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Generalized hedgehog ansatz and Gribov copies in regions with nontrivial topologies

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In this paper the arising of Gribov copies both in Landau and Coulomb gauges in regions with nontrivial topologies but flat metric, (such as closed tubes $S^1 \times D^2$, or $\mathbb{R} \times T^2$) will be analyzed. Using a novel generalization of the hedgehog ansatz beyond spherical symmetry, analytic examples of Gribov copies of the vacuum will be constructed. Using such ansatz, we will also construct the elliptic Gribov pendulum. The requirement of absence of Gribov copies of the vacuum satisfying the strong boundary conditions implies geometrical constraints on the shapes and sizes of the regions with nontrivial topologies.

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I. INTRODUCTION

One of the most important sectors of the standard model is Yang-Mills theory, which describes QCD and the electroweak theory. The degrees of freedom of a gauge theory are encoded in the connection $(A_\mu)^a$, which is a Lie algebra valued one form. The action functional is invariant under gauge transformations

$$A_\mu \to U^{-1} A_\mu U + U^{-1} \partial_\mu U, \tag{1}$$

whereas the physical observables are invariant under proper gauge transformations. The latter has to be everywhere smooth and it has to decrease at infinity in a suitable way (see, for instance, Ref. [1]).

Since it is still unknown how to use in practice gauge invariant variable in Yang-Mills case,[1] the most convenient practical choices for the gauge fixing are the Coulomb gauge and the Landau gauge.[2] The standard approach to fix the gauge and to use Feynman expansion around the trivial vacuum $A_\mu = 0$ provides one with excellent results when the coupling constant is small.

As it was for the first time discovered in Ref. [4], a *proper gauge fixing* is not possible globally due to the presence of Gribov copies.[3] In the QCD case, this effect is very important in the nonperturbative regime. Even if one chooses a gauge free of Gribov copies, the effects of Gribov ambiguities in other gauges generate a nonperturbative breaking of the BRST symmetry [6].

It has been suggested to exclude from the domain of the path-integral gauge potentials $A_{\mu,}$ which generate zero modes of the FP operator (see, in particular, Refs. [4,7–11]; two nice reviews are Refs. [12,13]). In this framework, which is called the (refined) *Gribov-Zwanziger approach* to QCD, The region Ω around $A_\mu = 0$ in which the FP operator is positive is called Gribov region:

$$\Omega \overset{\text{def}}{=} \{A_\mu | \partial^\mu A_\mu = 0 \quad \text{and} \quad \det \partial^\mu D(A)_\mu > 0\}, \tag{2}$$

$$D(A)_\mu = \partial_\mu + [A_{\mu,} \cdot], \tag{3}$$

where $D(A)_\mu$ is the covariant derivative corresponding to the field A_μ. In the case in which the space-time metric is flat and the *topology is trivial*, this approach coincides with usual perturbation theory when the gauge field A_μ is close to the origin (with respect to a suitable functional norm [11]). At the same time, this framework takes into account the infrared effects related to the partial[4] elimination of the Gribov copies [8,14,15]. When one takes into account the presence of suitable condensates [16–20] the agreement with lattice data is excellent [21,22]. Within the (refined) *Gribov-Zwanziger approach* to QCD, the *copy-free neighborhood* of the trivial vacuum $A_\mu = 0$ *has to be identified with perturbative region of the theory*. In this way, within the perturbative region, the standard recipe to fix the gauge and perform the perturbative expansion makes sense thanks to the lack of any overcounting. The appearance of copies of the vacuum satisfying the strong boundary conditions is one of the worst pathologies of perturbation theory.

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[1]In the cases of topological field theories in 2 + 1 dimensions [2] this goal has been partially achieved.

[2]The axial and light-cone gauge fixing choices are affected by quite nontrivial problems (see, for instance, Ref. [3]).

[3]Furthermore, it has been shown by Singer [5], that if Gribov ambiguities occur in Coulomb gauge, they occur in all the gauge fixing conditions involving derivatives of the gauge field. Other gauge fixings (such as the axial gauge, the temporal gauge, and so on) free from gauge fixing ambiguities are possible but these choices have their own problems (see, for instance, Ref. [3]).

[4]The condition to have a positive Faddeev-Popov operator is not enough to completely eliminate Gribov copies in the Coulomb and Landau gauges. It can be shown [11] that there exists a smaller region (called the *modular region*) contained in the Gribov region which is free of gauge fixing ambiguities. However, it is still not clear how to implement the restriction to the modular region in practice.

On the other hand, it is a well-established fact by now that Yang-Mills theory may have knotted excitations [23–25] (the simplest nontrivial example corresponding to a closed tube: the "donut" or *unknot* with topology $S^1 \times D^2$). Moreover, as it is well known (see, for instance, Ref. [26]), nontrivial topologies such as $\mathbb{R} \times T^2$ and $\mathbb{R} \times T^3$ are very important to understand the infinite volume limit of Yang-Mills theory as well as chiral symmetry breaking generated by the Casimir force (a related reference is Ref. [27]). In order to describe these situations one has to be able to define consistently Yang-Mills theory inside bounded regions with the nontrivial topologies.

The main goal of the present paper is to analyze whether or not Gribov copies of the vacuum can actually exist inside a space-time region of the topology such as $S^1 \times D^2$ (but more general topologies relevant, for instance, for the infinite volume limit of Yang-Mills as well as lattice QCD-such as T^3 and $\mathbb{R} \times T^2$, $\mathbb{R} \times T^3$-as well as the interior of an ellipse-will be considered). The absence of Gribov copies of the vacuum (satisfying strong boundary conditions) is a necessary condition for the existence of a perturbative region in the functional space around $A_\mu = 0$. It will be shown that such requirement implies nontrivial restrictions on the possible sizes and shapes of the corresponding regions. In a sense, these results are more surprising than the ones obtained in Refs. [28–30] in which it has been shown that the pattern of appearance of Gribov copies strongly depends on the space-time metric. All the examples of nontrivial copies of the vacuum will be in flat metrics (but with nontrivial topologies). The present analysis will also show that the Gribov phenomenon strongly depends on the shapes and sizes of the bounded regions where one wants to study gauge theories.

Unlike the equation for the zero-modes of the Faddeev-Popov operator, the issue of the appearance of Gribov copies of the vacuum is nonlinear in nature and therefore can be quite complicated when nonstandard topologies are considered. The technical tool necessary in order to analyze such an issue is a novel (to the best of authors knowledge) generalization of the *hedgehog ansatz* beyond spherical symmetry for the nonlinear sigma model. Such a generalization is quite interesting in itself since it allows one to reduce the nonlinear system of coupled partial differential equations corresponding to the equations of motion of the nonlinear sigma model to a single nonlinear partial differential equation which can be analyzed with the tools of solitons theory.

The paper is organized as follows. In Sec. II the notion of strong boundary conditions with nontrivial topologies will be analyzed. In Sec. III, the relations between the Gribov copies equation and the non linear sigma model will be discussed. In Sec. IV, the generalized hedgehog ansatz will be constructed. In Secs. V and VI, various nontrivial examples of copies of the vacuum will be described. Some conclusions will be drawn in the last section. In the Appendix , a novel way to implement both spherical and elliptical symmetries (which could be useful in the context of calorons) will be presented.

