Finite temperature current densities and Bose-Einstein condensation in topologically nontrivial spaces

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We investigate the finite temperature expectation values of the charge and current densities for a complex scalar field with nonzero chemical potential in the background of a flat spacetime with spatial topology $R^p \times (S^1)^q$. Along compact dimensions quasiperiodicity conditions with general phases are imposed on the field. In addition, we assume the presence of a constant gauge field which, due to the nontrivial topology of background space, leads to Aharonov-Bohm-like effects on the expectation values. By using the Abel-Plana-type summation formula and zeta function techniques, two different representations are provided for both the current and charge densities. The current density has nonzero components along the compact dimensions only and, in the absence of a gauge field, it vanishes for special cases of twisted and untwisted scalar fields. In the high-temperature limit, the current density and the topological part in the charge density are linear functions of the temperature. The Bose-Einstein condensation for a fixed value of the charge is discussed. The expression for the chemical potential is given in terms of the lengths of compact dimensions, temperature, and gauge field. It is shown that the parameters of the phase transition can be controlled by tuning the gauge field. The separate contributions to the charge and current densities coming from the Bose-Einstein condensate and from excited states are also investigated.

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I. INTRODUCTION

In recent years, there has been much interest in the physical problems with compact spatial dimensions. Several models of this sort appear in high-energy physics, in cosmology, and in condensed matter physics. In particular, many high-energy theories of fundamental physics, including supergravity and superstring theories, are formulated in spacetimes having extra compact dimensions which are characterized by extremely small length scales. These theories provide an attractive framework for the unification of gravitational and gauge interactions. The models of a compact universe with nontrivial topology may also play an important role by providing proper initial conditions for inflation [1].

In the models with compact dimensions, the nontrivial topology of background space can have important physical implications in classical and quantum field theories, which include instabilities in interacting field theories [2], topological mass generation [3,4], and symmetry breaking [4,5]. The periodicity conditions imposed on fields along compact dimensions allow only the normal modes with suitable wavelengths. As a result of this, the expectation values of various physical observables are modified. In particular, many authors have investigated the effects of vacuum or Casimir energies and stresses associated with the presence of compact dimensions (for reviews see Refs. [6,7]). The topological Casimir effect is a physical example of the connection between quantum phenomena and global properties of spacetime. The Casimir energy of bulk fields induces a nontrivial potential for the compactification radius of higher-dimensional field theories providing a stabilization mechanism for the corresponding moduli fields and thereby fixing the effective gauge couplings. The Casimir effect has also been considered as a possible origin for the dark energy in both Kaluza-Klein-type models and in braneworld scenarios [8].

The main part of the previous papers, devoted to the influence of the nontrivial topology on the properties of the quantum vacuum, considers the vacuum energy and stresses. These quantities are chosen because of their close connection with the structure of spacetime through the theory of gravitation. For charged fields another important characteristic, which is bilinear in the field, is the expectation value of the current density in a given state. In Ref. [9], we have investigated the vacuum expectation value of the current density for a fermionic field in spaces with an arbitrary number of toroidally compactified dimensions. We apply the general results to the electrons of a graphene sheet rolled into cylindrical and toroidal shapes. For the description of the relevant low-energy degrees of freedom, we have used the effective field theory treatment of graphene in terms of a pair of Dirac fermions. For this model one has the topologies $R^1 \times S^1$ and $(S^1)^2$ for cylindrical and toroidal nanotubes, respectively. Combined effects of compact spatial dimensions and boundaries on the vacuum expectation values of the fermionic current have

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been discussed recently in Ref. [10]. In the latter, the geometry of boundaries is given by two parallel plates on which the fermion field obeys bag boundary conditions. The effects of nontrivial topology around a conical defect on the current induced by a magnetic flux were investigated in Ref. [11] for scalar and fermion fields.

In the present paper we consider the finite temperature charge and current densities for a scalar field in background spacetime with spatial topology $R^p \times (S^1)^q$. In both types of models with compact dimensions used in the cosmology of the early Universe and in condensed matter physics, the effects induced by the finite temperature play an important role. The thermal corrections arise from thermal excitations of the fluctuation spectrum, and they depend strongly on the geometry. As a consequence of this, thermal modifications of quantum topological effects can differ qualitatively for different geometries. The thermal Casimir effect in cosmological models with nontrivial topology has been considered in Ref. [12]. A general discussion of the finite temperature effects for a scalar field in higher-dimensional product manifolds with compact subspaces is given in Ref. [13]. Specific calculations are presented for the cases when the internal space is a torus or a sphere. In Ref. [14] the corresponding results are extended to the case in which a chemical potential is present. In the previous discussions of the effects from nontrivial topology and finite temperature, the authors mainly consider periodicity and antiperiodicity conditions imposed on the field along compact dimensions. The latter correspond to untwisted and twisted configurations of fields, respectively. In this case the current density corresponding to a conserved charge associated with an internal symmetry vanishes. As it will be seen below, the presence of a constant gauge field, interacting with a charged quantum field, will induce a nontrivial phase in the periodicity conditions along compact dimensions. As a consequence of this, nonzero components of the current density appear along compact dimensions. This is a sort of Aharonov-Bohm-like effect related to the nontrivial topology of the background space.

The organization of the paper is as follows. In the next section the geometry of the problem is described and the thermal Hadamard function is evaluated for a complex scalar field in thermal equilibrium. In Sec. III, by using the expression for the Hadamard function, the expectation values of the charge and current densities are investigated. Various limiting cases are discussed. Alternative expressions for the charge and current densities are provided in Sec. V by making use of the zeta function renormalization approach. Section VI is devoted to the investigation of the Bose-Einstein condensation in the background under consideration. The properties of the vacuum expectation value of the charge density are discussed in the Appendix. Throughout the paper we use the units $\hbar = c = k_B = 1$, with k_B being the Boltzmann constant.

II. GEOMETRY OF THE PROBLEM AND THE HADAMARD FUNCTION

We consider the quantum scalar field $\varphi(x)$ on the background of $(D+1)$-dimensional flat spacetime with spatial topology $R^p \times (S^1)^q$, $p + q = D$. For the Cartesian coordinates along uncompactified and compactified dimensions, we use the notations $\mathbf{x}_p = (x^1, \ldots, x^p)$ and $\mathbf{x}_q = (x^{p+1}, \ldots, x^D)$, respectively. The length of the lth compact dimension we denote as L_l. Hence, for coordinates one has $-\infty < x^l < \infty$ for $l = 1, \ldots, p$, and $0 \leq x^l \leq L_l$ for $l = p+1, \ldots, D$. In the presence of a gauge field A_μ the field equation has the form

$$(g^{\mu\nu} D_\mu D_\nu + m^2)\varphi = 0, \tag{2.1}$$

where $D_\mu = \partial_\mu + ieA_\mu$ and e is the charge associated with the field. One of the characteristic features of field theory on backgrounds with nontrivial topology is the appearance of topologically inequivalent field configurations [15]. The boundary conditions should be specified along the compact dimensions for the theory to be defined. We assume that the field obeys generic quasiperiodic boundary conditions,

$$\varphi(t, \mathbf{x}_p, \mathbf{x}_q + L_l \mathbf{e}_l) = e^{i\alpha_l}\varphi(t, \mathbf{x}_p, \mathbf{x}_q), \tag{2.2}$$

with constant phases $|\alpha_l| \leq \pi$ and with $\mathbf{e}_l$ being the unit vector along the direction of the coordinate x^l, $l = p+1, \ldots, D$. The condition (2.2) includes the periodicity conditions for both untwisted and twisted scalar fields as special cases with $\alpha_l = 0$ and $\alpha_l = \pi$, respectively.

The geometry under consideration can be used to describe two types of models. The first one, with $p = 3$, $q \geqslant 1$, corresponds to the universe with Kaluza-Klein-type extra dimensions. For the second model one has $D = 3$, and the results given below describe how the properties of the universe are changed by one-loop quantum effects induced by the compactness of spatial dimensions. Another possible range for the applications of the results presented in the present paper could be graphene-made structures like cylindrical and toroidal carbon nanotubes. The long-wavelength description of the graphene excitations can be formulated in terms of the effective field theory in $(2+1)$-dimensional spacetime. In addition to the Dirac spinor field, this theory also contains complex scalar and gauge fields (see, for instance, Ref. [16]). For cylindrical and toroidal nanotubes the background space for the corresponding effective field theory has topologies $R^1 \times S^1$ and $(S^1)^2$, respectively.

In the discussion below we will assume a constant gauge field A_μ. Though the corresponding field strength vanishes, the nontrivial topology of the background spacetime leads to the Aharonov-Bohm-like effects on physical observables. In the case of constant A_μ, by making use of the gauge transformation

$$\varphi(x) = e^{-ie\chi}\varphi'(x), \qquad A_\mu = A'_\mu + \partial_\mu \chi, \tag{2.3}$$

with $\chi = A_\mu x^\mu$, we see that in the new gauge one has $A'_\mu = 0$ and the vector potential disappears from the equation for $\varphi'(x)$. For the new field we have the periodicity condition

$$\varphi'(t, \mathbf{x}_p, \mathbf{x}_q + L_l \mathbf{e}_l) = e^{i\tilde{\alpha}_l}\varphi'(t, \mathbf{x}_p, \mathbf{x}_q), \tag{2.4}$$

where

$$\tilde{\alpha}_l = \alpha_l + eA_l L_l. \tag{2.5}$$

In what follows we will work with the field $\varphi'(x)$, omitting the prime. Note that for this field $D_\mu = \partial_\mu$. As it is seen from Eq. (2.5), the presence of a constant gauge field shifts the phases in the periodicity conditions along compact dimensions. In particular, a nontrivial phase is induced for special cases of twisted and untwisted fields. As it will be shown below, this is crucial for the appearance of the nonzero current density along compact dimensions. Another interesting physical effect related to the presence of a constant gauge field is the topological generation of gauge field mass by toroidal spacetime (see Ref. [17] and references therein). Note that the term in Eq. (2.5) due to the gauge field may be written as

$$eA_l L_l = 2\pi\Phi_l/\Phi_0, \tag{2.6}$$

where Φ_l is a formal flux enclosed by the circle corresponding to the lth compact dimension and $\Phi_0 = 2\pi/e$ is the flux quantum.

