measure is desired.

[2] Studies based on fuzzy spheres were given in Refs. [12–14]. MM's for orbifolds and orientifolds were studied in Refs. [15,16]. Related works were given in Refs. [17,18].

where A_M and Ψ are $N \times N$ Hermitian matrices. They are also a ten-dimensional vector and a Majorana-Weyl spinor, respectively. Performing a kind of functional integration

$$\int dAd\Psi e^{-S_{\rm IIBMM}}, \tag{2.2}$$

as a statistical system, and taking a suitable large-N limit, one can obtain a nonperturbative formulation of string theory. Note that the measure as well as the action is defined definitely, so we can calculate everything in principle. Note also that the model can be formulated either as an Euclidean or as a Lorentzian system. It was shown in Ref. [6] that treating it as a Lorentzian system is important for obtaining a four-dimensional extended spacetime with a six-dimensional compactified space. Since we assume a compactification and focus on the extra-dimensional space in this paper, our results hold in either case.

We then consider compactifications to $M^4 \times X^6$ with X^6 carrying nontrivial topologies.[3] For concreteness, we consider toroidal compactifications of $M^4 \times T^6$. Toroidal compactifications were studied in Hermitian matrices [20,21] and in unitary matrices [22]. The unitary matrix formulations can be described by finite matrices. It is also considered that noncommutative (NC) spaces arise naturally from MM [21,23]. We thus use a unitary matrix formulation for NC tori in this paper. It can be defined by the twisted Eguchi-Kawai model [24,25] (see, for instance, Ref. [26]). Note, however, that such details of formulations—i.e., Hermitian or unitary, commutative or NC—are not relevant for obtaining chiral fermions and the SM. Any compactifications with nontrivial topologies can work as well. We then consider background configurations corresponding to

$$e^{iA_\mu} \sim e^{ix_\mu} \otimes \mathbb{1}, \qquad e^{iA_i} \sim \mathbb{1} \otimes V_i, \tag{2.3}$$

with $\mu = 0, \ldots, 3$ and $i = 4, \ldots, 9$. x_μ represents our spacetime M^4, and V_i represents T^6. A more precise correspondence between the IIB MM and the unitary MM will be given in Sec. IV.

We now focus on V_i in Eq. (2.3), i.e., NC T^6 with nontrivial topologies. It is well-known that nontrivial topological sectors are defined by the so-called modules in NC geometries (see, for instance, Ref. [27]). In the MM formulations, such modules are defined by imposing twisted boundary conditions on the matrices [26,28]. In fact, each theory with twisted boundary conditions yields a single topological sector specified by the boundary conditions [29,30], while in ordinary gauge theories on commutative spaces, a theory, for instance, with periodic boundary conditions, provides all the topological sectors. However, since we now want to derive everything from the IIB MM, those topological features of NC gauge theories are not desirable. We thus introduce nontrivial topological sectors by background matrix configurations, not by imposing twisted boundary conditions by hand. Nontrivial topologies can be given by block-diagonal matrices [10]. We then consider the following configurations:

$$\begin{aligned}
V_{3+j} &= \begin{pmatrix} \Gamma^1_{1,j} \otimes \mathbb{1}_{n^1_2} \otimes \mathbb{1}_{n^1_3} \otimes \mathbb{1}_{p^1} & & \\ & \ddots & \\ & & \Gamma^h_{1,j} \otimes \mathbb{1}_{n^h_2} \otimes \mathbb{1}_{n^h_3} \otimes \mathbb{1}_{p^h} \end{pmatrix}, \\
V_{5+j} &= \begin{pmatrix} \mathbb{1}_{n^1_1} \otimes \Gamma^1_{2,j} \otimes \mathbb{1}_{n^1_3} \otimes \mathbb{1}_{p^1} & & \\ & \ddots & \\ & & \mathbb{1}_{n^h_1} \otimes \Gamma^h_{2,j} \otimes \mathbb{1}_{n^h_3} \otimes \mathbb{1}_{p^h} \end{pmatrix}, \\
V_{7+j} &= \begin{pmatrix} \mathbb{1}_{n^1_1} \otimes \mathbb{1}_{n^1_2} \otimes \Gamma^1_{3,j} \otimes \mathbb{1}_{p^1} & & \\ & \ddots & \\ & & \mathbb{1}_{n^h_1} \otimes \mathbb{1}_{n^h_2} \otimes \Gamma^h_{3,j} \otimes \mathbb{1}_{p^h} \end{pmatrix},
\end{aligned} \tag{2.4}$$

with $j = 1, 2$. The number of blocks is denoted by h. Each block is a tensor product of four factors. The first three factors each represent T^2 of $T^6 = T^2 \times T^2 \times T^2$, and the last factor provides a gauge group structure. The configuration (2.4) gives the gauge group $U(p^1) \times U(p^2) \times \cdots \times U(p^h)$.

The matrices $\Gamma^a_{l,j}$ with $a = 1, \ldots, h$ and $l = 1, 2, 3$ in Eq. (2.4) are actually defined by using the Morita equivalence, which is well-known in NC geometries. For details, see, for instance, Refs. [10,26–28]. We follow the conventions used in Ref. [10]. $\Gamma^a_{l,j}$ are $U(n^a_l)$ matrices that satisfy the 't Hooft-Weyl algebra

$$\Gamma^a_{l,1}\Gamma^a_{l,2} = \mathrm{e}^{-2\pi i \frac{m^a_l}{n^a_l}} \Gamma^a_{l,2}\Gamma^a_{l,1}, \tag{2.5}$$

[3] Related works were given in Ref. [19].