II. STRONG BOUNDARY CONDITIONS WITH NON-TRIVIAL TOPOLOGIES

Here, we will first review the definition of strong boundary conditions in the case in which the metric is flat and the topology trivial. In the present paper we will mainly analyze the $SU(2)$ case but many of the present results also extend to other Lie groups. A useful starting point is the definition of non-Abelian charges Q^a (see Refs. [1,4,31,32]; see, for a detailed review, Ref. [33]):

$$Q^{(a)} = \int_M d^3x \partial_i E^{ia} = \lim_{R\to\infty} \int_{\Sigma_R} d^2\Sigma(n_i E^{ia}), \tag{4}$$

where M is the constant-time hypersurface in four-dimensional Minkowski space, Σ_R is the two-dimensional sphere of radius R, n_i is the outer pointing unit normal to Σ_R, the indices i, j, k refers to space-like directions while $a, b, c, \ldots$ are internal $SU(2)$ indices.

Thus, one can define a *proper gauge transformation U* as a smooth gauge transformation which does not change the value of the charge as a surface integral at infinity:

$$\lim_{R\to\infty} \int_{\Sigma_R} d^2\Sigma(n_i E^{ia}) = \lim_{R\to\infty} \int_{\Sigma_R} d^2\Sigma(n_i (U^{-1}EU)^{ia}). \tag{5}$$

Therefore, U is proper if it is smooth and approaches to an element of the center of the gauge group at spatial infinity.

A Gribov copy U on a flat and topologically trivial space-time satisfies *the strong boundary* conditions if U is proper. A copy of this type is particularly problematic since it would represent a failure of the whole gauge fixing procedure. Indeed, if the vacuum $A_\mu = 0$ possesses a copy fulfilling the strong boundary conditions, not even usual perturbation theory leading to the standard Feynman rules in the Landau or Coulomb gauge would be well defined.

A. Strong boundary conditions on $S^1 \times D^2$

The simplest topology corresponding to a closed knotted tube is $\Upsilon = S^1 \times D^2$: this case is relevant in the analysis of nontrivial topological excitations of glueballs. In particular, there is a sound evidence supporting the existence of excited glueball states with the topology of $\Upsilon = S^1 \times D^2$ (see, in particular, Refs. [24,25]). In this case, the spatial metric describing the region Υ reads:

$$ds^2 = (d\phi)^2 + (dr^2 + r^2 d\theta^2), \qquad 0 \le \phi \le 2\pi,$$
$$0 \le \theta \le 2\pi, \qquad 0 \le r \le R, \tag{6}$$

where the coordinate ϕ corresponds to the S^1 circle of Υ, while the coordinates r and θ describe the disk D^2 of Υ.

The radius of the disk is R and the boundary of Υ is:

$$\partial\Upsilon = S^1 \times \partial D^2.$$

Thus, in this case a smooth gauge transformation U is proper if

$$U|_{S^1\times\partial D^2} \in Z_2,$$

where Z_2 is the center[5] of $SU(2)$. In other words, a smooth gauge transformation $U = U(\phi, r, \theta)$ is proper if

$$U(\phi + 2m\pi, r, \theta + 2n\pi) = U(\phi, r, \theta), \qquad m, n \in \mathbb{Z}, \tag{7}$$

$$U(\phi, R, \theta) \in Z_2, \quad \forall\ \phi, \theta. \tag{8}$$

B. Strong boundary conditions on T^3

In the cases in which the spatial topology is T^3 (which, for instance, is relevant in the case of lattice QCD), the flat spatial metric describing T^3 reads:

$$ds^2 = \sum_{i=1}^{3} \lambda_i^2 (d\phi_i)^2, \qquad \lambda_i \in \mathbb{R}, \qquad 0 \le \phi_i \le 2\pi, \tag{9}$$

where the coordinate ϕ_i corresponds to the ith factor S^1 in T^3 while λ_i represents the size of the ith factor S^1 in T^3. In this case, due to the fact that $\partial T^3 = 0$, a smooth gauge transformation $U = U(\phi_1, \phi_2, \phi_3)$ is proper if

$$U(\phi_1 + 2m_1\pi, \phi_2 + 2m_2\pi, \phi_3 + 2m_3\pi) = U(\phi_1, \phi_2, \phi_3), \qquad m_i \in \mathbb{Z}. \tag{10}$$

C. Strong boundary conditions on $\mathbb{R} \times T^2$

The cases in which the spatial topology is $\mathbb{R} \times T^2$ may correspond to situations in which the bounded region in which we want to define perturbative Yang-Mills theory is much longer in one direction (the x axis) with respect to the other two.

It is worth emphasizing here that, in the Euclidean theory, the topology $\mathbb{R} \times T^3$ is also very important in order to understand both the infinite volume limit in QCD (see, for instance, Ref. [26]) and the effects of chiral symmetry breaking in QCD (see, for instance, Ref. [27]). In particular, it has been shown in Ref. [27] the relevance of the Casimir[6] force to understand chiral symmetry breaking. Hence, the importance of finite-sized effects to explain these phenomena and the necessity to have a well-defined perturbation theory in such bounded regions with nontrivial topology require a deeper understanding of the Gribov problem in bounded region as well. As it will be shown in the following, the present results on the Coulomb gauge can be trivially extended to the case of the Landau gauge defined on $\mathbb{R} \times T^3$. Thus, the gauge-fixing pathologies arising in this case have to be carefully taken into account.

The flat spatial metric describing $\mathbb{R} \times T^2$ reads:

$$ds^2 = dx^2 + \sum_{i=1}^{2} \lambda_i^2 (d\phi_i)^2, \qquad \lambda_i \in \mathbb{R}, \qquad 0 \le \phi_i \le 2\pi, \tag{11}$$

where the coordinate ϕ_i corresponds to the ith factor S^1 in T^2 while λ_i represents the size of the ith factor S^1 in T^2. In this case, $\partial(\mathbb{R} \times T^2)$ is nontrivial and can be identified with the two limit point $x \to \pm\infty$ of $\mathbb{R}$. Thus, a smooth gauge transformation $U = U(\phi_1, \phi_2, x)$ is proper if

$$U(\phi_1 + 2m_1\pi, \phi_2 + 2m_2\pi, x) = U(\phi_1, \phi_2, x), \qquad m_i \in \mathbb{Z}, \tag{12}$$

$$\lim_{x\to\pm\infty} U(\phi_1, \phi_2, x) \in Z_2, \quad \forall\ \phi_1, \phi_2. \tag{13}$$

III. GRIBOV COPIES AND NONLINEAR SIGMA MODEL

In the present paper we will analyze the copies of the vacuum in the Coulomb gauge but many of the present results can be easily extended to the Landau gauge. Since the Gribov copies of the vacuum in the Coulomb gauge can be seen as the Euler-Lagrange equations corresponding to a nonlinear sigma model, we will first review some basic features of this model. The nonlinear sigma model Lagrangian in D spacelike dimensions can be written in terms of group valued scalar field U. In the following we will consider the $SU(2)$ group, so that the corresponding Lagrangian is:

$$S = \frac{\kappa^2}{2} \int \sqrt{-g} d^D x \,\mathrm{Tr}[R^i R_i], \tag{14}$$

$$R_i = U^{-1}\partial_i U, \tag{15}$$

$$\begin{aligned} R_i &= R_i^a t_a, \\ t^a t^b &= -\delta^{ab}\mathbf{1} - \varepsilon^{abc} t^c, \\ [R_i, R_j]^c &= C^c_{ab} R_i^a R_j^b, \\ C^c_{ab} &= -2\varepsilon_{cab}, \\ \varepsilon^{abc}\varepsilon^{mnc} &= (\delta_a^m \delta_b^n - \delta_a^n \delta_b^m), \end{aligned}$$

where D is the number of space dimensions, κ is the sigma-model coupling constants, g is the determinant of the spacelike metric, $\mathbf{1}$ is the identity 2×2 matrix, and t^a the generator of $SU(2)$ (where the Latin letters a, b, c and so on corresponds to group indices). The equation

[5]Obviously, the center of $SU(2)$ is made of two elements: $\pm\mathbf{1}$.

