Isospin particle systems on quaternionic projective spaces

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We construct the isospin particle system on n-dimensional quaternionic projective spaces in the presence of the Belavin-Polyakov-Schwarz-Tyupkin instanton by the reduction from the free particle on $(2n+1)$-dimensional complex projective space. Then we add to this system a "quaternionic oscillator potential" and show that this oscillatorlike system is superintegrable. We show that, besides the analogs of quadratic constants of motion of the spherical (Higgs) and $\mathbb{C}P^n$ oscillators, it possesses the third-order constants of motion, which are functionally independent from the quadratic ones.

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I. INTRODUCTION

Hopf maps play a distinguished role in theoretical physics, appearing, sometimes in a hidden way, in most of the key models. However, even the constructions related with the second Hopf map, particularly, quaternionic projective spaces $\mathbb{H}P^n$, are not properly studied or/and used. In fact, explicitly quaternionic projective spaces appear in the construction of the multi-instanton of self-dual Yang-Mills theory only [1]. Even the classical and quantum mechanical systems on quaternionic projective spaces (except for the systems on $\mathbb{H}P^1$, i.e., the four-dimensional sphere) were not paid enough attention. On the other hand, there is no doubt that on these spaces, one can easily construct the integrable systems of isospin particles interacting with instantons: due to the existence of the well-known fibration $S^2 \to \mathbb{C}P^{2n+1} \to \mathbb{H}P^n$, the inclusion of instanton fields should not destroy the symmetries of the $Sp(n)$-invariant systems on $\mathbb{H}P^n$. This is similar to the well-known preservation of the symmetries of $U(N)$-invariant systems on complex projective spaces after inclusion of a constant magnetic field, which reflects the existence of the fibration $S^1 \to S^{2n+1} \to \mathbb{C}P^n$ related with the first Hopf map. Moreover, it is obvious that on $\mathbb{H}P^n$ one can define the $sp(n+1)$-invariant "quaternionic Landau problem," i.e., a free particle interacting with a constant [Belavin-Polyakov-Schwarz-Tyupkin (BPST)] instanton field, which is the quaternionic analog of the "Landau problem" on $\mathbb{C}P^n$: a $su(n+1)$-invariant system of particles interacting with a constant magnetic field. The simplest, one-dimensional quaternionic Landau problem on $\mathbb{H}P^1 = S^4$ [2] has been used previously for developing the model of the "four-dimensional Hall effect" [3] and, by this reason, it attracted much attention (see, e.g., Ref. [4] and the brief review [5]). Nevertheless, all these studies were restricted to the systems on $\mathbb{H}P^1 = S^4$, and there was no attempt to consider even the higher-dimensional quaternionic Landau problem. Although, technically this should not be a difficult problem, since the fibration $S^2 \to \mathbb{C}P^{2n+1} \to \mathbb{H}P^n$ allows one to construct the lift (or reformulate it) from the free particle systems on the complex projective space $\mathbb{C}P^{2n+1}$. For the $n=1$ case, this fibration was widely explored in the study of the four-dimensional Hall effect, while the S^4-Landau problem in itself was explicitly constructed by the Hamiltonian reduction of the free particle on $\mathbb{C}P^3$ in Ref. [6]. Below, we will fill the mentioned gap, presenting the detailed description of the Hamiltonian reduction of the free particle on $\mathbb{C}P^{2n+1}$ to the quaternionic Landau problem on $\mathbb{H}P^n$ (see the Sec. III). Besides, we will present the superintegrable analog of the oscillator on quaternionic projective spaces, which respects the inclusion of a constant instanton field. In contrast with spherical (Higgs) [7] and $\mathbb{C}P^n$ [8] oscillators, whose hidden symmetries are of the second order in momenta, our model has additional constants of motion, which are of the third order in momenta (see Sec. IV). In Sec. II we describe the fibration $S^2 \to \mathbb{C}P^{2n+1} \to \mathbb{H}P^n$.