The complete set of positive- and negative-energy solutions for the problem under consideration can be written in the form of plane waves:

$$\varphi^{(\pm)}_{\mathbf{k}}(x) = C_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}\mp i\omega t}, \qquad \omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + m^2}, \tag{2.7}$$

where $\mathbf{k} = (\mathbf{k}_p, \mathbf{k}_q)$, $\mathbf{k}_p = (k_1, \ldots, k_p)$, $\mathbf{k}_q = (k_{p+1}, \ldots, k_D)$, with $-\infty < k_i < +\infty$ for $i = 1, \ldots, p$. For the momentum components along the compact dimensions the eigenvalues are determined from the conditions (2.4):

$$k_l = (2\pi n_l + \tilde{\alpha}_l)/L_l, \quad n_l = 0, \pm 1, \pm 2, \ldots, \tag{2.8}$$

with $l = p+1, \ldots, D$. From Eq. (2.8) it follows that the physical results will depend on the fractional part of $\tilde{\alpha}_l/(2\pi)$ only. The integer part can be absorbed by the redefinition of n_l. Hence, without loss of generality, we can assume that $|\tilde{\alpha}_l| \leq \pi$. The normalization coefficient in (2.7) is found from the orthonormalization condition

$$\int d^D x \varphi^{(\lambda)}_{\mathbf{k}}(x)\varphi^{(\lambda')*}_{\mathbf{k}'}(x) = \frac{1}{2\omega_{\mathbf{k}}}\delta_{\lambda\lambda'}\delta_{\mathbf{k}\mathbf{k}'}, \tag{2.9}$$

where $\delta_{\mathbf{k}\mathbf{k}'} = \delta(\mathbf{k}_p - \mathbf{k}'_p)\delta_{n_{p+1},n'_{p+1}}\ldots\delta_{n_D,n'_D}$. Substituting the functions (2.7), for the normalization coefficient we find

$$|C_{\mathbf{k}}|^2 = \frac{1}{2(2\pi)^p V_q \omega_{\mathbf{k}}}, \tag{2.10}$$

with $V_q = L_{p+1}\ldots L_D$ being the volume of the compact subspace and

$$\omega_{\mathbf{k}} = \sqrt{\mathbf{k}_p^2 + \mathbf{k}_q^2 + m^2}, \qquad \mathbf{k}_q^2 = \sum_{l=p+1}^{D}\left(\frac{2\pi n_l + \tilde{\alpha}_l}{L_l}\right)^2. \tag{2.11}$$

We will denote the smallest value for the energy by ω_0. Assuming that $|\tilde{\alpha}_l| \leq \pi$, we have

$$\omega_0 = \sqrt{\sum_{l=p+1}^{D}\tilde{\alpha}_l^2/L_l^2 + m^2}. \tag{2.12}$$

We are interested in the expectation values of the charge and current densities for the field $\varphi(x)$ in thermal equilibrium at finite temperature T. These quantities can be evaluated by using the thermal Hadamard function

$$\begin{aligned} G^{(1)}(x, x') &= \langle \varphi(x)\varphi^+(x') + \varphi^+(x')\varphi(x)\rangle \\ &= \mathrm{tr}[\hat{\rho}(\varphi(x)\varphi^+(x') + \varphi^+(x')\varphi(x))], \end{aligned} \tag{2.13}$$

where $\langle\cdots\rangle$ means the ensemble average and $\hat{\rho}$ is the density matrix. For the thermodynamical equilibrium distribution at temperature T, the latter is given by

$$\hat{\rho} = Z^{-1}e^{-\beta(\hat{H}-\mu'\hat{Q})}, \tag{2.14}$$

where $\beta = 1/T$. In Eq. (2.14), $\hat{Q}$ denotes a conserved charge, μ' is the related chemical potential, and Z is the grand-canonical partition function,

$$Z = \mathrm{tr}[e^{-\beta(\hat{H}-\mu'\hat{Q})}]. \tag{2.15}$$

In order to evaluate the expectation value in Eq. (2.13) we expand the field operator over a complete set of solutions:

$$\varphi(x) = \sum_{\mathbf{k}}[\hat{a}_{\mathbf{k}}\varphi^{(+)}_{\mathbf{k}}(x) + \hat{b}^+_{\mathbf{k}}\varphi^{(-)}_{\mathbf{k}}(x)], \tag{2.16}$$

with $\sum_{\mathbf{k}} = \int d\mathbf{k}_p \sum_{\mathbf{n}_q}$ and $\mathbf{n}_q = (n_{p+1}, \ldots, n_D)$. Here and in what follows we use the notation

$$\sum_{\mathbf{n}} = \sum_{n_1=-\infty}^{+\infty}\cdots\sum_{n_l=-\infty}^{+\infty}, \tag{2.17}$$

for $\mathbf{n} = (n_1, \ldots, n_l)$. Substituting the expansion (2.16) into Eq. (2.13), we use the relations

$$\begin{aligned} \mathrm{tr}[\hat{\rho}\hat{a}^+_{\mathbf{k}}\hat{a}_{\mathbf{k}'}] &= \frac{\delta_{\mathbf{k}\mathbf{k}'}}{e^{\beta(\omega_{\mathbf{k}}-\mu)} - 1}, \\ \mathrm{tr}[\hat{\rho}\hat{b}^+_{\mathbf{k}}\hat{b}_{\mathbf{k}'}] &= \frac{\delta_{\mathbf{k}\mathbf{k}'}}{e^{\beta(\omega_{\mathbf{k}}+\mu)} - 1}, \end{aligned} \tag{2.18}$$

where $\mu = e\mu'$. Note that the chemical potentials have opposite signs for particles (μ) and antiparticles ($-\mu$).

The expectation values for $\hat{a}_{\mathbf{k}}\hat{a}^{+}_{\mathbf{k}'}$ and $\hat{b}_{\mathbf{k}}\hat{b}^{+}_{\mathbf{k}'}$ are obtained from (2.18) by using the commutation relations, and the expectation values for the other products are zero. For the Hadamard function we get

$$G^{(1)}(x,x') = G^{(1)}_0(x,x') + 2\sum_{\mathbf{k}}\sum_{s=\pm}\frac{\varphi^{(s)}_{\mathbf{k}}(x)\varphi^{(s)*}_{\mathbf{k}}(x')}{e^{\beta(\omega_{\mathbf{k}}-s\mu)}-1}, \tag{2.19}$$

where the first term in the right-hand side corresponds to the zero temperature Hadamard function:

$$G^{(1)}_0(x,x') = \langle 0|\varphi(x)\varphi^{+}(x') + \varphi^{+}(x')\varphi(x)|0\rangle = \sum_{\mathbf{k}}\sum_{s=\pm}\varphi^{(s)}_{\mathbf{k}}(x)\varphi^{(s)*}_{\mathbf{k}}(x'), \tag{2.20}$$

with $|0\rangle$ being the vacuum state. In order to ensure a positive-definite value for the number of particles, we assume that $|\mu| \leq \omega_0$, where ω_0 is the smallest value of the energy [see Eq. (2.12)].

By using the expressions (2.7) for the mode functions and the expansion $(e^{y}-1)^{-1} = \sum_{n=1}^{\infty} e^{-ny}$, the mode sum for the Hadamard function is written in the form

$$G^{(1)}(x,x') = \frac{1}{V_q}\int\frac{d\mathbf{k}_p}{(2\pi)^p}e^{i\mathbf{k}_p\cdot\Delta\mathbf{x}_p}\sum_{\mathbf{n}_q}\frac{e^{i\mathbf{k}_q\cdot\Delta\mathbf{x}_q}}{\omega_{\mathbf{k}}} \times\left[\cos(\omega_{\mathbf{k}}\Delta t) + \sum_{n=1}^{\infty}\sum_{s=\pm}e^{\omega_{\mathbf{k}}(si\Delta t - n\beta) - sn\mu\beta}\right], \tag{2.21}$$

where $\Delta\mathbf{x}_p = \mathbf{x}_p - \mathbf{x}'_p$, $\Delta\mathbf{x}_q = \mathbf{x}_q - \mathbf{x}'_q$, $\Delta t = t - t'$. For the evaluation of the Hadamard function we apply to the series over n_r the Abel-Plana-type summation formula [18,19] (for applications of the Abel-Plana formula and its generalizations in quantum field theory, see Refs. [6,20,21]),

$$\frac{2\pi}{L_r}\sum_{n_r=-\infty}^{\infty}g(k_r)f(|k_r|) = \int_0^{\infty}dz[g(z)+g(-z)]f(z) + i\int_0^{\infty}dz[f(iz)-f(-iz)]\sum_{\lambda=\pm1}\frac{g(i\lambda z)}{e^{zL_r+i\lambda\tilde{\alpha}_r}-1}, \tag{2.22}$$

where k_r is given by Eq. (2.8). For the Hadamard function we find the expression

$$G^{(1)}(x,x') = G^{(1)}_{p+1,q-1}(x,x') + \frac{L_r}{\pi V_q}\int\frac{d\mathbf{k}_p}{(2\pi)^p}\sum_{\mathbf{n}^r_{q-1}}e^{i\mathbf{k}_p\cdot\Delta\mathbf{x}_p + i\mathbf{k}_{q-1}\cdot\Delta\mathbf{x}_{q-1}} \times\sum_{n=-\infty}^{\infty}e^{\mu n\beta}\int_{\omega_{p,q-1}}^{\infty}dz\frac{\cosh\left[(\Delta t - in\beta)\sqrt{z^2-\omega^2_{p,q-1}}\right]}{\sqrt{z^2-\omega^2_{p,q-1}}}\sum_{\lambda=\pm1}\frac{e^{-\lambda z\Delta x^r}}{e^{zL_r+\lambda i\tilde{\alpha}_r}-1}, \tag{2.23}$$

where $\mathbf{n}^r_{q-1} = (n_{p+1},\ldots,n_{r-1},n_{r+1},\ldots,n_D)$, $\mathbf{k}_{q-1} = (k_{p+1},\ldots,k_{r-1},k_{r+1},\ldots,k_D)$, and

$$\omega_{p,q-1} = \sqrt{\mathbf{k}^2_p + \mathbf{k}^2_{q-1} + m^2}. \tag{2.24}$$

The first term in the right-hand side of Eq. (2.23), $G^{(1)}_{p+1,q-1}(x,x')$, comes from the first term on the right of Eq. (2.22), and it is the Hadamard function for the topology $R^{p+1}\times(S^1)^{q-1}$ with the lengths of the compact dimensions $(L_{p+1},\ldots,L_{r-1},L_{r+1},\ldots,L_D)$.

For further transformation of the expression (2.23) we use the expansion

$$\frac{e^{-\lambda z\Delta x^r}}{e^{zL_r+\lambda i\tilde{\alpha}_r}-1} = \sum_{l=1}^{\infty}e^{-z(lL_r+\lambda\Delta x^r)-\lambda il\tilde{\alpha}_r}. \tag{2.25}$$

With this expansion the z integral is expressed in terms of the Macdonald function of the zeroth order. Then the integral over $\mathbf{k}_p$ is evaluated by using the formula from Ref. [22]. For the Hadamard function we arrive at the final expression

$$G^{(1)}(x,x') = \frac{2L_rV_q^{-1}}{(2\pi)^{p/2+1}}\sum_{n=-\infty}^{\infty}\sum_{\mathbf{n}_q}e^{in_r\tilde{\alpha}_r + n\mu\beta}e^{i\mathbf{k}_{q-1}\cdot\Delta\mathbf{x}_{q-1}}\omega^p_{\mathbf{n}^r_{q-1}}f_{p/2}\left(\omega_{\mathbf{n}^r_{q-1}}\sqrt{|\Delta\mathbf{x}_p|^2 + (\Delta x^r - n_rL_r)^2 - (\Delta t - in\beta)^2}\right), \tag{2.26}$$

where

$$f_{\nu}(x) = x^{-\nu}K_{\nu}(x), \qquad \omega_{\mathbf{n}^r_{q-1}} = \sqrt{\mathbf{k}^2_{q-1} + m^2}. \tag{2.27}$$

Note that the $n_r = 0$ term in Eq. (2.26) corresponds to the function $G^{(1)}_{p+1,q-1}(x, x')$. Hence, the part of the Hadamard function in Eq. (2.26) with $n_r \neq 0$ is induced by the compactification of the rth direction to a circle with the length L_r.

An alternative expression for the Hadamard function is obtained directly from Eq. (2.21). We first integrate over the angular part of $\mathbf{k}_p$, and then the integral over $|\mathbf{k}_p|$ is expressed in terms of the Macdonald function. The corresponding expression is written in terms of the function (2.27) as

$$G^{(1)}(x, x') = \frac{2V_q^{-1}}{(2\pi)^{\frac{p+1}{2}}} \sum_{\mathbf{n}_q} e^{i\mathbf{k}_q \cdot \Delta \mathbf{x}_q} \omega_{\mathbf{n}_q}^{p-1} \sum_{n=-\infty}^{+\infty} e^{n\mu\beta} \times f_{\frac{p-1}{2}}\left(\omega_{\mathbf{n}_q} \sqrt{|\Delta \mathbf{x}_p|^2 - (\Delta t - in\beta)^2}\right), \quad (2.28)$$

with the notation

$$\omega_{\mathbf{n}_q} = \sqrt{\mathbf{k}_q^2 + m^2}, \quad (2.29)$$

and $\mathbf{k}_q^2$ is given by Eq. (2.11). Note that the explicit information contained in Eq. (2.26) is more detailed. Both representations (2.26) and (2.28) present the thermal Hadamard function as an infinite imaginary-time image sum of the zero temperature Hadamard function. This is the well-known result in finite temperature field theory (see, for instance, Ref. [23]).