[6]Indeed, Casimir force is a genuine finite-size effect which, obviously, is not visible if the theory is analyzed in unbounded regions.

characterizing the appearance of Gribov copies of the vacuum in the Coulomb gauge [which corresponds to the Euler-Lagrange equations of the action in Eq. (14)] read:

$$\nabla^i R_i = \nabla^i (U^{-1}\partial_i U) = 0, \tag{16}$$

where ∇^i is the Levi-Civita covariant derivative corresponding to the metric [which, in the present case will have one of the forms in Eqs. (6), (9), and (11)]. The case of the Landau gauge is analogous.

The following standard parametrization of the $SU(2)$-valued functions $U(x^i)$ is useful:

$$U(x^i) = Y^0\mathbf{1} + Y^a t_a, \qquad U^{-1}(x^i) = Y^0\mathbf{1} - Y^a t_a, \tag{17}$$

$$Y^0 = Y^0(x^i), \qquad Y^a = Y^a(x^i), \tag{18}$$

$$(Y^0)^2 + Y^a Y_a = 1, \tag{19}$$

where, of course, the sum over repeated indices is understood also in the case of the group indices (in which case the indices are raised and lowered with the flat metric δ_{ab}). Therefore, the R_i in Eq. (15) can be written as follows:

$$R_i^c = \varepsilon^{abc} Y_a \partial_i Y_b + Y^0 \partial_i Y^c - Y^c \partial_i Y^0. \tag{20}$$

IV. GENERALIZED HEDGEHOG ANSATZ

Due to the intrinsic nonlinear nature of the Gribov copies of the vacuum, it is necessary to introduce a suitable technical tool which allows us to study such phenomenon with nontrivial topologies.

Here we will first discuss the geometrical interpretation of *hedgehog ansatz* as an effective tool to reduce the field equations of the nonlinear sigma model (which, generically, are a *system of coupled nonlinear partial differential equations*) to a single scalar nonlinear partial differential equation. The geometrical analysis of such a very important feature of the hedgehog ansatz allows us to construct the natural generalization of the hedgehog ansatz for the nonlinear sigma model. This section is interesting in itself since, to the best of the authors' knowledge, this is the first systematic reduction of the equations of motion of the nonlinear sigma model (which is a system of coupled nonlinear partial differential equations) to a single nonlinear scalar equation beyond spherical symmetry. In terms of the group element U, the standard spherically symmetric hedgehog ansatz reads

$$\begin{aligned} U &= \mathbf{1}\cos\alpha(r) + \hat{n}^a t_a \sin\alpha(r), \\ U^{-1} &= \mathbf{1}\cos\alpha(r) - \hat{n}^a t_a \sin\alpha(r), \end{aligned} \tag{21}$$

$$\hat{n}^1 = \sin\theta\cos\phi, \quad \hat{n}^2 = \sin\theta\sin\phi, \quad \hat{n}^3 = \cos\theta, \tag{22}$$

where r, θ, and ϕ are spherical coordinates of the flat Euclidean metric, the $\hat{n}^a$ are normalized with respect to the internal metric δ_{ab}. In terms of the variables Y^0 and Y^a the hedgehog ansatz corresponds to the following choice:

$$Y^0 = \cos\alpha(r), \qquad Y^a = \hat{n}^a \sin\alpha(r), \qquad \delta_{ab}\hat{n}^a\hat{n}^b = 1. \tag{23}$$

Thus, the expression for R_i^a in Eq. (20) reads

$$R_i^c = \delta^{dc}(\sin^2\alpha)\varepsilon_{abd}\hat{n}^a\partial_i\hat{n}^b + \hat{n}^c\partial_i\alpha + \frac{\sin(2\alpha)}{2}\partial_i\hat{n}^c. \tag{24}$$

The equations of motion for the nonlinear sigma model in Eq. (16) corresponding to this ansatz read

$$(\nabla^i\partial_i\alpha) + L\sin 2\alpha = 0, \tag{25}$$

where L is a suitable function of the radial coordinate r (see below). Indeed, this is a very important and nontrivial characteristic of the spherically symmetric hedgehog ansatz which reduces a system of coupled nonlinear partial differential equations to a single scalar equation. It is easy to see that in the case of the flat three-dimensional Euclidean metric on $\mathbb{R}^3$ in spherical coordinates the above reduces to the standard Gribov pendulum of the vacuum (see Refs. [4,12]).

The previous analysis suggests to analyze which are the geometrical conditions which allow us to reduce a system of non-linear coupled partial differential equations (PDE) to a single scalar PDE. Let us consider the following generalization of the hedgehog ansatz:

$$\begin{aligned} U &= \mathbf{1}\cos\alpha + \hat{n}^a t_a \sin\alpha, \\ U^{-1} &= \mathbf{1}\cos\alpha - \hat{n}^a t_a \sin\alpha, \end{aligned} \tag{26}$$

$$\delta_{ab}\hat{n}^a\hat{n}^b = 1, \qquad \alpha = \alpha(x^i), \qquad \hat{n}^a = \hat{n}^a(x^i), \tag{27}$$

such that

$$(\partial_i\alpha)(\nabla^i\hat{n}^a) = 0, \tag{28}$$

$$(\nabla^i\partial_i)\hat{n}^c = 2L\hat{n}^c, \tag{29}$$

where L (which has to be the same for all the nonvanishing[7] $\hat{n}^c$) may depend on the space-time coordinates.

The important point is that, when the conditions (28) and (29) are satisfied, the function α (which in the usual spherically symmetric hedgehog ansatz can depend only on the radial coordinate r) can now depend on any set of coordinates and the functions $\hat{n}^a$ (which, in the usual case, have to coincide with the unit radial vector) can be adapted to cases in which there is no spherical symmetry (and, therefore, no natural radial coordinate). Indeed, the expression for R_i^a in Eq. (20) reads

$$R_i^c = \delta^{dc}(\sin^2\alpha)\varepsilon_{abd}\hat{n}^a\partial_i\hat{n}^b + \hat{n}^c\partial_i\alpha + \frac{\sin(2\alpha)}{2}\partial_i\hat{n}^c. \tag{30}$$

[7]Note that, in order for the solution to be nontrivial, at least two of the three $\hat{n}^c$ have to be nonvanishing.

Hence, provided Eqs. (28) and (29) are fulfilled and taking into account that the expression

$$(\nabla^i \hat{n}^a)(\partial_i \hat{n}^b) = g^{ij}(\partial_i \hat{n}^a)(\partial_j \hat{n}^b)$$

is symmetric under the exchange of a and b (so that its contraction with ε_{abd} vanishes) the coupled system of nonlinear PDE corresponding to the equations of motion of the nonlinear sigma model reduces to the single scalar nonlinear partial differential equation of sine-Gordon type:

$$(\nabla^i \partial_i \alpha) + L \sin 2\alpha = 0. \tag{31}$$

Indeed, it is easy to see that in the usual spherically symmetric flat ansatz in Eqs. (21) and (22) satisfies the conditions in Eqs. (28) and (29) with $L = -1/r^2$ and that, correspondingly, Eq. (31) reduces to the usual Gribov pendulum (see Refs. [4,12]). However, it is worth noting here that the present derivation is far simpler (just three lines of computation) and, furthermore, it discloses in a very clear way the geometry behind the hedgehog ansatz.

The usual Gribov pendulum equation on flat and topologically trivial three-dimensional spaces (see Refs. [4,12]) does not coincide with the sine-Gordon equation due to the radial dependence of the determinant of the three-dimensional Euclidean metric in spherical coordinates. For these reasons, there are no analytic examples of copies of the vacuum in this case. On the other hand, using the present generalized hedgehog ansatz, one can construct many analytic examples of copies of the vacuum. These examples are very useful since they allow us to express the requirement of absence of copies of the vacuum in terms of explicit constraints on the shapes and sizes of the corresponding bounded regions where one wants to define a perturbative region.