where the integers m_l^a and n_l^a are specified by

$$m_l^a = -s_l + k_l q_l^a, \qquad n_l^a = N_l - 2r_l q_l^a, \tag{2.6}$$

for each a and l. The integers N_l, r_l, s_l, and k_l for each l specify the original torus (of the Morita equivalence) for each T^2. Equations (2.6) can be inverted as

$$1 = 2r_l m_l^a + k_l n_l^a, \qquad q_l^a = N_l m_l^a + s_l n_l^a. \tag{2.7}$$

For a summary, the configuration (2.4) is specified by the integers p^a and q_l^a with $a = 1, \ldots, h$ and $l = 1, 2, 3$, once the original tori are specified. p^a gives the gauge group, and q_l^a specifies magnetic fluxes penetrating each T^2. The total matrix size is

$$\sum_{a=1}^{h} n_1^a n_2^a n_3^a p^a. \tag{2.8}$$

The fermionic matrix Ψ is similarly decomposed into blocks as

$$\Psi = \begin{pmatrix} \varphi^{11} \otimes \psi^{11} & \cdots & \varphi^{1h} \otimes \psi^{1h} \\ \vdots & \ddots & \vdots \\ \varphi^{h1} \otimes \psi^{h1} & \cdots & \varphi^{hh} \otimes \psi^{hh} \end{pmatrix}, \tag{2.9}$$

where φ^{ab} and ψ^{ab} represent spinor fields on M^4 and T^6, respectively. Each block $\varphi^{ab} \otimes \psi^{ab}$ is in a bi-fundamental representation $(p^a, \bar{p}^b)$ under the gauge group $U(p^a) \times U(p^b)$. It turns out [10] that ψ^{ab} has the topological charge on T^6 as

$$p^a p^b \prod_{l=1}^{3} (q_l^a - q_l^b) = p^a p^b \prod_{l=1}^{3} \left(-\frac{1}{2r} (n_l^a - n_l^b) \right). \tag{2.10}$$

Indeed, by defining an overlap-Dirac operator, which satisfies a Ginsparg-Wilson relation and an index theorem,[4] the Dirac index, i.e., the difference between the numbers of chiral zero modes, was shown to take the corresponding values.[5] In the present paper, we do not specify forms of the Dirac operator, and just assume that in the large-N limit the correct number of chiral zero modes arises.

III. CONFIGURATIONS FOR THE STANDARD MODEL

We now study matrix configurations that provide the SM matter content; more precisely speaking, the SM gauge group plus extra U(1)'s and the SM fermion species with generation number three.

A. Too-minimal case

We first consider the case with the number of blocks being four, i.e., $h = 4$. The integers p^a are taken to be 3, 2, 1, 1 for $a = 1, \ldots, h$, so that the gauge group is $U(3) \times U(2) \times U(1)^2 \simeq SU(3) \times SU(2) \times U(1)^4$.

The SM fermionic species are embedded in the fermionic matrix ψ as

$$\psi = \begin{pmatrix} o & q & u & d \\ & o & \bar{l} & o \\ & & o & e \\ & & & o \end{pmatrix}, \tag{3.1}$$

where q denotes the quark doublets, l the lepton doublets, u and d the quark singlets, and e the lepton singlets. They are in the correct representations under $SU(3) \times SU(2)$. Note that the singlet neutrino is not included here. The entries denoted as o give no massless fermions since, as we will see below, they are set to have a vanishing index. The lower triangle part can be obtained from the upper part by the charge conjugation transformation.

The hypercharge Y is given by a linear combination of the four U(1) charges as

$$Y = \sum_{i=1}^{4} x^i Q^i, \tag{3.2}$$

where $Q^i = \pm 1$ with $i = 1, \ldots, 4$ is the U(1) charge from the ith block. From the hypercharge of q, u, d, l, and e, the following constraints are obtained:

$$\begin{aligned} x^1 - x^2 &= 1/6, \\ x^1 - x^3 &= 2/3, \\ x^1 - x^4 &= -1/3, \\ -(x^2 - x^3) &= -1/2, \\ x^3 - x^4 &= -1. \end{aligned} \tag{3.3}$$

Their general solutions are given by

$$\begin{aligned} x^1 &= 1/6 + c, \qquad x^2 = c, \\ x^3 &= -1/2 + c, \qquad x^4 = 1/2 + c, \end{aligned} \tag{3.4}$$

with c being an arbitrary constant. Since Eqs. (3.3) depend only on the differences of x^i, the solution (3.4) is determined with an arbitrary constant shift c. The existence of a solution is not automatically ensured, since the number of independent variables is three while the number of equations is five.

As for the other U(1) charges, the baryon number B, left-handed charge Q_L, and another charge Q' can be considered. Their charge for q, u, d, l, and e, and the corresponding values for x^i are given as follows:

[4] These techniques were developed in the lattice gauge theories [31] and applied to MM and NC geometries [32].

[5] The same results were obtained in the fuzzy spheres [14,33].

$$\begin{array}{cccccccccc} & q & u & d & l & e & x^1 & x^2 & x^3 & x^4 \\ Y & 1/6 & 2/3 & -1/3 & -1/2 & -1 & 1/6 & 0 & -1/2 & 1/2 \\ B & 1/3 & 1/3 & 1/3 & 0 & 0 & 1/3 & 0 & 0 & 0 \\ Q_L & 1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ Q' & 0 & 1 & 1 & -1 & 0 & 0 & 0 & -1 & -1 \end{array}. \quad (3.5)$$

A linear combination of these four U(1) charges gives an overall U(1) and does not couple to the matter. Only three U(1) charges couple to the matter. Note that no lepton number L nor $B - L$ is included in this setting.