III. CHARGE DENSITY

Having the thermal Hadamard function we can evaluate the expectation value for the current density

$$j_l(x) = ie[\varphi^+(x)\partial_l \varphi(x) - (\partial_l \varphi^+(x))\varphi(x)], \quad (3.1)$$

$l = 0, 1, \ldots, D$, by using the formula

$$\langle j_l(x) \rangle = \frac{i}{2} e \lim_{x' \to x} (\partial_l - \partial'_l) G^{(1)}(x, x'). \quad (3.2)$$

By making use of the relation $\partial_z f_\nu(z) = -z f_{\nu+1}(z)$, from Eq. (2.26) for the charge density ($l = 0$) one finds

$$\langle j_0 \rangle = \frac{8e\beta L_r}{(2\pi)^{\frac{p}{2}+1} V_q} \sum_{n_r=0}^{\infty}{}' \cos(n_r \tilde{\alpha}_r) \sum_{n=1}^{\infty} n \sinh(n\mu\beta) \times \sum_{\mathbf{n}_{q-1}^r} \omega_{\mathbf{n}_{q-1}^r}^{p+2} f_{\frac{p}{2}+1}\left(\omega_{\mathbf{n}_{q-1}^r} \sqrt{n_r^2 L_r^2 + n^2\beta^2}\right), \quad (3.3)$$

where the prime on the sign of the sum means that the term $n_r = 0$ should be taken with the coefficient $1/2$.

As it is seen from Eq. (3.3), the charge density is an even function of the phases $\tilde{\alpha}_l$ and, for a fixed value of the chemical potential, it vanishes in the zero temperature limit. It is a periodic function of $\tilde{\alpha}_l$ with the period equal to 2π. In the case of zero chemical potential the charge density is zero. In Eq. (3.3), the term with $n_r = 0$ corresponds to the charge density for the topology $R^{p+1} \times (S^1)^{q-1}$ with the lengths of the compact dimensions $(L_{p+1}, \ldots, L_{r-1}, L_{r+1}, \ldots, L_D)$, and the contribution of the terms with $n_r \neq 0$ is the change in the charge density due to the compactification of the rth dimension to S^1 with the length L_r. By taking into account Eq. (2.6), we see that the charge density is a periodic function of fluxes Φ_l, with the period equal to the flux quantum. Note that the sign of the ratio $\langle j_0 \rangle / e$ coincides with the sign of the chemical potential.

An alternative expression for the charge density, more symmetric with respect to the compact dimensions, is obtained by applying the formula

$$\sum_{n=-\infty}^{+\infty} \cos(n\alpha) f_\nu\left(c\sqrt{b^2 + a^2 n^2}\right) = \frac{\sqrt{2\pi}}{ac^{2\nu}} \sum_{n=-\infty}^{+\infty} w_n^{2\nu-1} f_{\nu-1/2}(bw_n), \quad (3.4)$$

with $a, b, c > 0$, $w_n = \sqrt{(2\pi n + \alpha)^2/a^2 + c^2}$, to the series over n_r in Eq. (3.3). This leads to the expression

$$\langle j_0 \rangle = \frac{4e\beta V_q^{-1}}{(2\pi)^{\frac{p+1}{2}}} \sum_{n=1}^{\infty} n \sinh(n\mu\beta) \sum_{\mathbf{n}_q} \omega_{\mathbf{n}_q}^{p+1} f_{\frac{p+1}{2}}(n\beta\omega_{\mathbf{n}_q}), \quad (3.5)$$

with the notation (2.29). This formula could also be directly obtained from Eq. (3.2) using the expression (2.28) for the Hadamard function. The form (3.5) for the charge density in the case of topology $R^{p+1} \times (S^1)^{q-1}$ is also obtained from Eq. (3.3), taking the limit $L_r \to \infty$.

In the case of Minkowski spacetime one has $p = D$, $q = 0$, and from Eq. (3.5) we get

$$\langle j_0 \rangle_{(\mathrm{M})} = \frac{4e\beta m^{D+1}}{(2\pi)^{\frac{D+1}{2}}} \sum_{n=1}^{\infty} n \sinh(n\mu\beta) f_{\frac{D+1}{2}}(n\beta m), \quad (3.6)$$

with $|\mu| \leq m$. The thermodynamic properties of the relativistic Bose gas in this case have been considered in Refs. [24,25]. If all spatial dimensions are compactified, the corresponding formulas are obtained from Eqs. (3.3) and (3.5), taking $p = 0$. In particular, from Eq. (3.5) one has

$$\langle j_0 \rangle = \frac{2e}{V_q} \sum_{n=1}^{\infty} \sinh(n\mu\beta) \sum_{\mathbf{n}_q} e^{-n\beta\omega_{\mathbf{n}_q}}, \quad (3.7)$$

where we have used $f_{1/2}(x) = \sqrt{\pi/2} x^{-1} e^{-x}$.

Let us consider some limiting cases of Eq. (3.5). If the length of the lth compact dimensions is large compared to other length scales, in the sum over n_l in Eq. (3.5) the contribution from large values of n_l dominates and, to the leading order, we replace the summation by the integration. The corresponding integral is evaluated with the help of the formula

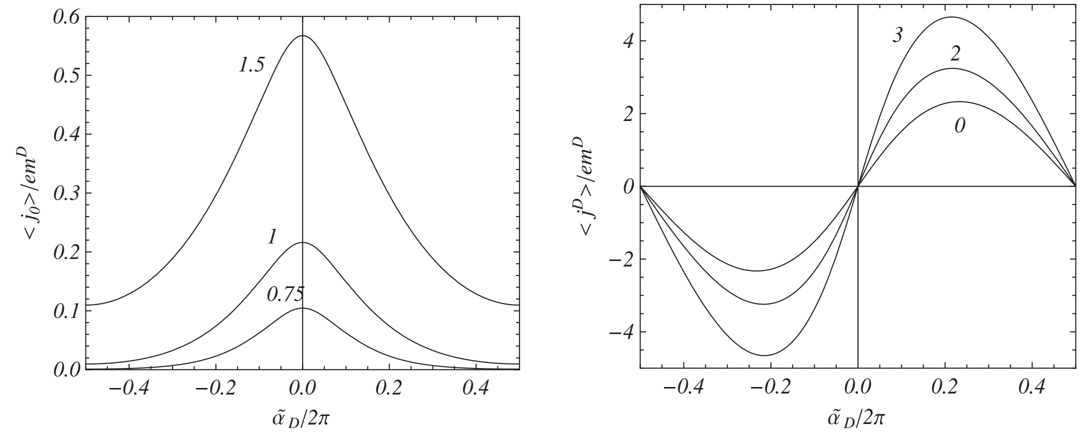

FIG. 1. The expectation values of the charge (left plot) and current (right plot) densities as functions of the parameter $\tilde{\alpha}_D/2\pi$ for the $D = 4$ model with a single compact dimension and for $\mu = 0.5m$, $mL_D = 0.5$. The numbers near the curves correspond to the values of T/m.

$$\int_0^\infty dy(y^2 + b^2)^{\frac{p+1}{2}} f_{\frac{p+1}{2}}\left(c\sqrt{y^2 + b^2}\right) = \sqrt{\frac{\pi}{2}} b^{p+2} f_{\frac{p}{2}+1}(cb), \tag{3.8}$$

and from Eq. (3.5) we obtain the expression of the charge density for the topology $R^{p+1} \times (S^1)^{q-1}$.

If the length of the lth compact dimension is small compared with the other length scales and $L_l \ll \beta$, under the assumption $|\tilde{\alpha}_l| < \pi$, the main contribution to the corresponding series in Eq. (3.5) comes from the term with $n_l = 0$. The behavior of the charge density is essentially different, depending on whether the phase $\tilde{\alpha}_l$ is zero or not. When $\tilde{\alpha}_l = 0$, we can see that, to the leading order, $L_l\langle j_0\rangle$ coincides with the charge density in $(D-1)$-dimensional space of topology $R^p \times (S^1)^{q-1}$ and with the lengths of the compact dimensions $L_{p+1}, \ldots, L_{l-1}, L_{l+1}, \ldots, L_D$. In particular, this is the case for an untwisted scalar field in the absence of a gauge field. For $\tilde{\alpha}_l \neq 0$ and for small values of L_l, the argument of the Macdonald function in Eq. (3.5) is large and the charge density is suppressed by the factor $e^{-|\tilde{\alpha}_l|\beta/L_l}$.

In the low-temperature limit the parameter β is large and the dominant contribution to the charge density comes from the term $n = 1$ in the series over n and from the term in the series over $\mathbf{n}_q$ with the smallest value of $\omega_{\mathbf{n}_q}$ which corresponds to $n_l = 0$, $l = p+1, \ldots, D$. To the leading order we find

$$\langle j_0\rangle \approx \frac{4eV_q^{-1}\mathrm{sgn}(\mu)}{(2\pi)^{p/2+1}\beta^{p/2}} \omega_0^{p/2} e^{-\beta\omega_0 + |\mu|\beta}, \tag{3.9}$$

with ω_0 given by Eq. (2.12).

From Eq. (3.5) it follows that the expectation value of the charge density is finite in the limit $|\mu| \to \omega_0$ for $p > 2$, and it diverges for $p \leq 2$. In order to find the asymptotic behavior near the point $|\mu| = \omega_0$, we note that for $p \leq 2$, under the condition $\beta(\omega_0 - |\mu|) \ll 1$, the main contribution to Eq. (3.5) comes from the term with $n_l = 0$ ($\omega_{\mathbf{n}_q} = \omega_0$), and in the corresponding series over n the contribution from large n dominates. In this case we can use the asymptotic expression for the Macdonald function for large values of the argument, and to the leading order this gives

$$\langle j_0\rangle \approx \mathrm{sgn}(\mu)\frac{e}{V_q}\left(\frac{\omega_0}{2\pi\beta}\right)^{p/2} \mathrm{Li}_{p/2}(e^{-\beta(\omega_0 - |\mu|)}), \tag{3.10}$$

where $\mathrm{Li}_s(x)$ is the polylogarithm function. For the latter one has $\mathrm{Li}_0(x) = x/(1-x)$, $\mathrm{Li}_1(x) = -\ln(1-x)$. By taking into account that $\mathrm{Li}_s(e^{-y}) \approx \Gamma(1-s)y^{s-1}$ for $|y| \ll 1$ and $s < 1$, one finds the following asymptotic expressions:

$$\langle j_0\rangle \approx eT\frac{\mathrm{sgn}(\mu)\Gamma(1-p/2)}{V_q(\omega_0 - |\mu|)^{1-p/2}}\left(\frac{\omega_0}{2\pi}\right)^{p/2}, \qquad p = 0, 1,$$
$$\langle j_0\rangle \approx -eT\frac{\omega_0\mathrm{sgn}(\mu)}{2\pi V_q}\ln[(\omega_0 - |\mu|)/T], \qquad p = 2. \tag{3.11}$$

In the left plot of Fig. 1 we present the charge density as a function of the parameter $\tilde{\alpha}_D/(2\pi)$ in the $D = 4$ model with a single compact dimension of length L_D. Note that for an untwisted scalar field this parameter is the flux measured in units of the flux quantum. For the chemical potential and for the length of the compact dimensions, we have taken the values corresponding to $\mu = 0.5m$ and $mL_D = 0.5$. The numbers near the curves correspond to the values of T/m.