A. Regularity conditions

Another important requirement to satisfy the strong boundary conditions is the regularity of the pure gauge field $U^{-1}dU$ everywhere. In the standard spherically-symmetric case, the only problematic point is the origin of the coordinates in which the spherical coordinates system is not well defined. One way to understand this condition in general is to analyze the behavior of the one-form

$$R^c = R_i^c \wedge dx^i \tag{32}$$

[where R_i^c is given by (30) and α satisfies (31)] when one changes coordinates from the Cartesian to a coordinates system which is singular somewhere. Indeed, in the following we will need to use non-Cartesian coordinates systems which are adapted to the symmetry of the problem (such as the elliptic coordinates systems in the next sections).

In the case of Cartesian coordinates, the regularity at the origin can be read by looking at the components of the one-form R_i^c since the coordinates system is regular everywhere. On the other hand, if one is in a spherical or elliptical coordinates system, the angular coordinates are not well defined at the origin and some extra care is required. For instance, to disclose the singularity of the one forms $d\theta$ and $d\phi$ at the origin one can analyze the Jacobian of the transformation from Cartesian coordinates (x'^i) to spherical coordinates (x^j):

$$dx'^i = J^i_j dx^j. \tag{33}$$

Due to the regularity of the Cartesian coordinates, the Jacobian itself encodes the information of the singularity at the origin. In the spherically symmetric case, for example, if one takes into account the Jacobian, the regularity condition at the origin coincides[8] with the usual one derived in Cartesian coordinates [12].

The requirement for regularity can be described as follows. Let us call x_s a point in which the coordinates are singular and let us consider the following behavior for α when $x \to x_s$

$$\alpha(x) \underset{x\to x_s}{\longrightarrow} n\pi + \beta f(x) \tag{34}$$

with $f(x) \underset{x\to x_s}{\rightrightarrows} 0$ and β an arbitrary constant. Then, a copy of the form (30) satisfies

$$R_i^c \underset{x\to x_s}{\longrightarrow} \beta^2 \delta^{dc} f^2 \varepsilon_{abd} \hat{n}^a \partial_i \hat{n}^b + \beta \hat{n}^c \partial_i f + \beta f \partial_i \hat{n}^c. \tag{35}$$

Then using (33) we see that $f(x)$ must be such that

$$R_i^c (J^{-1})^i_j \tag{36}$$

(where $(J^{-1})^i_j$ is the inverse Jacobian of the transformation from Cartesian to the curvilinear coordinates system of interest) are regular functions in $x = x_s$.

V. EXPLICIT EXAMPLES OF VACUUM COPIES IN NONTRIVIAL TOPOLOGIES

Here we will discuss many example of vacuum copies.

A. $S^1 \times D^2$ topology

Here we will consider the following ansatz corresponding to the metric in Eq. (6):

$$\alpha = \alpha(r),$$
$$\hat{n}^1 = \cos(\theta),$$
$$\hat{n}^2 = \sin(\theta),$$
$$\hat{n}^3 = 0.$$

It is easy to see that the above ansatz satisfies the conditions in Eqs. (28) and (29) with the following L:

[8] In the spherical case the factor $\sin\theta$, which appears in the denominator of many components of the Jacobian, is canceled out by a similar factor in the numerator of the spherically symmetric hedgehog ansatz in the normalized internal vector $\hat{n}^i$.

$$(\nabla^i \partial_i)\hat{n}^1 = -\frac{1}{r^2}\hat{n}^1,$$
$$(\nabla^i \partial_i)\hat{n}^2 = -\frac{1}{r^2}\hat{n}^2 \Rightarrow L = -\frac{1}{2r^2}.$$

Thus, in this case the Eq. (31) for the copy of the vacuum reduces to

$$r\partial_r(r\partial_r\alpha) - \gamma \sin 2\alpha = 0 \Leftrightarrow \quad (37)$$

$$\frac{\partial^2\alpha}{\partial\tau^2} = \gamma \sin 2\alpha, \quad \gamma = \frac{1}{2}, \quad \tau - \tau_0 = \log r, \quad (38)$$

$$E = \frac{1}{2}\left[\left(\frac{\partial\alpha}{\partial\tau}\right)^2 + \gamma\cos 2\alpha\right], \quad (39)$$

$$\tau - \tau_0 = \pm\int_{\alpha(\tau_0)}^{\alpha(\tau)} \frac{dy}{\sqrt{2E - \gamma\cos 2y}}, \quad (40)$$

where τ_0 and E are integration constants.

In terms of the radial coordinate r one has to require that the copy is regular at the origin and that it approaches an element of the center when $r = R$ which is the boundary of D^2. Without loss of generality one can require [see Eq. (26)]:

$$\alpha \underset{r\to 0}{\longrightarrow} 0, \quad \alpha(R) = n\pi, \quad n \in \mathbb{Z}. \quad (41)$$

In terms of the variable τ the above conditions read

$$\alpha \underset{(\tau-\tau_0)\to-\infty}{\longrightarrow} 0, \quad \alpha \underset{(\tau-\tau_0)\to\tau^*}{\longrightarrow} n\pi, \quad \tau^* = \log R. \quad (42)$$

Hence, we have to analyze under which conditions on the parameter γ and on the integration constants E and τ_0 Eq. (39) can have solutions satisfying the conditions in Eq. (42). The simplest way to answer this question is to interpret Eq. (39) as the energy conservation of the following one-dimensional problem in which τ plays the role of the effective time:

$$\bar{E} = \frac{1}{2}\left(\frac{\partial A}{\partial\tau}\right)^2 + V(A), \quad (43)$$

$$V(A) = 2\gamma\cos A, \quad A = 2\alpha, \quad \bar{E} = 4E. \quad (44)$$

The boundary conditions in Eq. (42) in terms of A read (we will consider the case $n = 1$)

$$A \underset{(\tau-\tau_0)\to-\infty}{\longrightarrow} 0, \quad A \underset{(\tau-\tau_0)\to\tau^*}{\longrightarrow} 2\pi. \quad (45)$$

The above conditions mean that 0 and 2π have to be two turning points and that the corresponding period has to diverge:

$$\frac{1}{\sqrt{2}}\int_0^{2\pi} \frac{dy}{\sqrt{\bar{E} - 2\gamma\cos y}} \to \infty. \quad (46)$$

In order to satisfy this condition, it is enough to choose

$$\bar{E} = 2\gamma = 1, \quad (47)$$

the explicit form for the copy being

$$\alpha = 2\arctan\left[\frac{r}{\bar{r}_0}\right] \quad (48)$$

with $\bar{r}_0$ an integration constant (whose role will be described in a moment).

In terms of the original radial variable r, the condition in Eq. (41) can only be satisfied[9] if $R \to \infty$. In physical terms, this means that the condition in Eq. (41) can be fulfilled when R is very large compared to $\bar{r}_0$:

$$R \gg \bar{r}_0.$$

Obviously, in the present case the only two natural lengths are the radius of the disk R and the perimeter of the S^1 factor (which has been set to 2π) of the donut $S^1 \times D^2$ and consequently $\bar{r}_0$ represents the size of S^1. Thus, the previous analysis tells us that when the radius of the disk is much larger than the perimeter of S^1 the donut is on the verge of supporting smooth copies of the vacuum satisfying the strong boundary conditions. Hence, in order to avoid this pathology the donut cannot be too "fat".