II. $\mathbb{C}P^{2n+1} \to \mathbb{H}P^n$ FIBRATION

In this section we formulate the fibration and define the mathematical objects we are going to deal with.

First, let us notice that the definition of projective spaces defines an infinite series of fibrations, the natural projections of which are called Hopf maps. Indeed, by definition, the projective space over a field $\mathbb{F}$ ($\mathbb{F} = \mathbb{C}, \mathbb{H}$) is the set of all the lines through the origin. A natural chart on this manifold is given by the formula

$$q_i^{(k)} = v_i v_k^{-1}, \qquad i, k = 1, \dots, n+1, \qquad (2.1)$$

where v defines coordinates of the corresponding $\mathbb{F}^{n+1}$ and $q_i^{(k)}$ is the ith coordinate of the kth chart of the $\mathbb{F}P^n$. These maps define two infinite families of tautological fibrations

$$S^{2^N-1} \to S^{2^N(n+1)-1} \to \mathbb{F}P^n, \qquad (2.2)$$

where 2^N $(N = 1,\ 2)$ is the dimensionality of the corresponding field $\mathbb{F}$. Each first element of these families is the famous Hopf fibration of sphere over sphere:

$$S^1 \to S^3 \to S^2, \qquad S^3 \to S^7 \to S^4. \tag{2.3}$$

In our research we are interested in the projectivization of the second fibrations in Eq. (2.2). Namely, it is possible to project the total space and the fiber of Eq. (2.2) with $N = 2$ using the projection of the corresponding fibration with $N = 1$ as a map, so that it will not affect the base. After projectivization we will arrive to the following fiber bundle:

$$S^2 \to \mathbb{C}P^{2n+1} \to \mathbb{H}P^n. \tag{2.4}$$

Now, let us pass to the explicit construction of these fibrations. We start from the $2n + 2$-dimensional complex plane $\mathbb{C}^{2n+2} \simeq \mathbb{H}^{n+1}$ with complex coordinates λ or quaternionic ones: $v_i = \lambda_{2i-1} + j\lambda_{2i}$ $(i = 1, \ldots, n + 1)$. By definition the coordinates

$$q_\alpha = v_\alpha v_{n+1}^{-1} \equiv v_\alpha \frac{\bar{v}_n}{\|v_n\|^2}, \qquad \alpha = 1, \ldots, n \tag{2.5}$$

define a chart on the quaternionic projective space $\mathbb{H}P^n$.

The inverse formulas look like as follows:

$$v_\alpha = q_\alpha v_{n+1} = q_\alpha(\lambda_{2n+1} + j\lambda_{2n+2}) = \lambda_{2\alpha-1} + j\lambda_{2\alpha}. \tag{2.6}$$

Multiplying the last equation by λ_{2n+2}^{-1}, one finds

$$q_\alpha(z_{2n+1} + j) = z_{2\alpha-1} + jz_{2\alpha}, \tag{2.7}$$

where the quantities $z_r = \lambda_r/\lambda_{2n+2}$ $(r = 1, \ldots, 2n + 1)$ define a chart on the complex projective space $\mathbb{C}P^{2n+1}$. It is clear that any coordinate of $\mathbb{C}P^{2n+1}$ by itself defines a chart on a $\mathbb{C}P^1 \simeq S^2$. In particular, one can consider as such the last coordinate z_{2n+1}.

We can rewrite Eq. (2.7) in the following form:

$$z_{2\alpha-1} + jz_{2\alpha} = q_\alpha(u + j), \qquad z_{2n+1} = u. \tag{2.8}$$

In this form those relations define a natural projection of the fibration (2.4)

The form of transition functions can be easily found from the construction described above.