IV. CURRENT DENSITY

Now we turn to the expectation value of the current density. As it can be easily seen, the components of the current density along the uncompactified dimensions vanish: $\langle j_r\rangle = 0$ for $r = 1, \ldots, p$. By making use of Eq. (3.2) and the expression (2.26) of the Hadamard function, for the current density along the rth compact dimension we get

$$\langle j^r \rangle = \frac{8eL_r^2 V_q^{-1}}{(2\pi)^{p/2+1}} \sum_{n=0}^{\infty}{}' \cosh(\mu n\beta) \sum_{n_r=1}^{\infty} n_r \sin(n_r \tilde{\alpha}_r) \times \sum_{\mathbf{n}_{q-1}^r} \omega_{\mathbf{n}_{q-1}^r}^{p+2} f_{\frac{p}{2}+1}\Big(\omega_{\mathbf{n}_{q-1}^r} \sqrt{n_r^2 L_r^2 + n^2\beta^2}\Big), \tag{4.1}$$

with $r = p+1, \ldots, D$ and, as before, the prime means that the term with $n = 0$ should be taken with the weight 1/2. Note that, unlike the case of the charge density, the current density does not vanish at zero temperature for a fixed value of the chemical potential. The zero temperature current density is given by the $n = 0$ term in Eq. (4.1):

$$\langle j^r \rangle_0 = \frac{4eL_r^2 V_q^{-1}}{(2\pi)^{p/2+1}} \sum_{n_r=1}^{\infty} n_r \sin(n_r \tilde{\alpha}_r) \times \sum_{\mathbf{n}_{q-1}^r} \omega_{\mathbf{n}_{q-1}^r}^{p+2} f_{p/2+1}\Big(n_r L_r \omega_{\mathbf{n}_{q-1}^r}\Big). \tag{4.2}$$

The features of this current are discussed in detail in the Appendix. For the model with a single compact dimension the general formula reduces to

$$\langle j^r \rangle = \frac{8eL_r m^{D+1}}{(2\pi)^{\frac{D+1}{2}}} \sum_{n=0}^{\infty}{}' \cosh(n\mu\beta) \sum_{n_r=1}^{\infty} n_r \sin(n_r \tilde{\alpha}_r) \times f_{\frac{D+1}{2}}\Big(m\sqrt{n_r^2 L_r^2 + n^2\beta^2}\Big). \tag{4.3}$$

An alternative expression of the current density is obtained by making use of the formula (2.28) for the Hadamard function in Eq. (3.2):

$$\langle j^r \rangle = \langle j^r \rangle_0 + \frac{4eV_q^{-1}}{(2\pi)^{\frac{p+1}{2}} L_r} \sum_{n=1}^{\infty} \cosh(n\mu\beta) \times \sum_{\mathbf{n}_q} (2\pi n_r + \tilde{\alpha}_r) \omega_{\mathbf{n}_q}^{p-1} f_{\frac{p-1}{2}}\big(n\beta\omega_{\mathbf{n}_q}\big). \tag{4.4}$$

From Eqs. (4.1) and (4.4) it follows that the current density along the rth compact dimension is an odd periodic function of $\tilde{\alpha}_r$ and an even periodic function of $\tilde{\alpha}_l$, $l \neq r$, with the period equal to 2π. The current density is an even function of the chemical potential, and it does not vanish in the limit of zero chemical potential. In the absence of uncompactified dimensions one has $p = 0$, and from Eq. (4.4) we get

$$\langle j^r \rangle = \langle j^r \rangle_0 + \frac{2e}{V_q L_r} \sum_{n=1}^{\infty} \cosh(n\mu\beta) \times \sum_{\mathbf{n}_q} (2\pi n_r + \tilde{\alpha}_r) \frac{e^{-n\beta\omega_{\mathbf{n}_q}}}{\omega_{\mathbf{n}_q}}, \tag{4.5}$$

where we have used $f_{-1/2}(x) = \sqrt{\pi/2}e^{-x}$. Here we assume that $\omega_0 > 0$. In the case $\omega_0 = 0$ there is a zero mode, and the contribution of this mode should be considered separately.

In a way similar to that for the case of the charge density, we can see that, in the limit when the length of the lth compact dimension is large ($l \neq r$), the leading term obtained from Eq. (4.4) coincides with the current density in the space with topology $R^{p+1} \times (S^1)^{q-1}$, with the lengths of the compact dimensions $L_{p+1}, \ldots, L_{l-1}, L_{l+1}, \ldots, L_D$. For small values of L_l, $l \neq r$, the behavior of the current density crucially depends on whether $\tilde{\alpha}_l$ is zero or not. For $\tilde{\alpha}_l = 0$ the dominant contribution comes from the term with $n_l = 0$, and from the expression given above we can see that, to the leading order, $L_l \langle j^r \rangle$ coincides with the corresponding quantity in $(D-1)$-dimensional space with topology $R^p \times (S^1)^{q-1}$ and with the lengths of the compact dimensions $L_{p+1}, \ldots, L_{l-1}, L_{l+1}, \ldots, L_D$. For $\tilde{\alpha}_l \neq 0$ and for small values of L_l, the current density $\langle j^r \rangle$ is exponentially suppressed.

If $L_r \gg \beta$, the dominant contribution to the series over n in Eq. (4.3) comes from large values of $n \sim L_r/\beta$. In this case we can replace the summation by the integration, and the corresponding integral is evaluated by using the formula from Ref. [22] (assuming that $|\mu| < \omega_{\mathbf{n}_{q-1}^r}$). To the leading order we get

$$\langle j^r \rangle \approx \frac{4eL_r^2 V_q^{-1} T}{(2\pi)^{(p+1)/2}} \sum_{l=1}^{\infty} n_r \sin(n_r \tilde{\alpha}_r) \sum_{\mathbf{n}_{q-1}^r} (\omega_{\mathbf{n}_{q-1}^r}^2 - \mu^2)^{\frac{p+1}{2}} \times f_{(p+1)/2}\Big(n_r L_r \sqrt{\omega_{\mathbf{n}_{q-1}^r}^2 - \mu^2}\Big). \tag{4.6}$$

For a fixed value of L_r this formula gives the leading term in the high-temperature asymptotic regime for the current density. If, in addition, $L_r \gg L_l$, $l \neq r$, the dominant contribution comes from the term with $n_r = 1$, $n_l = 0$, and the current density $\langle j^r \rangle$ is suppressed by a factor $e^{-L_r\sqrt{\omega_{0r}^2 - \mu^2}}$, where

$$\omega_{0r} = \sqrt{\sum_{l=p+1, \neq r}^{D} \tilde{\alpha}_l^2 / L_l^2 + m^2}. \tag{4.7}$$

In order to see the asymptotic behavior of the current density at low temperatures, it is more convenient to use Eq. (4.4). Assuming that $\beta(\omega_0 - |\mu|) \gg 1$, the dominant contribution to the temperature-dependent part comes from the mode with the smallest energy corresponding to $n_l = 0$, and one has

$$\langle j^r \rangle \approx \langle j^r \rangle_0 + e\tilde{\alpha}_r \frac{\omega_0^{p/2-1} e^{-\beta\omega_0 + |\mu|\beta}}{(2\pi)^{p/2} V_q L_r \beta^{p/2}}. \tag{4.8}$$

In this case the temperature corrections are exponentially small.

For $p \leq 2$ the current density, defined by Eq. (4.4), is divergent in the limit $|\mu| \to \omega_0$. The corresponding asymptotic regime is found in a way similar to that for the case of the charge density. To the leading order we have

$$\langle j^r \rangle \approx \frac{\tilde{\alpha}_r \mathrm{sgn}(\mu)}{L_r \omega_0} \langle j_0 \rangle, \quad (4.9)$$

where the asymptotic expressions for $\langle j_0 \rangle$ for separate values of p are given in Eq. (3.11).

In the right plot of Fig. 1 we display the current density along the compact dimension x^D as a function of $\tilde{\alpha}_D/(2\pi)$ for the $D = 4$ model, with a single compact dimension of the length corresponding to $mL_D = 0.5$. The numbers near the curves are the values of T/m, and for the chemical potential we have taken the value $\mu = 0.5m$.

V. ZETA FUNCTION APPROACH

The expectation values of the charge and current densities can be evaluated directly from Eq. (3.1) by using zeta function techniques (see, for instance, Ref. [26]). First we consider the current density.

A. Current density

Substituting the expansion (2.16) for the field operator and making use of the expression (2.7) for the mode functions, for the current density along compact dimensions one finds the following expression:

$$\langle j^r \rangle = \frac{e}{(2\pi)^p V_q} \sum_{\mathbf{k}} \frac{k_r}{\omega_{\mathbf{k}}} \left[1 + \sum_{s=\pm} \frac{1}{e^{\beta(\omega_{\mathbf{k}} - s\mu)} - 1} \right], \quad (5.1)$$

with $k_r = (2\pi n_r + \tilde{\alpha}_r)/L_r$ and $r = p+1, \ldots, D$. The first term in the square brackets corresponds to the current density at zero temperature. The $s = +/-$ terms are contributions coming from the particles and/or antiparticles. For further transformations it is convenient to write Eq. (5.1) in the form

$$\langle j^r \rangle = \frac{2e}{(2\pi)^p V_q} \sum_{\mathbf{k}} \frac{k_r}{\omega_{\mathbf{k}}} \sum_{n=0}^{\infty}{}' e^{-n\beta\omega_{\mathbf{k}}} \cosh(n\beta\mu). \quad (5.2)$$

In the special case $p = 0$ this formula is reduced to Eq. (4.5). In the representation (5.2), the zero temperature part corresponds to the $n = 0$ term. The divergences are contained in this part only. The components of the current density along uncompact dimensions vanish.

As the next step, in Eq. (5.2) we use the integral representation

$$\frac{e^{-n\beta\omega}}{\omega} = \frac{2}{\sqrt{\pi}} \int_0^\infty ds e^{-\omega^2 s^2 - n^2\beta^2/4s^2}. \quad (5.3)$$

This allows us to write the expectation value of the current density in the form

$$\langle j^r \rangle = \frac{2\pi^{-1/2} e}{(2\pi)^p V_q} \sum_{\mathbf{k}} k_r \int_0^\infty ds e^{-\omega_{\mathbf{k}}^2 s^2} \sum_{n=-\infty}^{\infty} e^{n\beta\mu - n^2\beta^2/4s^2}. \quad (5.4)$$

Now we apply to the sum of the series over n the Poison summation formula

$$\sum_{n=-\infty}^{+\infty} g(n\alpha) = \frac{1}{\alpha} \sum_{n=-\infty}^{+\infty} \tilde{g}(2\pi n/\alpha), \quad (5.5)$$

where $\tilde{g}(y) = \int_{-\infty}^{+\infty} dx e^{-iyx} g(x)$. For the function corresponding to the series in Eq. (5.4) one has $\tilde{g}(y) = \sqrt{\pi} e^{y^2/4 - iys\mu}$. After integration over s we get the expression

$$\langle j^r \rangle = \frac{2e\beta^{-1}}{(2\pi)^p V_q} \sum_{\mathbf{k}} \sum_{n=-\infty}^{\infty} \frac{k_r}{\omega_{\mathbf{k}}^2 + (2\pi n/\beta + i\mu)^2}. \quad (5.6)$$

The current density defined by Eq. (5.6) can be written as

$$\langle j^r \rangle = \frac{2e}{L_r^2} \sum_{n_r=-\infty}^{\infty} (2\pi n_r + \tilde{\alpha}_r) \zeta_r(s)|_{s=1}, \quad (5.7)$$

with the partial zeta function

$$\zeta_r(s) = \frac{L_r}{\beta V_q} \int \frac{d\mathbf{k}_p}{(2\pi)^p} \times \sum_{\mathbf{n}_q^r} \left[\mathbf{k}_p^2 + \mathbf{k}_q^2 + \left(\frac{2\pi n_{D+1}}{\beta} + i\mu \right)^2 + m^2 \right]^{-s}, \quad (5.8)$$

where $\mathbf{n}_q^r = (n_{p+1}, \ldots, n_{r-1}, n_{r+1}, \ldots, n_{D+1})$ and $\mathbf{k}_q^2$ is given by Eq. (2.11). Hence, in order to find the renormalized value for the current density, we need to have the analytic continuation of the zeta function (5.8) at the point $s = 1$.