It is easy to check that such copy is regular at the origin. The components of $U^{-1}dU$ in Cartesian coordinates for this case are given by

$$R_i^c(J^{-1})_j^i, \quad i = r, \theta, \quad j = 1, 2$$

with

$$J^{-1} = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\frac{\sin(\theta)}{r} & \frac{\cos(\theta)}{r} \end{pmatrix}.$$

For $r \to 0$ copies must decay as

$$\alpha \underset{r\to 0}{\longrightarrow} \beta r, \quad (49)$$

for some $\beta \in \mathbb{R}$ and the profile in Eq. (48) satisfies this condition.

B. T^3 topology

As it has been already emphasized, the T^3 topology is relevant, for instance, in the analysis of the thermodynamical limit. Here we will consider the following ansatz corresponding to the metric in Eq. (9):

$$\alpha = \alpha(\phi_1), \quad \hat{n}^1 = \cos(p\phi_2 + q\phi_3),$$
$$\hat{n}^2 = \sin(p\phi_2 + q\phi_3), \quad \hat{n}^3 = 0, \quad p, q \in \mathbb{Z}. \quad (50)$$

It is easy to see that the above ansatz satisfies the conditions in Eqs. (28) and (29) with the following L:

$$L = -\frac{1}{2}\left(\left(\frac{p}{\lambda_2}\right)^2 + \left(\frac{q}{\lambda_3}\right)^2\right).$$

[9]The reason is that the copy profile in Eq. (48) is an increasing function from 0 to ∞ and its maximum is π. Note that, however, the copy profile approaches the asymptotic value very rapidly.

Thus, in this case the Eq. (31) for the copy of the vacuum reduces to a flat elliptic sine-Gordon equation:

$$\left(\frac{\partial^2}{\partial\phi_1^2}\right)\alpha = \gamma\sin 2\alpha, \tag{51}$$

$$\gamma = \frac{\lambda_1^2}{2}\left(\left(\frac{p}{\lambda_2}\right)^2 + \left(\frac{q}{\lambda_3}\right)^2\right). \tag{52}$$

In order for a solution of Eq. (51) to define a copy of the vacuum satisfying the strong boundary condition [see Eqs. (26) and (10)] it is necessary to require

$$\alpha(\phi_1 + 2m\pi) = \alpha(\phi_1), \qquad m \in \mathbb{Z}, \tag{53}$$

where we will consider $m = 1$ in the following. As in the previous subsection, Eq. (51) can be reduced to a first order conservation law:

$$E = \frac{1}{2}\left[\left(\frac{\partial\alpha}{\partial\phi_1}\right)^2 + \gamma\cos 2\alpha\right],$$
$$\phi_1 - \phi_0 = \pm\int_{\alpha(\phi_0)}^{\alpha(\phi_1)}\frac{dy}{\sqrt{2E - \gamma\cos 2y}}, \tag{54}$$

where ϕ_0 and E are integration constants. Hence, we have to analyze under which conditions on the parameter γ and on the integration constants E and ϕ_0 Eq. (54) can have solutions satisfying the conditions in Eq. (53). As in the previous subsection, it is possible to interpret Eq. (54) as the energy conservation of a one-dimensional problem in Eqs. (43) and (44) in which ϕ_1 plays the role of the effective time $\tau = \phi_1$. However, in this case the boundary conditions in Eq. (53) become:

$$A(\tau + 2\pi) = A(\tau). \tag{55}$$

Thus, given two consecutive turning points A_0 and A_1:

$$A_0 = \arccos\frac{\bar{E}}{2\gamma}, \qquad A_1 = 2\pi - A_0,$$

one has to require that the the time needed to go from A_0 to A_1 is half of the period in Eq. (55), namely:

$$\tau - \tau_0 = \int_{A_0}^{A_1}\frac{dy}{\sqrt{2(\bar{E} - 2\gamma\cos y)}} = \pi.$$

The time for a particle to go from A_0 to A_1 runs from 0 to infinity as $\bar{E}$ runs from -2γ to 2γ. This means that, given a γ, there is always an $\bar{E}$ such that $\tau - \tau_0 = \pi$. Then, for $-2\gamma < \bar{E} < 2\gamma$, it is always possible to construct copies of the vacuum satisfying the strong boundary conditions.

Furthermore, it is easy to see that the norm of the copy is finite:

$$\begin{aligned}\|U^{-1}dU\| &= \int_{T^3} d^3x\sqrt{g}tr(U^{-1}dU)^2 \\ &= (2\pi)^2\lambda_1\lambda_2\lambda_3\int_{T^3} d\phi_1\left(\left(\frac{\partial_{\phi_1}\alpha}{\lambda_1}\right)^2 \right.\\ &\quad \left. + \sin^2\alpha\left(\left(\frac{p}{\lambda_2}\right)^2 + \left(\frac{q}{\lambda_3}\right)^2\right)\right) < \infty.\end{aligned}$$

On the other hand, it worth emphasizing that while there is a common agreement on the importance of the strong boundary conditions, it is not clear yet whether or not it is mandatory to require the finite norm condition. For instance, it is possible to construct configurations of the Yang-Mills gauge potential which have infinite norm but finite energy and/or action (see, for a recent discussion, Ref. [34]).

It is worth emphasizing that, at a first glance, one could think that the presence of Gribov copies of the vacuum prevents one from using the Gribov semiclassical approach whose aim is, of course, to eliminate Gribov copies from a suitable neighborhood of the vacuum itself. However, the copies which can be eliminated using the Gribov semiclassical approach are "small" copies, namely zero-modes of the Faddeev-Popov operator (which is a linear condition). On the other hand, the vacuum copies which have been constructed here are solutions of the full nonlinear equation for the copies and they would disappear if one would only consider the linearized equation (which reduces just to the Laplace equation in all the cases discussed in the present paper). Therefore, the Gribov semiclassical approach can be applied, for instance, to the $R \times T^3$ or T^4 cases. For instance, in the T^4 case in the Landau gauge, the inverse Gribov propagator $(\Delta_G)^{-1}$ would have, schematically, the usual form

$$(\Delta_G)^{-1} = \Box + \gamma\Box^{-1},$$

where γ is the nonperturbative Gribov mass parameter determined by the usual gap equation [4] and $\Box$ is the Laplacian on T^4. Hence, the only technical difference with respect the usual case would be that, in order to invert the Laplacian one should use the Fourier series instead of the Fourier transform. Indeed, in this way one can eliminate, from a suitable neighborhood of the vacuum $A_\mu = 0$, zero modes of the Faddeev-Popov operator but the copies of the vacuum discussed above cannot be eliminated with this procedure. The physical consequences of this fact are very interesting also from the point of view of lattice QCD and are actually under investigation.

C. $\mathbb{R} \times T^2$ topology

As it has been already explained, this topology is very important, for instance, in relation with the infinite volume limit of Yang-Mills theory as well as chiral symmetry breaking.

Actually, as far as the infinite volume limit and chiral symmetry breaking are concerned (see, for instance, Refs. [26,27]), the setting corresponds to the Euclidean theory defined on $\mathbb{R}\times T^3$. Thus, in this case the Landau gauge is to be preferred. However, the present construction trivially extends to this situation as well. One has just to interpret the coordinate x in the metric in Eq. (11) as the Euclidean time in order to apply the present analysis to this case.