Let us now determine the integers q_l^a specifying the magnetic fluxes. From Eq. (2.10), only the differences $q_l^a - q_l^b$ are relevant to the topology for the block ψ^{ab}. We thus define

$$q_l^{ab} = q_l^a - q_l^b, \quad (3.6)$$

$$q^{ab} = \prod_{l=1}^{3} q_l^{ab}. \quad (3.7)$$

In order for Eq. (3.1) to have the correct generation number, q^{ab} must have the values

$$q^{ab} = \begin{pmatrix} 0 & -3 & 3 & 3 \\ & 0 & 3 & 0 \\ & & 0 & 3 \\ & & & 0 \end{pmatrix}. \quad (3.8)$$

The lower triangle part is obtained from the upper part by the relation $q^{ab} = -q^{ba}$. The block component with a vanishing index gives no chiral zero modes, and thus no massless fermions on our spacetime. Unfortunately, however, there is no solution of q_l^{ab} that satisfies Eq. (3.7) with Eq. (3.8). (Proof: q_l^{12} and q_l^{23} must take ± 1 or ± 3. It follows that $q_l^{13} = q_l^{12} + q_l^{23}$ must take 0, ± 2, ± 4, or ± 6. Hence, q^{13} could not take 3.)

We therefore conclude that the present too-minimal case, which does not include the right-handed neutrino or the $B - L$ gauge field, has no solution.

B. Minimal case

We then consider the $h = 5$ case. The integers p^a are taken to be 3, 2, 1, 1, 1 for $a = 1, \ldots, h$, so that the gauge group is $U(3) \times U(2) \times U(1)^3 \simeq SU(3) \times SU(2) \times U(1)^5$.

The SM fermionic species are embedded in the fermionic matrix ψ as

$$\psi = \begin{pmatrix} o & q & u' & u & d \\ & o & \bar{l} & \bar{l}' & o \\ & & o & \nu(\bar{\nu}) & e \\ & & & o & e' \\ & & & & o \end{pmatrix}, \quad (3.9)$$

where q denotes the quark doublets, l the lepton doublets, u and d the quark singlets, and ν and e the lepton singlets. Note that the singlet neutrino ν is now included. In fact, Eq. (3.9) is the most general embedding, where all the block elements have the correct representations under the SM gauge group $SU_c(3) \times SU_L(2) \times U(1)_Y$ and the correct generation numbers. Since ν is a gauge singlet, either ν or $\bar{\nu}$ can be embedded.

The U(1) charges can be determined as in the previous subsection. By taking linear combinations of the five U(1) charges as $\sum_{i=1}^{5} x^i Q^i$, we can consider the hypercharge Y, baryon number B, lepton number L', left-handed charge Q_L, and right-handed charge Q'_R. Their charge for q, u, u', d, l, l', $\nu(\bar{\nu})$, e, and e', and the corresponding values for x^i are given as follows:

$$\begin{array}{cccccccccc} & q & u & u' & d & l & l' & \nu(\bar{\nu}) & e & e' \\ Y & 1/6 & 2/3 & 2/3 & -1/3 & -1/2 & -1/2 & 0 & -1 & -1 \\ B & 1/3 & 1/3 & 1/3 & 1/3 & 0 & 0 & 0 & 0 & 0 \\ L' & 0 & 0 & -1 & 0 & 1 & 0 & 1 & 1 & 0 \\ Q_L & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ Q'_R & 0 & 1 & 0 & 1 & 0 & -1 & 1 & 1 & 0 \end{array}, \quad (3.10)$$

$$\begin{array}{cccccc} & x^1 & x^2 & x^3 & x^4 & x^5 \\ Y & 1/6 & 0 & -1/2 & -1/2 & 1/2 \\ B & 1/3 & 0 & 0 & 0 & 0 \\ L' & 0 & 0 & 1 & 0 & 0 \\ Q_L & 0 & -1 & 0 & 0 & 0 \\ Q'_R & 0 & 0 & 0 & -1 & -1 \end{array}. \qquad (3.11)$$

A linear combination of these five U(1) charges gives an overall U(1) and does not couple to the matter. Only four U(1) charges couple to the matter.

The integers q_l^a specifying the magnetic fluxes can also be determined as before. In order for Eq. (3.9) to have the correct generation number, q^{ab}—which is defined in Eq. (3.7)—must take the values

$$q^{ab} = \begin{pmatrix} 0 & -3 & x & 3-x & 3 \\ & 0 & 3-y & y & 0 \\ & & 0 & \pm 3 & 3-z \\ & & & 0 & z \\ & & & & 0 \end{pmatrix}, \qquad (3.12)$$

with some integers x, y, and z. The double sign is chosen depending on whether ν or $\bar{\nu}$ is embedded in Eq. (3.9).

We now impose an extra condition: the extra U(1)'s should also have appropriate interpretations. While B and Q_L have the correct charge as the baryon number and the left-handed number in Eq. (3.10), L' and Q'_R do not unless u', l', and e' disappear, and ν, not $\bar{\nu}$, is chosen in Eq. (3.10), and thus in Eq. (3.9). Then, $x = y = z = 0$ is taken, and the upper sign in the double sign is chosen in Eq. (3.12). It thus becomes

$$q^{ab} = \begin{pmatrix} 0 & -3 & 0 & 3 & 3 \\ & 0 & 3 & 0 & 0 \\ & & 0 & 3 & 3 \\ & & & 0 & 0 \\ & & & & 0 \end{pmatrix}. \qquad (3.13)$$