The analytic continuation can be done in a way similar to what we have used in Ref. [9] for the zero temperature fermionic current. We first integrate over the momentum along the uncompactified dimensions:

$$\zeta_r(s) = \frac{\Gamma(s - p/2) L_r}{(4\pi)^{p/2} \Gamma(s) V_q \beta} \times \sum_{\mathbf{n}_q^r} \left[\mathbf{k}_q^2 + \left(\frac{2\pi n_{D+1}}{\beta} + i\mu \right)^2 + m^2 \right]^{\frac{p}{2} - s}. \quad (5.9)$$

Next, the direct application of the generalized Chowla-Selberg formula [27] to the series in Eq. (5.9) leads to the following expression:

$$\zeta_r(s) = \frac{m_r^{D-2s}}{(4\pi)^{D/2}} \frac{\Gamma(s - D/2)}{\Gamma(s)} + \frac{2^{1-s} m_r^{D-2s}}{(2\pi)^{D/2} \Gamma(s)} \times \sum_{\mathbf{n}_q^r}{}' \cos(\mathbf{n}_q^r \cdot \tilde{\boldsymbol{\alpha}}_q) f_{\frac{D}{2}-s}(m_r g_{\mathbf{n}_q^r}(\mathbf{L}_q^r)), \quad (5.10)$$

where $\mathbf{L}_q^r = (L_{p+1}, \ldots, L_{r-1}, L_{r+1}, \ldots L_{D+1})$, $\tilde{\boldsymbol{\alpha}}_q = (\tilde{\alpha}_{p+1}, \ldots, \tilde{\alpha}_{r-1}, \tilde{\alpha}_{r+1}, \ldots \tilde{\alpha}_{D+1})$, with

$$L_{D+1} = \beta, \qquad \tilde{\alpha}_{D+1} = i\mu\beta \quad (5.11)$$

and

$$m_r^2 = (2\pi n_r + \tilde{\alpha}_r)^2 / L_r^2 + m^2. \quad (5.12)$$

The prime on the summation sign in Eq. (5.10) means that the term $\mathbf{n}_q^r = 0$ should be excluded from the sum, and we use the notation

$$g_{\mathbf{c}}(\mathbf{b}) = \left(\sum_{i=1}^{l} c_i^2 b_i^2\right)^{1/2}, \tag{5.13}$$

for the vectors $\mathbf{c} = (c_1, \ldots, c_l)$ and $\mathbf{b} = (b_1, \ldots, b_l)$. Note that in Eq. (5.10), $\cos(\mathbf{n}_q^r \cdot \boldsymbol{\alpha})q$ can also be written as $\cosh(n_{D+1}\mu\beta)\prod_{l=p+1,\neq r}^{D}\cos(n_l\tilde{\alpha}_l)$.

The contribution of the second term on the right-hand side of Eq. (5.10) to the current density is finite at the physical point. The analytic continuation is required for the part with the first term only. This is done by applying the summation formula (2.22) to the series over n_r. Further transformations are similar to what we have used in deriving Eq. (2.26), and we get

$$\frac{\Gamma(s-D/2)}{(4\pi)^{D/2}\Gamma(s)} \sum_{n_r=-\infty}^{+\infty} \frac{2\pi n_r + \tilde{\alpha}_r}{L_r m_r^{2s-D}} = \frac{2^{2-s} m^{D+3-2s} L_r^2}{(2\pi)^{(D+1)/2}\Gamma(s)} \sum_{n=1}^{\infty} n \sin(n\tilde{\alpha}_r) f_{\frac{D+3}{2}-s}(nL_r m). \tag{5.14}$$

The right-hand side of Eq. (5.14) is finite at the point $s = 1$. Now, substituting Eq. (5.10) into Eq. (5.7) and using Eq. (5.14), we find the following expression for the current density:

$$\langle j^r \rangle = \frac{4em^{D+1}L_r}{(2\pi)^{\frac{D+1}{2}}} \sum_{n=1}^{\infty} n \sin(n\tilde{\alpha}_r) f_{\frac{D+1}{2}}(nL_r m) + \frac{2m_r^{D-2}}{(2\pi)^{\frac{D}{2}} L_r^2} \sum_{n_r=-\infty}^{+\infty} (\tilde{\alpha}_r + 2\pi n_r) \times \sum_{\mathbf{n}_q^r}{}' \cos(\mathbf{n}_q^r \cdot \tilde{\boldsymbol{\alpha}}_q) f_{\frac{D}{2}-1}\Big(m_r g_{\mathbf{n}_q^r}(\mathbf{L}_q^r)\Big). \tag{5.15}$$

Note that in the limit $T \to 0$ and $L_l \to \infty$, $l \neq r$, the second term in the right-hand side of this formula vanishes. The first term presents the current density at zero temperature in the model with a single compact dimension [see Eq. (4.2) for a special case, $p = D - 1$].

An alternative representation for the expectation value of the current density is obtained if we apply the formula (3.4) to the series over n_r in Eq. (5.15). Under the condition $|\mu| \leq m$, this leads to the following expression:

$$\langle j^r \rangle = \frac{4eL_r m^{D+1}}{(2\pi)^{(D+1)/2}} \sum_{n_r=1}^{\infty} n_r \sin(n_r\tilde{\alpha}_r) \sum_{\mathbf{n}_q^r} \cosh(n_{D+1}\mu\beta) \times \cos(\mathbf{n}_{q-1}^r \cdot \tilde{\boldsymbol{\alpha}}_{q-1}^r) f_{\frac{D+1}{2}}\Big(m\sqrt{g_{\mathbf{n}_q^2}(\mathbf{L}_q) + n_{D+1}^2\beta^2}\Big), \tag{5.16}$$

where $\tilde{\boldsymbol{\alpha}}_{q-1}^r = (\tilde{\alpha}_{p+1}, \ldots, \tilde{\alpha}_{r-1}, \tilde{\alpha}_{r+1}, \ldots \tilde{\alpha}_D)$, $\mathbf{L}_q = (L_{p+1}, \ldots, L_D)$, and $g_{\mathbf{n}_q^2}(\mathbf{L}_q)$ is defined by Eq. (5.13). In particular, for a massless field and for zero chemical potential, $\mu = 0$, from (5.16) we get

$$\langle j^r \rangle = 2eL_r \frac{\Gamma((D+1)/2)}{\pi^{(D+1)/2}} \sum_{n_r=1}^{\infty} n_r \sin(n_r\tilde{\alpha}_r) \times \sum_{\mathbf{n}_q^r} \frac{\cos(\mathbf{n}_{q-1}^r \cdot \tilde{\boldsymbol{\alpha}}_{q-1}^r)}{\big[g_{\mathbf{n}_q^2}(\mathbf{L}_q) + n_{D+1}^2\beta^2\big]^{(D+1)/2}}. \tag{5.17}$$

The equivalence of the two representations for the current density, Eqs. (4.1) and (5.16), can be seen by using the relation

$$\sum_{\mathbf{n}} \cos(\mathbf{n} \cdot \boldsymbol{\alpha}) f_\nu\left(c\sqrt{b^2 + \sum_{i=1}^{l} a_i^2 n_i^2}\right) = \frac{(2\pi)^{l/2}}{a_1 \ldots a_l c^{2\nu}} \sum_{\mathbf{n}} w_{\mathbf{n}}^{2\nu-l} f_{\nu-l/2}(bw_{\mathbf{n}}), \tag{5.18}$$

where $\mathbf{n} = (n_1, \ldots, n_l)$, $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_l)$, and $w_{\mathbf{n}}^2 = \sum_{i=1}^{l}(2\pi n_i + \alpha_i)^2/a_i^2 + c^2$. This relation has been proved in Ref. [19] by using the Poisson resummation formula. Note that the formula (3.4) is a special case of Eq. (5.18).

An expression for the current density, convenient for the discussion of the high-temperature limit, is obtained from Eq. (5.16), by applying to the series over n_{D+1} the formula (3.4), under the assumption $|\mu| \leq m$. This leads to the following expression:

$$\langle j^r \rangle = \frac{4eL_r}{(2\pi)^{D/2}\beta} \sum_{n_r=1}^{\infty} n_r \sin(n_r\tilde{\alpha}_r) \sum_{\mathbf{n}_q^r} \cos(\mathbf{n}_{q-1}^r \cdot \tilde{\boldsymbol{\alpha}}_{q-1}) \times [(2\pi n_{D+1}/\beta + i\mu)^2 + m^2]^{D/2} \times f_{D/2}\Big(g_{\mathbf{n}_q}(\mathbf{L}_q)\sqrt{(2\pi n_{D+1}/\beta + i\mu)^2 + m^2}\Big). \tag{5.19}$$

At high temperatures the dominant contribution comes from the $n_{D+1} = 0$ term, and to the leading order we have

$$\langle j^r \rangle \approx \frac{4eL_r T}{(2\pi)^{D/2}} \sum_{n_r=1}^{\infty} n_r \sin(n_r\tilde{\alpha}_r) \sum_{\mathbf{n}_{q-1}^r} \cos(\mathbf{n}_{q-1}^r \cdot \tilde{\boldsymbol{\alpha}}_{q-1}) \times (m^2 - \mu^2)^{D/2} f_{D/2}\Big(g_{\mathbf{n}_q}(\mathbf{L}_q)\sqrt{m^2 - \mu^2}\Big). \tag{5.20}$$

The corrections to this leading term are exponentially small. The equivalence of two representations, Eqs. (4.6) and (5.20), for the leading order term can be seen by using the relation (5.18).

B. Charge density

Now we turn to the evaluation of the charge density by using the zeta function approach. Similar to the case of Eq. (5.1), we have the following mode sum:

$$\langle j_0 \rangle = \frac{e}{(2\pi)^p V_q} \sum_{\mathbf{k}} \sum_{s=\pm} \frac{s}{e^{\beta(\omega_{\mathbf{k}} - s\mu)} - 1}. \tag{5.21}$$

The zero temperature part in the charge density vanishes due to the cancellation between the contributions from the virtual particles and antiparticles. The corresponding contributions to the finite temperature part have opposite signs due to the opposite signs of the charge for particles and antiparticles. Introducing the expectation values for the numbers of the particles and antiparticles (per unit volume of the uncompactified subspace),

$$\langle N_\pm \rangle = \frac{1}{(2\pi)^p} \sum_{\mathbf{k}} \frac{1}{e^{\beta(\omega_{\mathbf{k}} \mp \mu)} - 1}, \tag{5.22}$$

the charge density is written as $\langle j_0 \rangle = e\langle N_+ - N_- \rangle / V_q$. In Eq. (5.22), the upper (lower) sign corresponds to particles (antiparticles). Note that in the current density the contributions from particles and antiparticles have the same sign [see Eq. (5.1)]. This is due to the fact that, though the charges have opposite signs, the opposite signs have the velocities as well, $v_r^{(+)} = k_r/\omega$ for particles and $v_r^{(-)} = -k_r/\omega$ for antiparticles [see the phases in the expression (2.7) for the mode functions]. The expression for $\langle N_\pm \rangle$ is obtained from Eq. (3.5) by the replacement $2e \sinh(n\mu\beta)/V_q \to e^{\pm n\mu\beta}$.