Here we will consider the following ansatz corresponding to the metric in Eq. (11):

$$\alpha=\alpha(x),\qquad \hat{n}^1=\cos(p\phi_1+q\phi_2),$$
$$\hat{n}^2=\sin(p\phi_1+q\phi_2),\qquad \hat{n}^3=0,\qquad p,q\in\mathbb{Z}. \tag{56}$$

It is easy to see that the above ansatz satisfies the conditions in Eqs. (28) and (29) with the following L:

$$L=-\frac{1}{2}\left(\left(\frac{p}{\lambda_2}\right)^2+\left(\frac{q}{\lambda_3}\right)^2\right).$$

Thus, in this case the Eq. (31) for the copy of the vacuum reduces to a flat elliptic sine-Gordon equation:

$$\left(\frac{\partial^2}{\partial x^2}\right)\alpha=\gamma\sin 2\alpha, \tag{57}$$

$$\gamma=\frac{\lambda_1^2}{2}\left(\left(\frac{p}{\lambda_2}\right)^2+\left(\frac{q}{\lambda_3}\right)^2\right). \tag{58}$$

In order for a solution of Eq. (57) to define a copy of the vacuum satisfying the strong boundary condition [see Eqs. (26), (12), and (13)] it is necessary to require

$$\lim_{x\to+\infty}\alpha(x)=m\pi,\qquad \lim_{x\to-\infty}\alpha(x)=n\pi,\qquad m,n\in\mathbb{Z}. \tag{59}$$

Similarly to the previous subsections, Eq. (57) can be reduced to:

$$E=\frac{1}{2}\left[\left(\frac{\partial\alpha}{\partial x}\right)^2+\gamma\cos 2\alpha\right],$$

$$x-x_0=\pm\int_{\alpha(x_0)}^{\alpha(x)}\frac{dy}{\sqrt{2E-\gamma\cos 2y}}, \tag{60}$$

where x_0 and E are integration constants. Hence, we have to analyze under which conditions on the parameter γ and on the integration constants E and x_0 Eq. (60) can have solutions satisfying the conditions in Eq. (59) for some m and n. As in the previous subsections, (60) has the form of energy conservation of a one-dimensional problem (43) with the identifications (44). In this case the strong boundary conditions in Eq. (59) become:

$$\lim_{x\to+\infty}A(x)=2m\pi,\qquad \lim_{x\to-\infty}A(x)=2n\pi,\qquad m,n\in\mathbb{Z}. \tag{61}$$

For $m=1$ and $n=0$ this means that 0 and 2π have to be two turning points with infinite period. As in (46), this condition is ensured for $\bar{E}=2\gamma$. Furthermore, in this case as well it is easy to see that the norm of the copy is finite:

$$\|U^{-1}dU\|=\int_{T^2\times\mathbb{R}}\sqrt{g}d^3xtr(U^{-1}dU)^2<\infty.$$

where

$$\alpha(x)=2\arctan[\exp(x-\bar{x}_0)],$$

($\bar{x}_0$ being an integration constant) is the solution of Eq. (60).

VI. ELLIPTIC GRIBOV PENDULUM

In this section, we will analyze the elliptic generalization of the Gribov pendulum equation. It is easy to see that if one would try a naive generalization of the spherical Gribov pendulum [4] to elliptic coordinates (which reduces to the usual case in the spherical limit) then the system of equations for the Gribov copies of the vacuum do not reduce to a single scalar equation as it happens in the spherical case and, consequently, it would be extremely difficult to analyze the corresponding system of equations and the corresponding boundary conditions. On the other hand, within the present framework, one is lead to the two ansatzs in Eqs. (63) and (76) which *do reduce* the equations for the Gribov copies of the vacuum to a *single scalar equation*: this allows us to study the strong boundary conditions in the usual way. To the best of the authors' knowledge, this is the first nontrivial elliptic generalization of the Gribov pendulum.

This analysis is particularly important since it sheds light on how sensible the Gribov phenomenon is with respect to deformation of the spherical symmetry. Even if the usual Gribov pendulum equation[10] is analyzed in the unbounded region $r\in[0,\infty[$ [4,12], one implicitly assumes that similar results hold in a spherical bounded region. The reason is that, of course, experimentally gluons can live only within baryons and glueballs which are bounded regions. Therefore, it makes sense to study the arising of gauge copies of the vacuum in bounded region as well. Indeed, the results that no copy of the vacuum appear in the case of the usual Gribov pendulum also holds in the case of a bounded spherical region: as it will be now discussed, the elliptic case is more complicated.

A. Prolate spheroid

The line element for of flat three-dimensional Euclidean space in prolate spheroidal coordinates is given by

$$ds^2=a^2(\sinh^2\mu+\sin^2\nu)(d\mu^2+d\nu^2)$$
$$+a^2\sinh^2\mu\sin^2\nu d\phi^2. \tag{62}$$

For a prolate spheroidal bounded region the coordinate ranges are given by $0\leq\mu<R$, $\nu\in[0,\pi]$, $\phi\in[0,2\pi)$. The coordinate μ represents the elliptic radius since

[10] Actually, the whole issue of gauge copies is usually analyzed only in unbounded region.

$\mu = \text{const}$ surfaces are ellipses. Such $\mu = \text{const}$ ellipses have different eccentricities: large eccentricities (namely, large deformations from spherical symmetries) correspond to small μ while small eccentricities (namely, small deviations from spherical symmetry) correspond to large μ. Thus, in the above coordinates system, if one wants to analyze the limit of "large deformations from spherical symmetry" then one has to consider the small μ region. On the other hand, if one wants to consider the almost spherical case, then the large μ limit has to be considered. As it will be shown in the following, the large deformation limit is the most interesting case.

The Laplacian in this coordinates is given by

$$\nabla^i \partial_i = \frac{1}{a^2(\sinh^2\mu + \sin^2\nu)}[\partial_\mu^2 + \partial_\nu^2 + \coth\mu\,\partial_\mu + \cot\nu\,\partial_\nu] + \frac{1}{a^2\sinh^2\mu\sin^2\nu}\partial_\phi^2.$$

Then, the following ansatz satisfies the conditions in Eqs. (28) and (29)

$$\alpha = \alpha(\mu, \nu), \qquad n^1 = \cos\phi, \qquad n^2 = \sin\phi, \qquad n^3 = 0, \tag{63}$$

with the following L:

$$L = -\frac{1}{2a^2\sinh^2\mu\sin^2\nu}.$$

1. The strong boundary conditions

Due to the fact that μ represents the elliptic radius, the strong boundary conditions in an unbounded region correspond to the following conditions on α:

$$\alpha(\mu, \nu) \underset{\mu\to 0}{\longrightarrow} n\pi + f(\mu, \nu), \tag{64}$$

$$\alpha(\mu, \nu) \underset{\mu\to \infty}{\longrightarrow} m\pi, \qquad n, m \in \mathbb{Z}, \tag{65}$$

where $f(\mu, \nu)$ ensures regularity of the copies. On the other hand, if one is analyzing the theory in a bounded region of elliptic radius R then the condition in Eq. (65) has to be replaced by

$$\alpha(\mu, \nu) \underset{\mu\to R}{\longrightarrow} m\pi, \qquad m \in \mathbb{Z}, \tag{66}$$

since the boundary of the region is the surface $\mu = R$ and one has to require that the gauge copy belongs to the center of the gauge group on the boundary of the region itself.