For our further consideration, it is convenient, instead of the quaternionic coordinates q, to use complex coordinates w which we introduce by the following formula:

$$q_\alpha = w_{2\alpha-1} + jw_{2\alpha}. \tag{2.9}$$

In these coordinates Eq. (2.8) takes the following form:

$$z_{2\alpha-1} = w_{2\alpha-1}u - \bar{w}_{2\alpha}, \qquad z_{2_\alpha} = w_{2\alpha}u + \bar{w}_{2\alpha-1}, \qquad z_{2n+1} = u. \tag{2.10}$$

In order to unify the first two expressions, we introduce a matrix Ω by the following formula:

$$(\Omega_{\mu\nu}) = \begin{pmatrix} \varepsilon & 0 & 0 & 0 & \ldots \\ 0 & \varepsilon & 0 & 0 & \ldots \\ 0 & 0 & \varepsilon & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix}, \qquad \varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2.11}$$

With this matrix we can rewrite Eq. (2.10) in the following form:

$$z^\mu = uw^\mu + \Omega^{\mu\nu}\bar{w}_\nu, \qquad z_{2n+1} = z^{2n+1} = u \qquad \mu, \nu = 1, \ldots, 2n. \tag{2.12}$$

Remark. From this point we will make a difference between the upper and lower indices. We define w^μ with the upper index, while its complex conjugate has a lower one: $\bar{w}_\mu$. The rule of raising and lowering the indices is given via the matrix $\Omega_{\mu\nu}$ and its inverse $\Omega^{\mu\nu}$:

$$\Omega_{\mu\nu}\Omega^{\nu\lambda} = \delta_\mu^\lambda. \tag{2.13}$$

Thus, we define

$$w_\mu = \Omega_{\mu\nu}w^\nu, \qquad \bar{w}^\mu = \Omega^{\mu\nu}\bar{w}_\nu. \tag{2.14}$$

The contraction is done, as usual, between upper and lower indices. So,

$$z\bar{z} = z^\mu\bar{z}_\mu = -z_\mu\bar{z}^\mu. \tag{2.15}$$

Now, using the above established relations between inhomogeneous coordinates of complex and quaternionic spaces, let us relate metrics on these spaces.

It is known that the natural metric on $S^{2^N(n+1)-1}$ induces the Fubini-Study metric on the corresponding projective space:

$$ds^2 = \frac{dz d\bar{z}}{1 + z\bar{z}} - \frac{(\bar{z}dz)(zd\bar{z})}{(1 + z\bar{z})^2}. \tag{2.16}$$

The nondegenerate transformation (2.12) defines a connection on the fibration (2.4). Indeed, replacing the coordinates z_i with (q, u) transforms the Fubini-Study metric on $\mathbb{C}P^{2n+1}$ to the form

$$ds^2 = \frac{dq d\bar{q}}{1 + q\bar{q}} - \frac{(\bar{q}dq)(d\bar{q}q)}{(1 + q\bar{q})^2} + \frac{(du + A)(d\bar{u} + \bar{A})}{(1 + u\bar{u})^2}, \tag{2.17}$$

where

$$A = j\frac{(\bar{u} - j)\bar{q}dq(u + j)}{1 + w\bar{w}}\bigg|_{\mathbb{C}} \equiv \frac{u(\bar{w}dw - wd\bar{w}) - (\bar{w}\Omega d\bar{w}) - u^2(w\Omega dw)}{1 + w\bar{w}}. \tag{2.18}$$

Here, $q|_{\mathbb{C}} \equiv 1/2(q - \imath q \imath)$ denotes the complex part of the quaternion q. In complex coordinates w, the metrics of $\mathbb{H}P^n$ reads

$$g_{\mu}{}^{\nu} = \frac{\delta^{\nu}_{\mu}}{1 + w\bar{w}} - \frac{\bar{w}_{\mu} w^{\nu} + w_{\mu}\bar{w}^{\nu}}{(1 + w\bar{w})^2}. \qquad (2.19)$$