We then solve Eq. (3.7) with Eq. (3.13) to obtain q_l^{ab}. (See the Appendix for detailed calculations.) Here we note two comments. First, Eq. (3.7) is invariant under the permutations and the sign flips of q_l^{ab}. Using these symmetries we can fix the order of q_1^{ab}, q_2^{ab}, and q_3^{ab}, and the overall signs for two of them. Secondly, if $q_l^{ab} = 0$ for all l, which is equivalent to $q_l^a = q_l^b$ for all l, the ath block and the bth block of the bosonic matrix V_i in Eq. (2.4) become identical, and the gauge group is enhanced from $U(p^a) \times U(p^b)$ to $U(p^a + p^b)$. We thus exclude this case. Within these constraints, the solutions for Eq. (3.7) are determined almost uniquely. We have two solutions:

$$q_1^{ab} = \begin{pmatrix} 0 & 1 & 0 & \pm 1 & \mp 1 \\ & 0 & -1 & -1 \pm 1 & -1 \mp 1 \\ & & 0 & \pm 1 & \mp 1 \\ & & & 0 & \mp 2 \\ & & & & 0 \end{pmatrix},$$
$$q_2^{ab} = \begin{pmatrix} 0 & -1 & 0 & \pm 1 & \mp 1 \\ & 0 & 1 & 1 \pm 1 & 1 \mp 1 \\ & & 0 & \pm 1 & \mp 1 \\ & & & 0 & \mp 2 \\ & & & & 0 \end{pmatrix}, \qquad (3.14)$$
$$q_3^{ab} = \begin{pmatrix} 0 & 3 & 0 & 3 & 3 \\ & 0 & -3 & 0 & 0 \\ & & 0 & 3 & 3 \\ & & & 0 & 0 \\ & & & & 0 \end{pmatrix},$$

where all the double signs correspond.

IV. PROBABILITY OF THE STANDARD MODEL APPEARANCE

We now study the dynamics of MM semiclassically, and estimate the probabilities for the appearance of the topological configurations, and in particular, the SM configurations obtained in the previous section.

We first specify the model. We here consider a ten-dimensional torus with an anisotropy of sizes between four and six dimensions, namely, a NC $T^2 \times T^2 \times T^2 \times T^2 \times T^2$ with an anisotropy between two T^2's and three T^2's. The bosonic part is described by the twisted Eguchi-Kawai model [24,25], which can be seen by expanding the matrices in terms of bases (see, for instance, Ref. [26]). The action is written as

$$\begin{aligned} S_b = & -\beta\mathcal{N}\sum_{i\neq j} \mathcal{Z}_{ji}\mathrm{tr}(\mathcal{V}_i\mathcal{V}_j\mathcal{V}_i^\dagger\mathcal{V}_j^\dagger) \\ & - \beta'\mathcal{N}\sum_{\mu\neq\nu} \mathcal{Z}_{\nu\mu}\mathrm{tr}(\mathcal{V}_\mu\mathcal{V}_\nu\mathcal{V}_\mu^\dagger\mathcal{V}_\nu^\dagger) \\ & - \beta''\mathcal{N}\sum_{i\mu}[\mathcal{Z}_{\mu i}\mathrm{tr}(\mathcal{V}_i\mathcal{V}_\mu\mathcal{V}_i^\dagger\mathcal{V}_\mu^\dagger) \\ & + \mathcal{Z}_{i\mu}\mathrm{tr}(\mathcal{V}_\mu\mathcal{V}_i\mathcal{V}_\mu^\dagger\mathcal{V}_i^\dagger)], \end{aligned} \qquad (4.1)$$

with $\mu, \nu = 0, \ldots, 3$ and $i, j = 4, \ldots, 9$. $\mathcal{V}_\mu$ and $\mathcal{V}_i$ are $U(\mathcal{N})$ matrices, and are written as

$$\mathcal{V}_\mu = V_\mu \otimes \mathbb{1}, \qquad \mathcal{V}_i = \mathbb{1} \otimes V_i, \qquad (4.2)$$

where V_μ are $U(N'^2)$ matrices and V_i are $U(kN^3)$ matrices. The size of our spacetime is $\epsilon N'$ and that of the extra six dimensions is ϵN, where ϵ is a lattice spacing. There must be a huge anisotropy between N' and N. If the extra dimensions have size of the order of the Planck scale and our spacetime is bigger than the current horizon, they must satisfy

$$\frac{N'}{N} > 10^{60}. \tag{4.3}$$

The total matrix size $\mathcal{N}$ is related to N' and N as

$$\mathcal{N} = N'^2 N^3 k. \tag{4.4}$$

We now consider the following twists Z_{MN} in the action (4.1):

$$Z_{01} = Z_{23} = \exp\left(2\pi i \frac{s'}{N'}\right),$$
$$Z_{45} = Z_{67} = Z_{89} = \exp\left(2\pi i \frac{s}{N}\right). \tag{4.5}$$

The other twists are taken to be zero. Note that the matrix size (4.4) is k times larger than is usually expected from the integers that specify the twists (4.5).

Next, we consider the matrix configurations (2.4). In fact, they are classical solutions for the action (4.1) (see, for instance, Ref. [34]). In order to match the matrix size,

$$\sum_{a=1}^{h} n_1^a n_2^a n_3^a p^a = N^3 k \tag{4.6}$$

is required. Plugging Eq. (2.4) into Eq. (4.1), we obtain the classical action as

$$S_b = -2\beta \mathcal{N} N'^2 \sum_{l=1}^{3} \sum_{a=1}^{h} n_1^a n_2^a n_3^a p^a \cos\left(2\pi\left(\frac{s}{N} + \frac{m_l^a}{n_l^a}\right)\right), \tag{4.7}$$

where we have written only the contributions form the first term in Eq. (4.1). If the integers n_l^a, m_l^a are related to N, s by Eqs. (2.6) or (2.7), with N_l, s_l, r_l, and k_l set to be independent of l, we can find the relation

$$\frac{s}{N} + \frac{m_l^a}{n_l^a} = \frac{q_l^a}{N n_l^a} = -\frac{1}{2r}\left(\frac{1}{N} - \frac{1}{n_l^a}\right). \tag{4.8}$$

By plugging Eq. (4.8) into Eq. (4.7), we find that the classical action (4.7) takes the minimum value if and only if

$$q_l^a = 0 \Leftrightarrow n_l^a = N \tag{4.9}$$

for $\forall a$ and $\forall l$. Then, the constraint (4.6) becomes

$$\sum_{a=1}^{h} p^a = k. \tag{4.10}$$

Therefore, if we choose the parameters of the model, i.e., the matrix sizes and the twists, as in Eqs. (4.4) and (4.5), *block diagonal configurations*, where the total number of the blocks is specified by Eq. (4.10), *are dynamically favored*.