2. Regularity conditions

In the prolate spheroidal case, the inverse of the Jacobian in the transformation (33) reads

$$J^{-1} = \left(\frac{\partial(\mu, \nu, \phi)}{\partial(x, y, z)}\right) = \frac{1}{a}\begin{pmatrix} \frac{\cosh\mu\sin\nu\cos\phi}{\sinh^2\mu+\sinh^2\nu} & \frac{\cosh\mu\sin\nu\sin\phi}{\sinh^2\mu+\sinh^2\nu} & \frac{\sinh\mu\cos\nu}{\sinh^2\mu+\sinh^2\nu} \\ \frac{\sinh\mu\cos\nu\cos\phi}{\sinh^2\mu+\sinh^2\nu} & \frac{\sinh\mu\cos\nu\sin\phi}{\sinh^2\mu+\sinh^2\nu} & -\frac{\cosh\mu\sin\nu}{\sinh^2\mu+\sinh^2\nu} \\ -\frac{\sin\phi}{\sinh\mu\sin\nu} & \frac{\cos\phi}{\sinh\mu\sin\nu} & 0 \end{pmatrix}. \tag{67}$$

Thus, singularities could appear for $\mu = 0$ and $\nu = 0$. Following the prescription in Eq. (35), near points with $\mu = 0$ and/or $\nu = 0$ copies behave as

$$R_\mu^c \to \beta\hat{n}^c\partial_\mu f \qquad R_\nu^c \to \beta\hat{n}^c\partial_\nu f$$
$$R_\phi^c \to \beta^2\delta^{dc}f^2\varepsilon_{abd}\hat{n}^a\partial_\phi\hat{n}^b + \beta f\partial_\phi\hat{n}^c,$$

where f must be such that functions (36) are regular. A sufficient condition which ensures regularity is:

$$f(\mu, \nu) \underset{\mu,\nu\to 0}{\longrightarrow} \mu^3\nu^3.$$

Besides the singularity at $\mu = 0$, there are singularities for $\nu = 0$ and $\nu = \pi$ as well (which are similar to the $1/\sin\theta$ singularity in the spherical case). However, in the elliptic case, the $\sin\nu$ factor is not automatically canceled by the internal vector $\hat{n}^i$ of the hedgehog ansatz (which only depends on ϕ in the present case). Therefore the profile function $\alpha(\mu, \nu)$ has to take care of this divergence.

3. Prolate pendulum

Equation (31) in the prolate elliptic coordinates reduces to the following elliptic prolate Gribov pendulum:

$$\frac{1}{\sinh^2\mu + \sin^2\nu}[\partial_\mu^2 + \partial_\nu^2 + \coth\mu\,\partial_\mu + \cot\nu\,\partial_\nu]\alpha = \frac{1}{2\sinh^2\mu\sin^2\nu}\sin 2\alpha. \tag{68}$$

Now we consider the limits $\mu \to 0$ and $\mu \to \infty$ which correspond to the cases of large and small deformation from spherical symmetry respectively.

(i) For $\mu \to 0$, Eq. (68) takes the form:

$$[\partial_\mu^2 + \partial_\nu^2 + \coth\mu\,\partial_\mu + \cot\nu\,\partial_\nu]\alpha = \frac{1}{2\sinh^2\mu}\sin 2\alpha. \tag{69}$$

Interestingly enough, in this limit the ansatz $\alpha = \alpha(\mu)$ is consistent and the above equation reduces to

$$\frac{d^2\alpha}{d\mu^2} + \coth\mu\frac{d\alpha}{d\mu} = \frac{1}{2\sinh^2\mu}\sin 2\alpha. \tag{70}$$

With the following change of coordinate

$$\tau = \ln\left|\tanh\left(\frac{\mu}{2}\right)\right| \tag{71}$$

$$\mu = 2\,\mathrm{arcth}\, e^{\tau} \tag{72}$$

$$\sinh(\mathrm{arcth}\, x) = \frac{x}{\sqrt{1-x^2}} \tag{73}$$

the Eq. (70) can be written as

$$\frac{d^2\alpha(\tau)}{d\tau^2} = \frac{1}{2}\sin 2\alpha(\tau).$$

Of course, in this case, the natural boundary conditions correspond to the analysis within a bounded region in Eqs. (64) and (66) in which, in suitable units, the radius R is very small

$$R \ll 1.$$

In terms of the coordinates τ in Eq. (71), (64), and (66)

$$\alpha(\tau) \underset{\tau\to-\infty}{\longrightarrow} 2n\pi, \qquad \alpha(\tau) \underset{\tau\to\tau^*}{\longrightarrow} 2m\pi, \qquad n, m \in \mathbb{Z}.$$

In the limit of large prolate deformations, the equation can be integrated analytically:

$$E = \frac{1}{2}\left[\left(\frac{d\alpha}{d\tau}\right)^2 + \frac{1}{2}\cos 2\alpha\right], \tag{74}$$

$$\tau - \tau_0 = \pm\int_{\alpha(\tau_0)}^{\alpha(\tau)} \frac{dy}{\sqrt{2E - \frac{1}{2}\cos 2y}}. \tag{75}$$

This result discloses in a very clear way how sensible the Gribov phenomenon is not only to topology but also to the shapes of the region where it is analyzed. Indeed, even if the present analytic solution cannot satisfy the regularity conditions in $\nu = 0$ and $\nu = \pi$ (which are the north and south poles of the ellipse) since it is ν-independent, a small deformation of the ellipse at the poles could eliminate the necessity to require regularity of the solution at $\nu = 0$ and $\nu = \pi$ and consequently could give rise to the sudden appearance of a copy of the vacuum. Hence, the present analysis strongly suggests that large prolate deformations from spherical symmetry can be quite pathological. These results can be relevant in the cases in which one is analyzing Yang-Mills theory in bounded regions.

(ii) For $\mu \to \infty$, equation (68) reduces to:

$$[\partial_\mu^2 + \partial_\nu^2 + \partial_\mu + \cot\nu\,\partial_\nu]\alpha = \frac{1}{2}\csc^2\nu\,\sin 2\alpha.$$

Since the large μ limit corresponds to small deviations from spherical symmetry, in this case one should recover the standard results on the absence of vacuum copies.

B. Oblate spheroid

The line element for a flat three-dimensional Euclidean space in oblate spheroidal coordinates is given by

$$ds^2 = a^2(\sinh^2\mu + \sin^2\nu)(d\mu^2 + d\nu^2) + a^2\cosh^2\mu\cos^2\nu\, d\phi^2.$$

For a oblate spheroidal bounded region the coordinate ranges are given by $0 \le \mu < \mu^*$, $\nu \in [0, \pi]$, $\phi \in [0, 2\pi)$. Also in this case μ is the elliptic radius since $\mu =$ const surfaces are ellipses with eccentricities which decrease with μ. As in the prolate case, if one wants to analyze the limit of "large deformations" then one has to consider the small μ region. On the other hand, if one wants to consider the almost spherical case, then the large μ limit has to be considered.

The Laplacian in these coordinates reads

$$\nabla^i\partial_i = \frac{1}{a^2(\sinh^2\mu + \sin^2\nu)}[\partial_\mu^2 + \partial_\nu^2 + \tanh\mu\,\partial_\mu + \tan\nu\,\partial_\nu] + \frac{1}{a^2\cosh^2\mu\cos^2\nu}\partial_\phi^2.$$

Thus, the following ansatz satisfies the conditions in Eqs. (28) and (29)

$$\alpha = \alpha(\mu,\nu), \quad n^1 = \cos\phi, \quad n^2 = \sin\phi, \quad n^3 = 0 \tag{76}$$

with the following L:

$$L = -\frac{1}{2a^2\cosh^2\mu\cos\nu}.$$

1. Strong boundary conditions

As in the prolate case, the strong boundary conditions in an unbounded region correspond to the following conditions on α:

$$\alpha(\mu,\nu) \underset{\mu\to 0}{\longrightarrow} 2n\pi, \tag{77}$$

$$\alpha(\mu,\nu) \underset{\mu\to\infty}{\longrightarrow} 2m\pi, \qquad n, m \in \mathbb{Z}. \tag{78}$$

On the other hand, if one is analyzing the theory in a bounded region of elliptic radius R then the condition in Eq. (78) has to be replaced by

$$\alpha(\mu,\nu) \underset{\mu\to R}{\longrightarrow} 2m\pi, \qquad n, m \in \mathbb{Z}. \tag{79}$$