The expression (5.21) for the charge density may be written in the form

$$\langle j_0 \rangle = \frac{2e}{(2\pi)^p V_q} \sum_{\mathbf{k}} \sum_{n=1}^{\infty} e^{-n\beta\omega_{\mathbf{k}}} \sinh(n\beta\mu). \tag{5.23}$$

For further transformation of this expression we use the relation

$$\frac{\sin(n\beta\mu)}{e^{n\beta\omega}} = \left(\mu - \int_0^\mu d\mu \partial_\beta \beta\right) \frac{e^{-n\beta\omega}}{\omega} \cosh(n\beta\mu). \tag{5.24}$$

As a result, the expectation value of the charge density is presented in the form

$$\langle j_0 \rangle = \frac{2e}{(2\pi)^p V_q} \left(\mu - \int_0^\mu d\mu \partial_\beta \beta\right) \times \sum_{\mathbf{k}} \sum_{n=1}^{\infty} \frac{e^{-n\beta\omega_{\mathbf{k}}}}{\omega_{\mathbf{k}}} \cosh(n\beta\mu). \tag{5.25}$$

Substituting Eq. (5.3) by the transformations similar to what we have used in the case of the current density, one finds

$$\langle j_0 \rangle = 2e\left(\mu - \int_0^\mu d\mu \partial_\beta \beta\right) \zeta(s) \Big|_{s=1}, \tag{5.26}$$

where the corresponding zeta function is defined as

$$\zeta(s) = \frac{1}{V_q \beta} \int \frac{d\mathbf{k}_p}{(2\pi)^p} \sum_{\mathbf{n}_q} \sum_{n=-\infty}^{\infty} \left[\omega_{\mathbf{k}}^2 + \left(\frac{2\pi n}{\beta} + i\mu\right)^2\right]^{-s}, \tag{5.27}$$

with $\omega_{\mathbf{k}}$ defined by Eq. (2.11). After integration over the momentum along uncompact dimensions, the function (5.27) is written in the form

$$\zeta(s) = \frac{\Gamma(s - p/2)}{(4\pi)^{\frac{p}{2}} \Gamma(s) V_q \beta} \sum_{\mathbf{n}_{q+1}} \left[\sum_{l=p+1}^{D+1} \left(\frac{2\pi n_l + \tilde{\alpha}_l}{L_l}\right)^2 + m^2\right]^{\frac{p}{2} - s}, \tag{5.28}$$

where $\mathbf{n}_{q+1} = (n_{p+1}, \ldots, n_{D+1})$ and L_{D+1}, $\tilde{\alpha}_{D+1}$ are defined by Eq. (5.11). The application of the generalized Chowla-Selberg formula [27] to Eq. (5.28) gives

$$\zeta(s) = m^{D+1-2s} \frac{\Gamma(s - (D+1)/2)}{(4\pi)^{(D+1)/2} \Gamma(s)} + \frac{2^{1-s} m^{D+1-2s}}{(2\pi)^{(D+1)/2} \Gamma(s)} \times \sum_{\mathbf{n}_{q+1}}{}' \cos(\mathbf{n}_{q+1} \cdot \tilde{\boldsymbol{\alpha}}_{q+1}) \times f_{\frac{D+1}{2} - s}\left(m\sqrt{g_{\mathbf{n}_q}^2(\mathbf{L}_q) + n_{D+1}^2 \beta^2}\right), \tag{5.29}$$

with $\mathbf{L}_{q+1} = (L_{p+1}, \ldots, L_{D+1})$ and $\tilde{\boldsymbol{\alpha}}_{q+1} = (\tilde{\alpha}_{p+1}, \ldots, \tilde{\alpha}_{D+1})$. The prime on the summation sign in Eq. (5.29) means that the term with $n_l = 0$, $l = p+1, \ldots, D+1$, should be excluded from the sum.

Substituting Eq. (5.29) into Eq. (5.26), for the charge density one finds the expression

$$\langle j_0 \rangle = \frac{4em^{D+1}\beta}{(2\pi)^{(D+1)/2}} \sum_{n=1}^{\infty} n \sinh(\mu\beta n) \times \sum_{\mathbf{n}_q} \cos(\mathbf{n}_q \cdot \tilde{\boldsymbol{\alpha}}_q) f_{\frac{D+1}{2}}\left(m\sqrt{g_{\mathbf{n}_q}^2(\mathbf{L}_q) + n^2\beta^2}\right). \tag{5.30}$$

Note that the first term in the right-hand side of Eq. (5.29) does not depend on temperature, and the corresponding contribution in Eq. (5.25) vanishes. This expression for the charge density is valid for the region $|\mu| \leq m$. The equivalence of the representations (3.5) and (5.30) in this region is proved by using the formula (5.18). In Eq. (5.30) the term with $n_l = 0$, $l = p+1, \ldots, D$, coincides with the corresponding charge density in Minkowski spacetime ($p = D$, $q = 0$) given by Eq. (3.6). Note that, by the replacement $2e \sinh(n\mu\beta)/V_q \to e^{\pm n\mu\beta}$ in Eq. (5.30), we can obtain the corresponding formula for $\langle N_\pm \rangle$.

An alternative expression for the charge density, convenient for the investigation of the high-temperature limit, is obtained from Eq. (5.30) if we first separate the part corresponding to $\langle j_0\rangle_{(\mathrm{M})}$ and then apply formula (3.4) to the series over n in the remaining part. This leads to the following expression:

$$\langle j_0\rangle = \langle j_0\rangle_{(\mathrm{M})} - \frac{2ieT}{(2\pi)^{D/2}}\sum_{\mathbf{n}_q\neq 0}\cos(\mathbf{n}_q\cdot\tilde{\boldsymbol{\alpha}}_q) \times \sum_{n=-\infty}^{+\infty}(2\pi nT+i\mu)[(2\pi nT+i\mu)^2+m^2]^{\frac{D}{2}-1} \times f_{\frac{D}{2}-1}\Big(g_{\mathbf{n}_q}(\mathbf{L}_q)\sqrt{(2\pi nT+i\mu)^2+m^2}\Big). \quad (5.31)$$

As before, the prime means that the term with $n_l=0$, $l=p+1,\ldots,D$, should be excluded from the sum. At high temperatures the dominant contribution to the second term in the right-hand side comes from the term with $n=0$:

$$\langle j_0\rangle \approx \langle j_0\rangle_{\mathrm{M}} + \frac{2e\mu T}{(2\pi)^{D/2}}\sum_{\mathbf{n}_q}{}'\cos(\mathbf{n}_q\cdot\tilde{\boldsymbol{\alpha}}_q)(m^2-\mu^2)^{\frac{D}{2}-1} \times f_{\frac{D}{2}-1}\Big(g_{\mathbf{n}_q}(\mathbf{L}_q)\sqrt{m^2-\mu^2}\Big). \quad (5.32)$$

The higher order corrections to this asymptotic expression are exponentially small. Hence, similar to the case of the current density, the topological part of the charge density is a linear function of the temperature in the high-temperature limit.

In order to find the asymptotic expression for the part $\langle j_0\rangle_{(\mathrm{M})}$ at high temperatures, we use the integral representation

$$f_\nu(z) = \frac{2^{-\nu}\sqrt{\pi}}{\Gamma(\nu+1/2)}\int_1^\infty dt(t^2-1)^{\nu-1/2}e^{-zt}. \quad (5.33)$$

Substituting this into Eq. (3.6) and changing the order of integration and summation, the summation is done explicitly, and to the leading order we get

$$\langle j_0\rangle_{(\mathrm{M})} \approx 2e\mu T^{D-1}\frac{\Gamma((D+1)/2)}{\pi^{(D+1)/2}}\zeta_{\mathrm{R}}(D-1), \quad (5.34)$$

where $\zeta_{\mathrm{R}}(x)$ is the Riemann zeta function. This result has been obtained in Ref. [25]. As can be seen from Eq. (5.32), for $D>2$ the Minkowskian part dominates in the high-temperature limit, and one has $\langle j_0\rangle \approx \langle j_0\rangle_{(\mathrm{M})}$.

VI. BOSE-EINSTEIN CONDENSATION

In this section we consider the application of the formulas given before for the investigation of Bose-Einstein condensation (BEC). This phenomenon for a relativistic Bose gas of scalar particles in topologically trivial flat spacetime has been discussed in Refs. [24,25,28]. The investigation of the critical behavior of an ideal Bose gas confined to the background geometry of a static Einstein universe is given in Ref. [29] for scalar and vector fields. BEC in higher-dimensional spacetime with S^N as a compact subspace has been considered in Ref. [30]. The case of an ultrastatic (3 + 1)-dimensional manifold with a hyperbolic spatial part is analyzed in Ref. [31]. The background geometry of closed Robertson-Walker spacetime is discussed in Ref. [32]. Thermodynamics of ideal boson and fermion gases in anti-de Sitter spacetime and in the static Taub universe have been considered in Refs. [33,34]. In the high-temperature limit, BEC in a general background has been discussed in Refs. [35–37]. Recently, BEC on product manifolds, when the gas of bosons is confined by the anisotropic harmonic oscillator potential, has been investigated in Ref. [38].

Note that in the literature two criteria have been considered for BEC (see, for instance, the discussion in Ref. [39]). In the first one the existence of critical temperature, $T_c>0$, is assumed for which the chemical potential becomes equal to the single particle ground state energy. The derivative $\partial_T\mu$ for a fixed value of a conserved charge is discontinuous at the critical temperature, and the condensation corresponds to a phase transition. In the second criterion, one assumes the existence of a finite fraction of particle density in the ground state and in states in its neighborhood at $T>0$. In this case the presence of a phase transition is not required and thermodynamical functions can be continuous. In particular, in Ref. [25] it has been shown that for massive particles there is no BEC in dimensions $D\leq 2$ if one follows the first criterion.

In the discussion of the previous sections we have considered the charge and current densities as functions of the temperature, chemical potential, and the lengths of compact dimensions. From the physical point of view it is more important to consider the behavior of the system for a fixed value of the charge. We will denote by Q the charge per unit volume of the uncompactified subspace, $Q=V_q\langle j_0\rangle$. From Eq. (3.5) for this quantity one has

$$Q=\frac{4e\beta}{(2\pi)^{\frac{p+1}{2}}}\sum_{n=1}^{\infty}n\sinh(n\mu\beta)\sum_{\mathbf{n}_q}\omega_{\mathbf{n}_q}^{p+1}f_{\frac{p+1}{2}}(n\beta\omega_{\mathbf{n}_q}). \quad (6.1)$$

For a fixed value of the charge, this relation implicitly determines the chemical potential as a function of the temperature, lengths of the compact dimensions, and the charge.

For high temperatures the chemical potential determined from Eq. (6.1) tends to zero. Hence, at high temperatures we always have a solution with $|\mu|<\omega_0$. The further behavior of the function $\mu(T)$ with decreasing temperature is essentially different in the cases $p>2$ and $p\leq 2$. For $p>2$ the expression (6.1) is finite in the limit $|\mu|\to\omega_0$. We denote by T_c the temperature at which one has $|\mu(T_c)|=\omega_0$ for a fixed value of the charge. This is the critical temperature for BEC. The formula

$$|Q| = \frac{4|e|\beta_c}{(2\pi)^{\frac{p+1}{2}}}\sum_{n=1}^{\infty}n\sinh(n\omega_0\beta_c)\sum_{\mathbf{n}_q}\omega_{\mathbf{n}_q}^{p+1}f_{\frac{p+1}{2}}(n\beta_c\omega_{\mathbf{n}_q}), \quad (6.2)$$

with $\beta_c = 1/T_c$, determines the critical temperature as a function of the charge, of the lengths of the compact dimensions, and of the parameters $\tilde{\alpha}_l$.

Simple asymptotic formulas for the critical temperature are obtained for low and high temperatures. At low temperatures, $\omega_0\beta_c \gg 1$, the dominant contribution in Eq. (6.2) comes from the mode with the smallest energy corresponding to $n_l = 0$. By using the asymptotic expression for the Macdonald function for large values of the argument, from Eq. (6.2) to the leading order one finds

$$T_c \approx \frac{2\pi}{\omega_0}\left[\frac{|Q/e|}{\zeta_{\rm R}(p/2)}\right]^{2/p}. \tag{6.3}$$

This regime is realized for values of the charge corresponding to $|Q/e| \ll \omega_0^p$. For an untwisted scalar field and in the absence of a gauge field, the expression in the right-hand side of Eq. (6.3) coincides with the standard expression for the critical temperature of the nonrelativistic Bose gas in p-dimensional space. Note that for $\tilde{\alpha}_l \neq 0$, the presence of compact dimensions decreases the critical temperature. At high temperatures, by taking into account that to the leading order $\langle j_0\rangle \approx \langle j_0\rangle_{\rm (M)}$ and by using the asymptotic expression (5.34), one finds

$$T_c \approx \left[\frac{\pi^{(D+1)/2}|Q/e|V_q^{-1}}{2\omega_0\Gamma((D+1)/2)\zeta_R(D-1)}\right]^{1/(D-1)}. \tag{6.4}$$

This asymptotic expression corresponds to $|Q/e| \gg V_q\omega_0^D$. In Fig. 2 we display the critical temperature as a function of the charge density in the $D = 4$ model with a single compact dimension ($p = 3$, $q = 1$). The graphs are plotted for $mL_D = 0.5$ and for different values of $\tilde{\alpha}_D/(2\pi)$ (numbers near the curves). As we show, for fixed lengths of the compact dimensions, the critical temperature for the phase transition can be controlled by tuning the value for the gauge potential.