We then consider small fluctuations around the minimum: configurations with $|q_l^a| \ll N$. The condition (4.6), with the use of Eq. (2.6), requires Eq. (4.10) and also

$$\sum_{a=1}^{h} p^a (q_1^a + q_2^a + q_3^a) = 0,$$
$$\sum_{a=1}^{h} p^a (q_1^a q_2^a + q_2^a q_3^a + q_3^a q_1^a) = 0, \tag{4.11}$$
$$\sum_{a=1}^{h} p^a q_1^a q_2^a q_3^a = 0.$$

For $h \geq 2$, these conditions can be satisfied by a nonvanishing q_l^a. The classical action (4.7) is approximated as

$$\Delta S_b \simeq 4\pi^2 \beta \mathcal{N} \frac{N'^2 N^3}{N^4} \sum_{l=1}^{3} \sum_{a=1}^{h} p^a (q_l^a)^2, \tag{4.12}$$

where we have written the difference from the minimum value. For comparison, let us consider cases with large fluctuations: configurations where the total number of blocks is different from Eq. (4.10), and in particular, the configurations with $n_l^a = kN / \sum_{b=1}^{h} p^b$ for $\forall a$ and $\exists l$, and with $n_l^a = N$ for the other l. In this case, the action (4.7) receives an enhancement factor of order N^2, compared to Eq. (4.12).

A. T^2

Before going on to the case in the IIB MM, we first study the dynamics in T^2 as an exercise. In this case, Eq. (4.12) reduces to

$$\Delta S_b = 4\pi^2 \beta k \sum_{a=1}^{h} p^a \left(\frac{q^a}{N}\right)^2. \tag{4.13}$$

This result contrasts to the case where the topologies are defined by the total matrix [30], not by the blocks, as in the present case. There, the action became

$$\Delta S_t \sim \beta N, \tag{4.14}$$

and thus only a single topological sector survived in the continuum limit. In the present case, however, the result (4.13) agrees rather well with the commutative case.

Now, let us consider two continuum limits. The first one is to fix the dimensionful NC parameter

$$\theta \sim \frac{1}{N}(N\epsilon)^2, \tag{4.15}$$

and the dimensionful gauge coupling constant

$$g_{\mathrm{YM}_2}^2 \sim \frac{1}{\beta \epsilon^2}. \tag{4.16}$$

This leads to a double scaling limit: β, $N \to \infty$ with β/N fixed. Indeed, by Monte Carlo simulations, various correlation functions were shown to scale in this limit [35]. In this continuum limit, the action (4.13) vanishes for finite q^a. Then, all of the topological sectors with different q^a appear with equal probabilities.

The second continuum limit is to fix the dimensionful gauge coupling constant (4.16) and the torus size $N\epsilon$. This gives another double scaling limit: β, $N \to \infty$ with β/N^2 fixed. In this limit, the action (4.13) takes finite values for finite q^a. Then, topologically nontrivial sectors appear with finite probabilities, though they are suppressed compared to the trivial sector.

If we consider yet another double scaling limit by fixing β/N^α with $\alpha > 2$, the action (4.13) becomes infinite for finite q^a. In this limit, only a single topological sector appears.

B. T^d

Let us apply the analysis to a d-dimensional torus T^d, although in higher-dimensional gauge theories quantum corrections become larger, and such a semiclassical analysis is not ensured to be valid. In this case, the classical action (4.12) becomes

$$\Delta S_b = 4\pi^2 \beta k N^{d-4} \sum_{l=1}^{d/2} \sum_{a=1}^{h} p^a (q_l^a)^2, \tag{4.17}$$

where we have assumed that d is even.

If the continuum limit is taken by fixing the dimensionful gauge coupling constant

$$g_{\mathrm{YM}_d}^2 \sim \frac{\epsilon^{d-4}}{\beta}, \tag{4.18}$$

and the torus size ϵN, it gives a double scaling limit with a fixed βN^{d-4}. In this limit, the action (4.17) takes finite values for finite q_l^a. Then, topologically nontrivial sectors appear with finite probabilities, but they are suppressed compared to the trivial sector. Similarly, the limit of fixing Eq. (4.18) and the dimensionful NC parameter $N\epsilon^2$ leads to a double scaling limit with a fixed $\beta N^{(d-4)/2}$. The action (4.17) vanishes for finite q_l^a in $d < 4$, and diverges in $d > 4$. Moreover, a limit of fixing Eq. (4.18) and $N\epsilon^\delta$ gives a double scaling limit with a fixed $\beta N^{(d-4)/\delta}$.

C. The IIB MM compactified on a torus

We now study the case of the IIB MM compactified on a torus, assuming that the semiclassical analyses are somehow justified.