Regularity conditions As it has been explained in the previous subsection, the information about the singularities at the origin are encoded in the Jacobian. Also in this case, besides the singularity for $\mu = 0$, there are singularities for $\nu = 0$ and $\nu = \pi$ as well. Since the $\sin\nu$ factor is not automatically canceled by the internal vector $\hat{n}^i$ of the hedgehog ansatz, the profile function $\alpha(\mu,\nu)$ has to take care of this divergence. A sufficient condition to ensure regularity is

$$f(\mu,\nu) \underset{\mu,\nu\to 0}{\longrightarrow} \mu^3\nu^3.$$

2. Oblate pendulum

Hence, the equation for copies of the vacuum (31) in the oblate case now takes the form of the following oblate Gribov pendulum

$$\frac{1}{\sinh^2\mu+\sin^2\nu}[\partial_\mu^2+\partial_\nu^2+\tanh\mu\partial_\mu+\tan\nu\partial_\nu]\alpha = \frac{1}{2\cosh^2\mu\cos^2\nu}\sin(2\alpha). \tag{80}$$

Now we consider the limits $\mu\to 0$ and $\mu\to\infty$ corresponding to large and small deformation from spherical symmetry, respectively.

(i) For $\mu\to 0$ (80) reduces to

$$[\partial_\mu^2+\partial_\nu^2+\tan\nu\partial_\nu]\alpha=\frac{1}{2}\tan^2\nu\sin(2\alpha).$$

Unlike the prolate case, in the present case it is not possible to find analytic solutions in the limit of large deformations.

(ii) For $\mu\to\infty$ (80) reduces to

$$[\partial_\mu^2+\partial_\nu^2+\partial_\mu+\tan\nu\partial_\nu]\alpha=\frac{\tanh^2\mu}{2\cos^2\nu}\sin(2\alpha). \tag{81}$$

Also in this case, since the large μ limit corresponds to small deviations from spherical symmetry, one should recover the standard results on the absence of vacuum copies.

It is worth emphasizing that in the large deformation limit the prolate and oblate Gribov pendulum equations (68) and (80) differ significantly. In particular, unlike the spherical or oblate cases, in the prolate case the Gribov pendulum equation can be integrated exactly. This strongly suggests that prolate deformations from spherical symmetry are more pathological than oblate deformations. Indeed, the present results strongly suggest that the strong deformation limit of Eq. (68) may support copies of the vacuum. This analysis appears to be quite relevant as far as the issue of gauge copies in a bounded region is concerned.

VII. CONCLUSIONS AND FURTHER COMMENTS

In this paper the arising of Gribov copies in regions with nontrivial topologies (such as closed tubes $S^1\times D^2$, or $\mathbb{R}\times T^3$) but flat metric has been analyzed. The technical tool has been a generalization of the hedgehog ansatz beyond spherical symmetry. Such a generalization of the hedgehog ansatz is very interesting in itself since, in the case of the nonlinear sigma model, it provides one with a geometrical recipe to reduce the the field equations of the nonlinear sigma model (which are a *system of coupled nonlinear partial differential equations*) to a single scalar nonlinear partial differential equation even when there is no spherical symmetry. This ansatz allows us to construct many analytic examples of Gribov copies of the vacuum. Moreover, the elliptic Gribov pendulum has also been derived (to the best of the authors' knowledge, for the first time) both in the prolate and oblate cases. Our results suggest that large prolate deformations from spherical symmetry are likely to be more pathological than the oblate deformations. The requirement of absence of Gribov copies of the vacuum satisfying the strong boundary conditions implies geometrical constraints on the topology, on the shapes, and on the sizes of the regions with nontrivial topologies (such as upper bounds on the deviations from spherical symmetry or constraint on the shape of the donut $S^1\times D^2$). Moreover, we have shown that in the case of a flat metric *but with the topology of* T^3 it is possible to construct copies of the vacuum satisfying the strong boundary conditions and with finite norm. The present results are interesting in relation to the infinite volume limit of Yang-Mills theory (which is related to the $\mathbb{R}\times T^3$ topology). Indeed, one of the main points of the present analysis has been to show that when the T^3 (as well as T^4) topology *explodes* to R^3 (or to R^4) the vacuum copies disappear since neither R^3 nor R^4 support copies of the vacuum. Hence, as far as the Gribov copies are concerned, such limit is not smooth and should be studied more carefully. This analysis is also relevant in all the cases in which gluons are confined in regions of finite sizes with nontrivial topologies such as in the cases of knotted flux tubes, lattice QCD and so on. Due to the close relation between Gribov ambiguity and confinement the issue of the Gribov copies in bounded regions (both on flat and on curved space-times) is very important and worthwhile to be investigated.

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APPENDIX: LANDAU GAUGE

In this appendix we will present two applications of the present generalized hedgehog ansatz which lead to a novel realization of spherical and elliptical symmetries respectively. The new way to implement spherical and elliptical symmetry corresponds to a configuration in which the internal vectors n^i of the generalized hedgehog ansatz depend only on the Euclidean time. This realization of the spherical symmetry is only possible within the present "generalized hedgehog" framework and it could be useful in the context of calorons which are instantons which are periodic in Euclidean time. In both cases we will construct the Landau gauge pendulum.

1. A novel spherical case

Let us consider the following line element corresponding to a four-dimensional Euclidean space

$$ds^2 = d\tau^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$
$$0 \leq \tau \leq 2\pi,$$

where τ plays the role of Euclidean time (which is a periodic coordinate of period 2π) and the spatial section is written in spherical coordinates. This situation is relevant in the cases in which one wants to describe finite temperature effects. An ansatz satisfying conditions (28) and (29) suited to deal this situation is

$$\alpha = \alpha(r), \quad n^1 = \cos(\omega\tau), \quad n^2 = \sin(\omega\tau), \quad n^3 = 0$$

with

$$L = -\frac{\omega^2}{2}$$

Then, the equation for Gribov copies (31) takes the form

$$\partial_r^2\alpha + \frac{2}{r}\partial_r\alpha = \frac{\omega^2}{2}\sin(2\alpha), \tag{82}$$

which obviously *is not equivalent* to the usual spherically symmetric Gribov pendulum [4,12]. Defining $x = \ln r$ we can write the above equation as

$$\partial_x^2\alpha + \partial_x\alpha = \frac{\omega^2}{2}e^{2x}\sin(2\alpha),$$

another useful form is, defining $y = -1/r$, the following

$$\partial_y^2\alpha = \frac{\omega^2}{2y^4}\sin(2\alpha).$$

The regularity condition at the origin can be analyzed as in the previous sections.

2. Prolate spheroidal case

Let us consider the line element of four-dimensional Euclidean space-time (in which the Euclidean time has been compactified to describe finite-temperature effects) in prolate elliptic coordinates

$$ds^2 = d\tau^2 + a^2(\sinh^2\mu + \sin^2\nu)(d\mu^2 + d\nu^2) + a^2\sinh^2\mu\sin^2\nu d\phi^2.$$

An ansatz satisfying conditions (28) and (29) suited to deal this situation is

$$\alpha = \alpha(\mu,\nu), \quad n^1 = \cos(\omega\tau), \quad n^2 = \sin(\omega\tau), \quad n^3 = 0$$

with

$$L = -\frac{\omega^2}{2}.$$

Then, Eq. (31) reduces to

$$\frac{1}{a^2(\sinh^2\mu + \sin^2\nu)}[\partial_\mu^2 + \partial_\nu^2 + \tanh\mu\partial_\mu + \tan\nu\partial_\nu]\alpha = \frac{\omega^2}{2}\sin(2\alpha).$$

Also in this case, the regularity conditions can be studied as in the previous sections.

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