The complex projective space is a Riemannian symmetric space. Indeed, each sphere of the total space in Eq. (2.4) can be represented as a coset space

$$U(n + 1)/U(n) \simeq S^{2n+1}. \qquad (2.20)$$

Reducing this by the global factor $U(1)$, we find

$$SU(n + 1)/U(n) \simeq \mathbb{C}P^{2n+1}. \qquad (2.21)$$

Thus, the isometries of the complex projective space form the $su(n + 1)$ algebra. These isometries are defined, in the given parametrization, by the following vector fields:

$$R_i = \partial_i + \bar{z}_i(\bar{z}\,\bar{\partial}),$$
$$J_i{}^j = \imath(z^j\partial_i - \bar{z}_i\bar{\partial}^j) + \imath\delta^j_i((z\partial) - (\bar{z}\,\bar{\partial})). \qquad (2.22)$$

These are all we need to know about the transformation of coordinates of the $\mathbb{C}P^n$.

Now, we are ready to construct a mechanical system of a particle on $\mathbb{H}P^n$ in the vector potential (2.18) by the reduction of the free particle on $\mathbb{C}P^{2n+1}$.

III. LANDAU PROBLEM ON $\mathbb{H}P^n$ FROM A FREE PARTICLE ON $\mathbb{C}P^{2n+1}$

Let us show that the free particle on $\mathbb{C}P^{2n+1}$ is immediately reduced to the particle on $\mathbb{H}P^n$ in the presence of a BPST instanton field (which is natural to call the "Landau problem on $\mathbb{H}P^n$").

In accordance with Eq. (2.16), the free particle on $\mathbb{C}P^{2n+1}$ is defined by the Lagrangian

$$L_0 = \frac{\dot{\bar{z}}\cdot\dot{z}}{1 + z\bar{z}} - \frac{(\dot{\bar{z}}z)(\dot{z}\,\bar{z})}{(1 + z\bar{z})^2}. \qquad (3.1)$$

In terms of Eq. (2.12), it reads

$$L = \frac{\dot{q}\,\dot{\bar{q}}}{1 + q\bar{q}} - \frac{(\bar{q}\,\dot{q})(\dot{\bar{q}}q)}{(1 + q\bar{q})^2} + \frac{(\dot{u} + A)(\dot{\bar{u}} + \bar{A})}{(1 + u\bar{u})^2}. \qquad (3.2)$$

In order to reduce it to the system on $\mathbb{H}P^n$, it is convenient to give the Hamiltonian formulation of this system and then perform the Hamiltonian reduction associated with the Hopf map. Precisely, in the Hamiltonian language, the free particle system on $\mathbb{C}P^n$ is defined by the triple

$$(H_0 = (g^{-1})_i{}^j p_i\bar{p}^j,$$
$$\omega = dp_i \wedge dz^i + d\bar{p}_i \wedge d\bar{z}^i, T^*\mathbb{C}P^{2n+1}), \qquad (3.3)$$

where $(g^{-1})_i{}^j = (1 + z\bar{z})(\delta^j_i + \bar{z}_i z^j)$ are the components of the inverse Fubini-Study metric (2.16). This system possesses the $su(2n + 2)$ symmetry algebra given by the generators (2.22). These generators define the following Noether constants of motion:

$$R_i = p_i + \bar{z}_i(\bar{z}\,\bar{p}),$$
$$J_i{}^j = \imath(z^j p_i - \bar{z}_i\bar{p}^j) + \imath\delta^j_i((zp) - (\bar{z}\,\bar{p})). \qquad (3.4)$$

We can extend the transformation (2.12) to the canonical one by adding the following transformation rule for the conjugated momenta:

$$p_{\mu} = \frac{\bar{u}}{1 + u\bar{u}}\pi_{\mu} + \frac{1}{1 + u\bar{u}}\bar{\pi}_{\mu},$$
$$p_{2n+1} = p_u - \frac{1}{1 + u\bar{u}}(\bar{u}w^{\mu}\pi_{\mu} - w^{\mu}\bar{\pi}_{\mu}). \qquad (3.5)$$