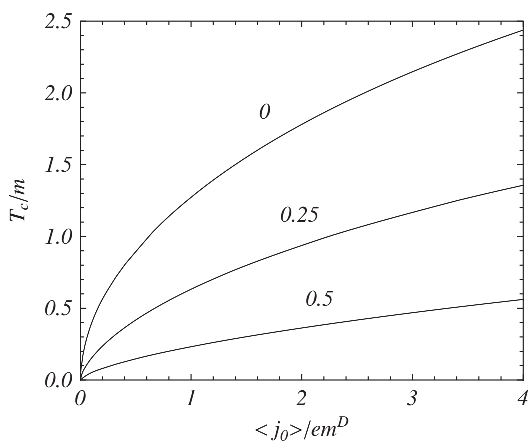

FIG. 2. The critical temperature as a function of the charge density in the $D = 4$ model with a single compact dimension. The graphs are plotted for $L_Dm = 0.5$ and for different values of $\tilde{\alpha}_D$, and the numbers near the curves correspond to the values of the parameter $\tilde{\alpha}_D/(2\pi)$.

At temperatures $T < T_c$, Eq. (6.1) has no solutions with $|\mu| < \omega_0$. The consideration in this region of temperature is similar to the standard one for the BEC in topologically trivial spaces. We note that the expression (6.1) does not include the charge corresponding to the states with $\mathbf{k}_p = 0$. At temperatures $T < T_c$ the expression (6.1) with $|\mu| = \omega_0$ determines the charge corresponding to the states with $\mathbf{k}_p \neq 0$. We denote this charge by Q_1:

$$Q_1 = \frac{4e\beta{\rm sgn}(\mu)}{(2\pi)^{\frac{p+1}{2}}}\sum_{n=1}^{\infty} n\sinh(n\omega_0\beta)\sum_{\mathbf{n}_q}\omega_{\mathbf{n}_q}^{p+1} f_{\frac{p+1}{2}}(n\beta\omega_{\mathbf{n}_q}). \tag{6.5}$$

For the charge corresponding to the Bose-Einstein condensate at $\mathbf{k}_p = 0$, one has $Q_c = Q - Q_1$. This charge vanishes at $T = T_c$. At low temperatures, by making use of Eq. (6.3), for the corresponding charges below the critical temperature one finds

$$Q_1 = Q(T/T_c)^{p/2}, \qquad Q_c = Q[1-(T/T_c)^{p/2}]. \tag{6.6}$$

For high temperatures we use the asymptotic formula (6.4), with the results

$$Q_1 = Q(T/T_c)^{D-1}, \qquad Q_c = Q[1-(T/T_c)^{D-1}]. \tag{6.7}$$

In particular, Eq. (6.7) coincides with the corresponding result in Minkowski spacetime [25]. In Fig. 3 we plot the chemical potential as a function of the temperature in the $D = 4$ model, with a single compact dimension ($p = 3$, $q = 1$). The left and right plots correspond to $\tilde{\alpha}_D = 0$ and $\tilde{\alpha}_D = \pi/2$, respectively. For the length of the compact dimension we have taken the value corresponding to $mL_D = 0.5$, and the numbers near the curves correspond to the values of the parameter $m^{1-D}Q/e$.

Note that for $T < T_c$ the scalar field acquires a nonzero ground-state expectation value φ_c. The latter can be found in a way similar to Ref. [35] (see also Refs. [24,25]):

$$|\varphi_c|^2 = \frac{|(Q-Q_1)/e|}{2\omega_0 V_q}, \tag{6.8}$$

with Q and Q_1 given by Eqs. (6.1) and (6.5).

Having the chemical potential from Eq. (6.1) for $T > T_c$ and taking $|\mu| = \omega_0$ for $T < T_c$, we can evaluate the current density for a fixed value of the charge by using Eq. (4.4). Note that at high temperatures the chemical potential tends to zero, and the leading term in the corresponding asymptotic expansion for the current density is obtained from Eq. (5.20), with $\mu = 0$. For $T < T_c$ the formula (4.4) gives a part of the current density due to the excited states only. In addition to this, there is a contribution due to the condensate, given by the expression

$$j_c^r = \frac{\tilde{\alpha}_r Q_c}{L_r\omega_0 V_q}, \tag{6.9}$$

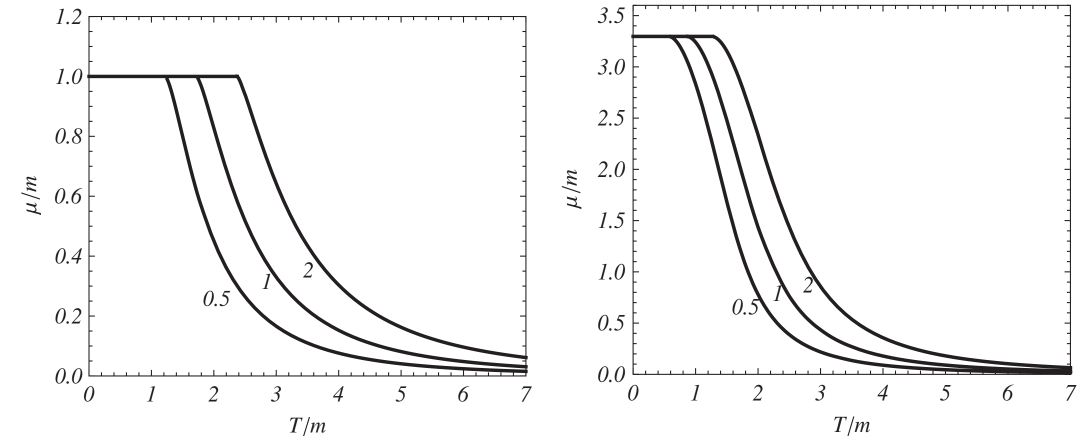

FIG. 3. The chemical potential as a function of the temperature in the $D = 4$ model, with a single compact dimension for $mL_D = 0.5$. The left and right plots correspond to $\tilde{\alpha}_D = 0$ and $\tilde{\alpha}_D = \pi/2$, respectively. The numbers near the curves correspond to the values of the parameter $m^{1-D}Q/e$.

with $r = p + 1, \ldots, D$ and $|\tilde{\alpha}_r| \leq \pi$. For the expectation value of the total current density one has $\langle j^r \rangle_t = j_c^r + \langle j^r \rangle$.

In Fig. 4 we plot the expectation values of the total current density (left plot, full curve) and of the particle and antiparticle numbers in the states with $\mathbf{k}_p \neq 0$ $\langle N_\pm \rangle$ (right plot) versus the temperature for $mL_D = 0.5$, $\tilde{\alpha}_D = \pi/2$ and $Q/e = 0.5m^{D-1}$ in the $D = 4$ model with a single compact dimension. For the corresponding critical temperature from Eq. (6.2) one finds $T_c \approx 0.63m$. In the left plot we have separately presented the contributions to the current density coming from particles (dot-dashed line) and antiparticles (large dashed line). The linear dependence at high temperatures is clearly seen. On the left panel we have also plotted the separate contributions to the current density from the excited states, $\langle j^D \rangle/(em^D)$ (dashed line), and from the condensate, $j_c^D/(em^D)$ (dotted line). Note that for the current density given by Eq. (4.2) we have $\langle j^D \rangle_0 = 2.32em^D$. The total current and its first derivative with respect to the temperature are continuous functions at the point of the phase transition. This is not the case for the separate parts coming from the condensate and from the excited states.

For $p \leq 2$ the charge defined by Eq. (6.1) diverges in the limit $|\mu| \to \omega_0$, and for a finite value of charge density the point $|\mu| = m$ cannot be reached. The corresponding asymptotic expressions for the charge and current densities are given by Eqs. (3.11) and (4.9). Thus, for $p \leq 2$ there is no BEC by the first criterion given above. In particular, this is the case in the model with compact space corresponding to $p = 0$. This result, for spaces of finite volume, in general, has been obtained in Ref. [37]. As before, for $p \leq 2$ and for a fixed value of the charge Q, the dependence of the chemical potential on the temperature is determined by Eq. (6.1). In the limit $\beta \to \infty$ (low temperatures) and for a fixed value of $|\mu| \neq \omega_0$, the expression on the right-hand side of Eq. (6.1) tends to zero. From this we conclude that for a fixed value of Q we should have $|\mu| \to \omega_0$ for $\beta \to \infty$. The corresponding asymptotic behavior is found in a way similar to Eq. (3.10) and is given by the same expression. Solving with respect to the

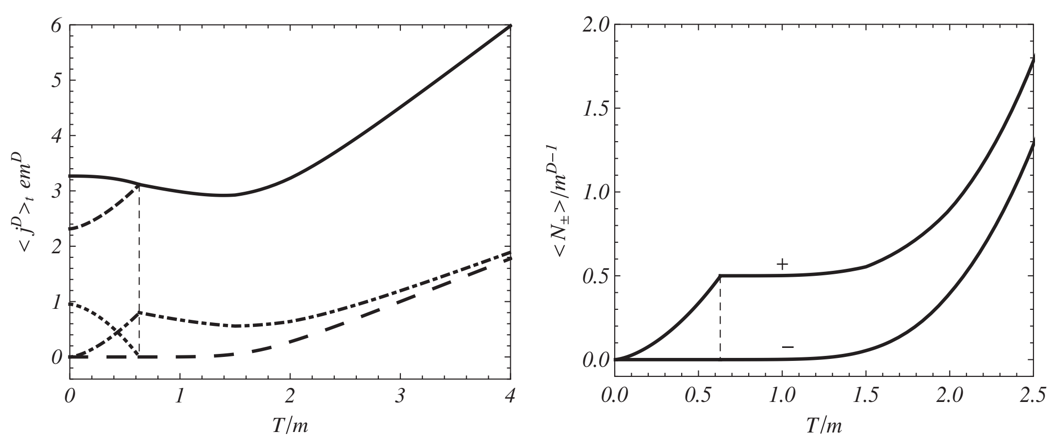

FIG. 4. The expectation values of the current density along the compact dimension (left plot) and of the particle or antiparticle numbers (right plot) as functions of the temperature in the $D = 4$ model for $mL_D = 0.5$, $\tilde{\alpha}_D = \pi/2$ and $Q/e = 0.5m^{D-1}$.

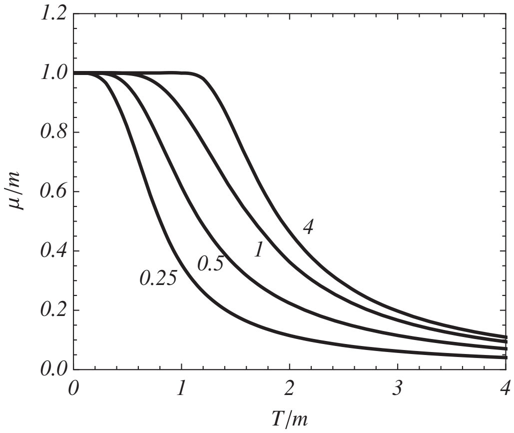

FIG. 5. The chemical potential as a function of the temperature in the $D = 3$ model with a single compact dimension for $\tilde{\alpha}_D = 0$ and $m^{1-D}\langle j_0\rangle/e = 0.5$. The numbers near the curves correspond to the values of the parameter $L_D m$.

chemical potential, we find the following asymptotic expressions:

$$|\mu| \approx \omega_0 - \left(\frac{\omega_0}{2\pi}\right)^{\frac{p}{2-p}}\left[\frac{|e|}{|Q|}\Gamma\left(1-\frac{p}{2}\right)T\right]^{\frac{2}{2-p}}, \qquad p = 0, 1,$$
$$|\mu| \approx \omega_0 - T\exp\left[-\frac{|Q|}{|e|}\left(\frac{2\pi}{\omega_0 T}\right)^{p/2}\right], \qquad p = 2, \tag{6.10}$$

in the limit $T \to 0$. In Fig. 5 we have plotted the chemical potential versus temperature in the $D = 3$ model with a single compact dimension ($p = 2$, $q = 1$) for $\tilde{\alpha}_D = 0$ and for a fixed value of the charge density corresponding to $m^{1-D}\langle j_0\rangle/e = 0.5$. The numbers near the curves correspond to the values of the parameter $L_D m$. As can be seen, although for a finite value of L_D the derivative $\partial_T\mu(T)$ is a continuous function, in the limit $L_D \to \infty$ it tends to the corresponding function in the $D = 3$ model with trivial topology ($p = 3$, $q = 0$) for which the function $\partial_T\mu(T)$ is discontinuous at the point $T = T_c$.