We first compare the IIB MM action (2.1) and the unitary version of it, Eq. (4.1). We consider a correspondence between the Hermitian matrices and the unitary matrices as

$$\mathcal{V}_\mu \sim \exp\left(2\pi i \frac{A_\mu}{\epsilon N'}\right), \qquad \mathcal{V}_i \sim \exp\left(2\pi i \frac{A_i}{\epsilon N}\right), \tag{4.19}$$

where the Hermitian matrices A_M are assumed to be constrained to satisfy some conditions (as in Refs. [20,21]) so that the size of the matrices, $\mathcal{N}$, is considered to be the one used after those constraints and quotients are applied. By plugging Eq. (4.19) into Eq. (4.1), and comparing it with Eq. (2.1), we find a relation among the coupling constants in Eqs. (4.1) and (2.1) as

$$\begin{aligned}\frac{1}{2}\beta\mathcal{N}\left(\frac{2\pi}{\epsilon N}\right)^4 &= \frac{1}{2}\beta'\mathcal{N}\left(\frac{2\pi}{\epsilon N'}\right)^4 \\ &= \frac{1}{2}\beta''\mathcal{N}\left(\frac{2\pi}{\epsilon}\right)^4 \frac{1}{N^2 N'^2} \\ &= \frac{1}{g_{\mathrm{IIBMM}}^2}.\end{aligned} \tag{4.20}$$

We then study how to take the large-$\mathcal{N}$ limit. From Eq. (4.20), by defining a combination as

$$\frac{g_{\mathrm{IIBMM}}^2}{\epsilon^4 \mathcal{N}} \equiv \frac{1}{A}, \tag{4.21}$$

the action (4.12) becomes

$$\Delta S_b = \frac{A}{2\pi^2 k} \sum_{l=1}^{3} \sum_{a=1}^{h} p^a (q_l^a)^2. \tag{4.22}$$

It then follows that scaling limits of fixing $g_{\mathrm{IIBMM}}^2 \mathcal{N}^\alpha/\epsilon^4$ with $\alpha > -1$, $\alpha = -1$, and $\alpha < -1$ give drastically different results. Together with fixing the torus size $\epsilon\mathcal{N}^{1/5}$, those scaling limits correspond to fixing $g_{\mathrm{IIBMM}}^2 \mathcal{N}^\gamma$ with $\gamma = \alpha + 4/5$.

Before going on, let us make a small digression. While in Eq. (4.12) we took the topological contributions only from T^6, we can consider the situations where T^4 also has fluxes, specified by integers $q_{l'}^a$ with $l' = 1, 2$. The contribution from T^4 becomes

$$\Delta S_b' = 4\pi^2 \beta' \mathcal{N} \frac{N^3 N'^2}{N'^4} \sum_{l'=1}^{2} \sum_{a=1}^{h} p^a (q_{l'}^a)^2, \tag{4.23}$$

$$= \frac{A}{2\pi^2 k} \sum_{l'=1}^{2} \sum_{a=1}^{h} p^a (q_{l'}^a)^2, \tag{4.24}$$

where again, Eq. (4.21) is used in the second line. Comparing this with Eq. (4.22), this shows that T^4 and T^6 give the same order of contributions. It may imply that topological phenomena on our spacetime, such as the baryon asymmetry of the universe and the strong CP problem, and topological phenomena in the extra dimensions, which determine matter content on our spacetime, are physics of the same order and can be discussed on the same footing. However, Eq. (4.23) is a naive three-level result, which might be interpreted to give phenomena at the Planck scale in our spacetime T^4. Due to large quantum corrections, phenomena at the low energies would not be so simply related to those in the extra dimensions.

We then come back to Eqs. (4.12) and (4.22), focusing on the extra dimensions T^6. If we take a large-$\mathcal{N}$ limit by fixing $g_{\mathrm{IIBMM}}^2 \mathcal{N}^\alpha/\epsilon^4$ with $\alpha > -1$, or by fixing $g_{\mathrm{IIBMM}}^2 \mathcal{N}^\gamma$ with $\gamma > -1/5$, the classical action (4.12) diverges for finite q_l^a, and only a single topological sector survives. While in the present model setting the topologically trivial sector, $q_l^a = 0$, is chosen, in more elaborated models desirable sectors—such as the SM configurations—may be chosen uniquely by the dynamics. This is drastically different from the situations where physicists usually consider the landscape.

In a limit with $\alpha < -1$ or $\gamma < -1/5$, the action (4.12) vanishes for finite q_l^a, and all the topological sectors appear with equal probabilities. Then, the estimation for the probability distribution over the string vacuum space reduces to the number counting of the classical solutions. Moreover, in a limit with $\alpha < -1 - 2/5$, a still larger number of configurations—where the block number is different from the value specified in Eq. (4.10)—can also appear, as can be seen from the study for large fluctuations given below Eq. (4.12).

In a limit with $\alpha = -1$ or $\gamma = -1/5$, the action (4.12) takes the finite values (4.22) for finite q_l^a, and the topologically nontrivial sectors appear with finite but suppressed probabilities. We now estimate the probabilities for the appearance of the SM configurations obtained in the previous section. By solving Eq. (3.6) for Eq. (3.14), the q_l^a are determined as

$$\begin{aligned} q_1^a &= (q_1, q_1 - 1, q_1, q_1 \mp 1, q_1 \pm 1), \\ q_2^a &= (q_2, q_2 + 1, q_2, q_2 \mp 1, q_2 \pm 1), \\ q_3^a &= (q_3, q_3 - 3, q_3, q_3 - 3, q_3 - 3), \end{aligned} \tag{4.25}$$

for $a = 1, \ldots, h$. Since only the differences are specified in Eq. (3.6), the q_l^a are determined with arbitrary integer shifts q_1, q_2, and q_3.[6]

We can lower the values of the classical action (4.22) by shifting the twists in the action (4.1) from Eq. (4.5). If we choose the twists as

$$\begin{aligned} Z_{45} &= \exp\left(2\pi i\left(\frac{s_1}{N_1} + \frac{-q_1 + 1/4}{N_1^2}\right)\right), \\ Z_{67} &= \exp\left(2\pi i\left(\frac{s_2}{N_2} + \frac{-q_2 - 1/4}{N_2^2}\right)\right), \\ Z_{89} &= \exp\left(2\pi i\left(\frac{s_3}{N_3} + \frac{-q_3 + 3/2}{N_3^2}\right)\right), \end{aligned} \tag{4.26}$$

the action (4.22) takes the minimum value

$$\Delta S_b = \frac{A}{2\pi^2 k} 25 \tag{4.27}$$

for either sign in the double signs in Eq. (4.25). The probability of the SM appearance is semiclassically given as $e^{-\Delta S_b}$, multiplied by a factor coming from quantum corrections. There exist configurations with the action (4.27), but with p^a and q_l^a different from Eq. (4.25), and thus the probability of the SM appearance must also be divided by this numerical factor. While we have considered the minimal case of $h = 5$ here, cases with $h > 5$ would lead to larger values of ΔS_b and be more suppressed. Since Eq. (4.27) is a result from the unitary MM (4.1), if we start from Eq. (2.1) and follow the procedures mentioned at the beginning of this subsection, Eq. (4.27) would receive some corrections.

[6] Unfortunately, the condition (4.11) can not be satisfied by Eq. (4.25) with any integers q_1, q_2, and q_3. However, by considering the cases where the three original T^2's are taken to be different, i.e., the integers N_l, r_l, s_l, k_l depend on l, the condition (4.11) is extended, and then satisfied by some integers. For instance, $q_1 = q_2 = 0$, $q_3 = 3$, $r_1 = 7$, $r_2 = 1$, $r_3 = 1$, and $N_1 = N_2 = N_3$ satisfy it.

V. CONCLUSIONS AND DISCUSSION

In this paper, we considered the situations where the IIB MM is compactified on a torus with fluxes, and found matrix configurations that yield the SM matter content. The configurations that provide the SM gauge group plus the minimum number of the extra U(1)'s and the SM fermion species are determined almost uniquely. We then studied the dynamics of the unitary MM semiclassically. We found that in an MM where the matrix sizes and the twists of the action are suitably chosen, block diagonal configurations are favored dynamically.

We also argued how to take large-N limits. In a large-$\mathcal{N}$ limit of fixing $g_{\text{IIBMM}}^2 \mathcal{N}^\alpha/\epsilon^4$ with $\alpha > -1$, or $g_{\text{IIBMM}}^2 \mathcal{N}^\gamma$ with $\gamma > -1/5$, only a single topological sector appears. This suggests that in some more elaborated models the SM may be chosen uniquely by the dynamics. This is drastically different from the situations where the landscape is usually considered. In a limit with $\alpha < -1$ or $\gamma < -1/5$, all the topological sectors appear with equal probabilities. Then, the estimation for the probability distribution reduces to the number countings of the classical solutions. In a limit with $\alpha = -1$ or $\gamma = -1/5$, all the topological sectors appear with finite but different probabilities. In this case, we estimated the probabilities of the appearance of the SM configurations.

There remain some important problems. One is about compactifications. In this paper, we assumed toroidal compactifications, and worked in a unitary matrix formulation. If we start from Hermitian matrices, however, we need to impose some conditions on the matrices to realize toroidal compactifications [20,21]. Those special configurations seem unlikely to appear dynamically. Note, however, that fluctuations around the background may not need to be restricted in the large-N limit [24], and that the backgrounds of the special forms may be chosen dynamically by the mechanism mentioned in this paper.

We should also study how the anisotropy between our large spacetime and the small compactified space arises, as in Refs. [4–6]. Moreover, our spacetime is commutative and local fields live on it. If we start from MM, however, those important properties are rather difficult to realize (see, for instance, arguments in Refs. [6,18]). On the other hand, the extra-dimensional spaces are free from those constraints, and need not have even a geometrical interpretation, which can broaden the possibilities of phenomenological model constructions. After all, the problems of compactification in MM will be clarified by understanding both our spacetime and the extra-dimensional space together.

A second issue is about anomaly cancellations. The model we considered in the present paper has extra U(1) gauge groups and is anomalous within the gauge dynamics. This anomaly may be canceled via the Green-Schwarz mechanism by the exchanges of the Ramond–Ramond fields. The exchange of Ramond–Ramond fields also makes the extra U(1) gauge fields massive. In order to realize this, the model should be modified (see, for instance, Ref. [36]). By these studies of comparing various phenomenological models in string theories and MM, we can also make progress for both string theories and MM.

A third issue is about the Higgs particles. While the gauge fields in the extra dimensions give scalar fields and candidates for the Higgs fields, it is difficult to keep them massless against quantum corrections, which is well-known as the naturalness or the hierarchy problem. In the gauge-Higgs unifications [37], higher-dimensional gauge symmetries protect the scalar mass from the quadratic divergences of the cutoff order, but it still can receive quantum corrections of the order of the Kaluza-Klein scale (see also Ref. [38]).

We will come back to these issues in future publications. Ultimately, we hope to analyze the full dynamics in the MM, and survey the probability distribution over the whole of the landscape.

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APPENDIX: SOLUTIONS OF q_l^{ab}

In this appendix, we find all the solutions of q_l^{ab} that satisfy Eq. (3.7) for Eq. (3.13). We first note that Eq. (3.7) is invariant under the permutations among q_1^{ab}, q_2^{ab}, and q_3^{ab}, and also under the sign flips: $q_1^{ab} \to -q_1^{ab}$, $q_2^{ab} \to -q_2^{ab}$, $q_3^{ab} \to q_3^{ab}$; $q_1^{ab} \to -q_1^{ab}$, $q_2^{ab} \to q_2^{ab}$, $q_3^{ab} \to -q_3^{ab}$; $q_1^{ab} \to q_1^{ab}$, $q_2^{ab} \to -q_2^{ab}$, $q_3^{ab} \to -q_3^{ab}$ (two of which are independent). By using these symmetries, we can fix the order of q_1^{ab}, q_2^{ab}, and q_3^{ab}, and the overall sign for two of them.