VII. CONCLUSION

In the present paper we have investigated the finite temperature expectation values of the charge and current densities for a complex scalar field, induced by nontrivial spatial topology. As an example, for the latter, we have considered a flat spacetime with an arbitrary number of toroidally compactified dimensions. This allowed us to escape the problems related to the curvature and to extract pure topological effects. The periodicity conditions along compact dimensions are taken in the form (2.2) with general constant phases. As special cases the latter includes the periodicity conditions for untwisted and twisted fields. In addition, we have assumed the presence of a constant gauge field. By performing a gauge transformation, the gauge field is excluded from the field equation. However, this leads to the shift in the phases appearing in the periodicity conditions given by Eq. (2.5).

In the evaluation of the expectation values for the charge and current densities, we have used two different approaches, which allowed us to obtain alternative representations for the corresponding expectation values. In the first approach we evaluated the thermal Hadamard function by using the Abel-Plana-type summation formula for the series over the momentum along a compact dimension. The corresponding expression is given by Eq. (2.26). The $n_r = 0$ term in that formula corresponds to the Hadamard function for the topology $R^{p+1} \times (S^1)^{q-1}$ and, hence, the $n_r \neq 0$ part is the change in the Hadamard function due to the compactification of the rth direction. An alternative representation for the Hadamard function is given by Eq. (2.28).

Given the Hadamard function, the expectation values of the charge and current densities are evaluated by making use of Eq. (3.2). The charge density is given by two equivalent representations, Eqs. (3.3) and (3.5). The explicit information contained in Eq. (3.3) is more detailed. The term with $n_r = 0$ in this representation corresponds to the charge density for the topology $R^{p+1} \times (S^1)^{q-1}$, with the lengths of the compact dimensions $(L_{p+1}, \ldots, L_{r-1}, L_{r+1}, \ldots, L_D)$, and the contribution of the terms with $n_r \neq 0$ is the change in the current density induced by the compactification of the rth dimension. The charge density is an even periodic function of the phases $\tilde{\alpha}_l$, with the period equal to 2π. The sign of the ratio $\langle j_0\rangle/e$ coincides with the sign of the chemical potential. If the length of the lth compact dimension is small compared with the other length scales and $L_l \ll \beta$, the behavior of the charge density is essentially different, depending on whether the parameter $\tilde{\alpha}_l$ is zero or not. For $\tilde{\alpha}_l = 0$, to the leading order, $L_l\langle j_0\rangle$ coincides with the charge density in $(D-1)$-dimensional space with topology $R^p \times (S^1)^{q-1}$ and with the lengths of the compact dimensions $L_{p+1}, \ldots, L_{l-1}, L_{l+1}, \ldots, L_D$. For $\tilde{\alpha}_l \neq 0$ the charge density is suppressed by the factor $e^{-|\tilde{\alpha}_l|\beta/L_l}$. At low temperatures and for a fixed value of $|\mu| < \omega_0$, the charge density is suppressed by the factor $e^{-(\omega_0-|\mu|)/T}$. For a fixed temperature and in the limit $|\mu| \to \omega_0$, the charge density is finite for $p > 2$ and it diverges for $p \leq 2$. The corresponding asymptotic behavior in the latter case is given by Eq. (3.11). In the high-temperature limit and for $D > 2$, the Minkowskian part dominates in the charge density with the leading term given by Eq. (5.34). In the same limit, the topological part of the charge density is a linear function of the temperature.

For the expectation value of the current density along the rth compact dimension, we have derived representations given by Eqs. (4.1) and (4.4). The components along uncompactified dimensions vanish. The current density along the rth compact dimension is an odd periodic function of

$\tilde{\alpha}_r$ and an even periodic function of $\tilde{\alpha}_l$, $l \neq r$, with the period equal to 2π. The current density is an even function of the chemical potential. Unlike the case of the charge density, the current density does not vanish at zero temperature for a fixed value of the chemical potential. The corresponding expression is given by Eq. (4.2), and the properties are discussed in the Appendix. For small values of L_l, $l \neq r$, and for $\tilde{\alpha}_l = 0$, to the leading order, the quantity $L_l\langle j^r\rangle$ coincides with the rth component of the current density in $(D-1)$-dimensional space, with topology $R^p \times (S^1)^{q-1}$ and with the lengths of the compact dimensions $L_{p+1}, \ldots, L_{l-1}, L_{l+1}, \ldots, L_D$. For $\tilde{\alpha}_l \neq 0$ and for small values of L_l, $l \neq r$, the current density $\langle j^r\rangle$ is exponentially suppressed. At a fixed temperature and for $p \leq 2$, the current density is divergent in the limit $|\mu| \to \omega_0$. The leading term in the corresponding asymptotic expansion is related to the charge density by Eq. (4.9). For a fixed value of the chemical potential $|\mu| < \omega_0$ and at low temperatures, the finite temperature corrections are given by Eq. (4.8), and they are exponentially small. In the limit of high temperatures, the current density is a linear function of the temperature.

In Sec. V we have derived alternative representations for the expectation values of the charge and current densities by using the zeta function approach. In both cases, by applying to the corresponding zeta functions the generalized Chowla-Selberg formula, Eqs. (5.16), (5.19), (5.30), and (5.31) are obtained for the current and charge densities, respectively. At high temperatures, the leading term in the asymptotic expansion of the current density is given by Eq. (5.20), with the linear dependence on the temperature, and the next corrections are exponentially small. For the charge density, for $D > 2$ the leading term in the high-temperature expansion coincides with the corresponding charge density in $(D+1)$-dimensional Minkowskian spacetime. The leading term in the correction induced by nontrivial topology linearly depends on the temperature, and the following corrections are exponentially suppressed.

The Bose-Einstein condensation is discussed in Sec. VI. For a fixed value of the charge, the relation (6.1) determines the chemical potential as a function of the temperature, of the lengths of compact directions, and of the phases in the periodicity conditions. For high temperatures the chemical potential tends to zero. With decreasing temperature the chemical potential increases, and for $p > 2$ one has $|\mu(T)| = \omega_0$ at some finite temperature $T = T_c$. The critical temperature for BEC, T_c, is determined by Eq. (6.2). Simple expressions are obtained for low and high temperatures, Eqs. (6.3) and (6.4), respectively. At temperatures $T < T_c$ one has $|\mu| = \omega_0$; Eq. (6.1) determines the charge corresponding to the states with $\mathbf{k}_p \neq 0$, and the remaining charge corresponds to the charge of the condensate. At low and high temperatures the charges are given by simple expressions (6.6) and (6.7). Similar to the charge density, for $T < T_c$ the current density is the sum of two parts. The first one is the contribution of excited states and is given by Eq. (4.4) with $|\mu| = \omega_0$. The second part is due to the condensate, and it is presented by Eq. (6.9). The total current and its first derivative with respect to the temperature are continuous functions at the critical temperature. For $p \leq 2$ the point $|\mu| = \omega_0$ cannot be reached for a finite value of charge density. For a fixed value of the charge, we have $|\mu| \to \omega_0$ in the limit $T \to 0$. The corresponding asymptotic behavior is given by Eq. (6.10). In this case the thermodynamical functions are continuous and there is no phase transition at finite temperature.

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APPENDIX: VACUUM EXPECTATION VALUE OF THE CURRENT DENSITY

In this appendix we give some properties of the zero temperature current density given by Eq. (4.2). An alternative expression is obtained from Eq. (5.16), taking the limit $\beta \to \infty$. In this limit only the term with $n = 0$ survives, and we get

$$\langle j^r\rangle_0 = \frac{4eL_r m^{D+1}}{(2\pi)^{(D+1)/2}} \sum_{n_r=1}^{\infty} n_r \sin(n_r\tilde{\alpha}_r) \times \sum_{\mathbf{n}_{q-1}^r} \cos(\mathbf{n}_{q-1}^r \cdot \boldsymbol{\alpha}_{q-1}) f_{\frac{D+1}{2}}(m g_{\mathbf{n}_q}(\mathbf{L}_q)). \tag{A1}$$

Let us consider the behavior of the zero temperature current density in some limiting cases. First we consider the limit where the length of the rth compact dimension, L_r, is much larger than the other length scales. The behavior of the current density in this limit crucially depends on whether ω_{0r}, defined by (4.7), is zero or not. In the first case, which is realized for $\tilde{\alpha}_l = 0$, $l \neq r$, and $m = 0$, the dominant contribution in Eq. (4.2) for large values of L_r comes from the modes with $n_l = 0$, $l \neq r$, for which $\omega_{\mathbf{n}_{q-1}^r} = \omega_{0r} = 0$. The corresponding expression is obtained from Eq. (4.2), taking the limit $\omega_{\mathbf{n}_{q-1}^r} \to 0$, and to the leading order we have

$$\langle j^r\rangle_0 \approx \frac{2e\Gamma(p/2+1)}{\pi^{p/2+1} L_r^p V_q} \sum_{n_r=1}^{\infty} \frac{\sin(n_r\tilde{\alpha}_r)}{n_r^{p+1}}. \tag{A2}$$

For $\omega_{0r} \neq 0$ and for large values of L_r, the main contribution to the zero temperature current density comes from the mode $n_r = 1$, $n_l = 0$, $l \neq r$, and from Eq. (4.2) one finds

$$\langle j^r\rangle_0 \approx \frac{2eV_q^{-1} \sin(\tilde{\alpha}_r)\omega_{0r}^{(p+1)/2}}{(2\pi)^{(p+1)/2} L_r^{(p-1)/2}} e^{-L_r\omega_{0r}}. \tag{A3}$$

In this case we have an exponential suppression.

Now we discuss the asymptotic behavior of the current density for small values of L_r. In this limit it is more convenient to use Eq. (A1). First we separate the term $n_l = 0$, $l \neq r$, in Eq. (A1) and use the asymptotic expression of the Macdonald function for small values of the argument. For the remaining part in Eq. (A1), the dominant contribution comes from large values of n_r and, to the leading order, we replace the summation over n_r by the integration. The corresponding integral involving the Macdonald function is evaluated by using the formula from Ref. [22]. In this way it can be seen that the contribution of the mode with a given $\mathbf{n}_{q-1}^r$ is suppressed by the factor $\exp(-g_r\sqrt{\tilde{\alpha}_r^2/L_r^2 + m^2})$, where $g_r = \sum_{l=p+1,\neq r}^{D} n_l^2 L_l^2$. As a result, we see that the dominant contribution to the current density is due to the modes with $n_l = 0$, $l \neq r$, and to the leading order we get

$$\langle j^r \rangle_0 \approx \frac{2e\Gamma((D+1)/2)}{\pi^{(D+1)/2} L_r^D} \sum_{n_r=1}^{\infty} \frac{\sin(n_r \tilde{\alpha}_r)}{n_r^D}. \quad \text{(A4)}$$

This leading term does not depend on the mass or on the lengths of the other compact dimensions. As can be seen from Eq. (A1), the expression in the right-hand side of Eq. (A4) coincides with the current density for a massless scalar field in the space with topology $R^{D-1} \times S^1$.

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