1. $h = 4$ case

For a preparation, we first consider the case with $h = 4$ and

$$q^{ab} = \begin{pmatrix} 0 & -3 & 0 & 3 \\ & 0 & 3 & 0 \\ & & 0 & 3 \\ & & & 0 \end{pmatrix}. \tag{A1}$$

Note that this is different from Eq. (3.8). In order to save space, we will omit the diagonal elements and write it as

$$\hat{q}^{ab} = \begin{pmatrix} -3 & 0 & 3 \\ & 3 & 0 \\ & & 3 \end{pmatrix}, \tag{A2}$$

and solve the equation $\prod_{l=1}^{3} \hat{q}_l^{ab} = \hat{q}^{ab}$.

One of the $\hat{q}_l^{11}$ must be ± 3 and the other two of $\hat{q}_l^{11}$ must be ± 1. The same is true for $\hat{q}_l^{22}$ and $\hat{q}_l^{33}$. We then classify all the possibilities into three cases: the case where all the three 3's are gathered in a single l; the case where the two 3's are in an l and the other 3 is in another l; the case where the three 3's are completely split into different l's.

In the first case, there exist six solutions:

$$\begin{array}{ccc} \hat{q}_1^{ab} & \hat{q}_2^{ab} & \hat{q}_3^{ab} \\ \begin{pmatrix} \pm 1 & -1 \pm 1 & \pm 1 \\ & -1 & 0 \\ & & 1 \end{pmatrix} & \begin{pmatrix} \pm 1 & -1 \pm 1 & \pm 1 \\ & -1 & 0 \\ & & 1 \end{pmatrix} & \begin{pmatrix} -3 & 0 & 3 \\ & 3 & 6 \\ & & 3 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 & \pm 1 \\ & -1 & -1 \pm 1 \\ & & \pm 1 \end{pmatrix} & \begin{pmatrix} -1 & 0 & \pm 1 \\ & 1 & 1 \pm 1 \\ & & \pm 1 \end{pmatrix} & \begin{pmatrix} 3 & 0 & 3 \\ & -3 & 0 \\ & & 3 \end{pmatrix} \\ \begin{pmatrix} 1 & 1 \pm 1 & \pm 1 \\ & \pm 1 & -1 \pm 1 \\ & & -1 \end{pmatrix} & \begin{pmatrix} -1 & -1 \pm 1 & \pm 1 \\ & \pm 1 & 1 \pm 1 \\ & & 1 \end{pmatrix} & \begin{pmatrix} 3 & 6 & 3 \\ & 3 & 0 \\ & & -3 \end{pmatrix} \end{array}. \tag{A3}$$

The double signs correspond in each row of the table. In the second case, there are four solutions:

$$
\hat{q}_1^{ab} \quad \hat{q}_2^{ab} \quad \hat{q}_3^{ab}
$$

$$
\begin{pmatrix} \pm 1 & 0 & \pm 1 \\ & \mp 1 & 0 \\ & & \pm 1 \end{pmatrix} \begin{pmatrix} \mp 1 & 0 & 3 \\ & \pm 1 & 3 \pm 1 \\ & & 3 \end{pmatrix} \begin{pmatrix} 3 & 0 & \pm 1 \\ & -3 & -3 \pm 1 \\ & & \pm 1 \end{pmatrix}. \tag{A4}
$$

$$
\begin{pmatrix} \mp 1 & 0 & \pm 1 \\ & \pm 1 & \pm 2 \\ & & \pm 1 \end{pmatrix} \begin{pmatrix} 3 & 3 \pm 1 & 3 \\ & \pm 1 & 0 \\ & & \mp 1 \end{pmatrix} \begin{pmatrix} \pm 1 & 3 \pm 1 & \pm 1 \\ & 3 & 0 \\ & & -3 \end{pmatrix}
$$

The third case has no solution. There are ten solutions total, taking the double signs into account.

2. $h = 5$ case

We now come back to the case with $h = 5$ and Eq. (3.13). Again, we omit diagonal elements and write it as

$$
\hat{q}^{ab} = \begin{pmatrix} -3 & 0 & 3 & 3 \\ & 3 & 0 & 0 \\ & & 3 & 3 \\ & & & 0 \end{pmatrix}. \tag{A5}
$$

The analysis for $\hat{q}_l^{ab}$ with $1 \le a, b \le 3$ is the same as in the $h = 4$ case of the previous subsection.

If $\hat{q}_l^{44} = 0$ for all l, which is equivalent to $q_l^4 = q_l^5$ for all l, the fourth and fifth blocks of the bosonic matrix V_i in Eq. (2.4) become identical, and the corresponding gauge group is enhanced from $U(1) \times U(1)$ to $U(2)$. We then exclude this case. Hence, we must find the solution where some of $\hat{q}_l^{44}$ are zero and some of $\hat{q}_l^{44}$ are nonzero. This can be achieved by using the second solution in Eq. (A3). We then obtain

$$
\hat{q}_1^{ab} \quad \hat{q}_2^{ab} \quad \hat{q}_3^{ab}
$$

$$
\begin{pmatrix} 1 & 0 & \pm 1 & \mp 1 \\ & -1 & -1 \pm 1 & -1 \mp 1 \\ & & \pm 1 & \mp 1 \\ & & & \mp 2 \end{pmatrix} \begin{pmatrix} -1 & 0 & \pm 1 & \mp 1 \\ & 1 & 1 \pm 1 & 1 \mp 1 \\ & & \pm 1 & \mp 1 \\ & & & \mp 2 \end{pmatrix} \begin{pmatrix} 3 & 0 & 3 & 3 \\ & -3 & 0 & 0 \\ & & 3 & 3 \\ & & & 0 \end{pmatrix}. \tag{A6}
$$

All the double signs correspond. As is clear from our calculations, these exhaust the solutions for Eq. (A5) under the conditions mentioned above.

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