It is an exercise to check that the canonical transformation (2.10) and (3.5) leads to the following form of the Hamiltonian:

$$H_0 = (g^{-1})_{\mu}{}^{\nu}\bar{P}^{\mu}P_{\nu} + (1 + u\bar{u})^2 p_u\bar{p}_u, \qquad (3.6)$$

where we introduced the inverse metric to Eq. (2.19),

$$(g^{-1})_{\mu}{}^{\nu} = (1 + w\bar{w})(\delta^{\nu}_{\mu} + \bar{w}_{\mu}w^{\nu} + w_{\mu}\bar{w}^{\nu}), \qquad (3.7)$$

and the covariant momenta

$$P_{\mu} = \pi_{\mu} - \imath\bar{w}_{\mu}\frac{I_3}{1 + w\bar{w}} - w_{\mu}\frac{I_+}{1 + w\bar{w}}, \qquad (3.8)$$

with the $su(2)$ generators $I_{\pm}$, I_3 defining the isometries of S^2,

$$I_3 = -\imath(up_u - \bar{u}\bar{p}_u), \qquad I_+ = \bar{p}_u + u^2 p_u,$$
$$I_- = p_u + \bar{u}^2\bar{p}_u, \qquad (3.9)$$

$$\{I_3, I_{\pm}\} = \pm\imath I_{\pm}, \qquad \{I_+, I_-\} = 2\imath I_3. \qquad (3.10)$$

The Poisson brackets between the quantities P_{μ} read

$$\{w^{\mu}, P_{\nu}\} = \delta^{\mu}_{\nu}, \qquad \{P_{\mu}, P_{\nu}\} = -2\frac{\Omega_{\mu\nu}}{1 + w\bar{w}}I_+,$$
$$\{P_{\mu}, \bar{P}^{\nu}\} = \imath\frac{\delta^{\nu}_{\mu}I_3}{(1 + w\bar{w})^2}. \qquad (3.11)$$

Besides, we have

$$\{P_{\mu}, I_+\} = \frac{\bar{w}_{\mu}I_+}{1 + w\bar{w}}, \quad \{P_{\mu}, I_-\} = -\frac{\bar{w}_{\mu}I_-}{1 + w\bar{w}} - 2\imath\frac{w_{\mu}I_3}{1 + w\bar{w}},$$
$$\{P_{\mu}, I_3\} = \frac{\imath w_{\mu}I_+}{1 + w\bar{w}}. \qquad (3.12)$$

Let us also present, for completeness, the some other relations as well:

$$\{I_3, p_u\} = -\imath p_u, \qquad \{I_+, p_u\} = 2up_u, \qquad \{I_-, p_u\} = 0,$$
$$\{P_{\mu}, p_u\} = -\frac{p_u}{1 + w\bar{w}}(\bar{w}_{\mu} + 2uw_{\mu}). \qquad (3.13)$$

It is easy to see that the Casimir operator of these $su(2)$ generators is precisely the Hamiltonian of a free particle moving on S^2, in Eq. (3.6) as a second summand:

$$I^2 = I_+I_- + I_3^2 = (1 + u\bar{u})^2 p_u\bar{p}_u. \qquad (3.14)$$

Obviously, it commutes with the Hamiltonian and defines an integral of motion of the system.

Our goal is to perform the Hamiltonian reduction by this constant of motion. For this purpose we should fix the $(4(2n+1)-1)$-dimensional level surface of I^2, by putting

$$I^2 = s^2, \tag{3.15}$$

and then factorize it, by the vector flow $\{I^2, \}$, to the $(8n+2) = (2\cdot 4n+2)$-dimensional phase space. It is clear that $(8n+2)$ functions commuting with I will play the role of local coordinates on the reduced coordinates. In order to find the Poisson brackets on the reduced phase space, we should simply calculate the Poisson brackets between these "I-invariant" functions and then restrict them on the level surface (3.15). Similarly, for the construction of the reduced Hamiltonian, we should express the initial Hamiltonian in terms of I and these I-invariant functions and then restrict the Hamiltonian to the level surface as well (see, e.g., Ref. [9]). Let us find these I-invariant functions, playing the role of coordinates of the reduced phase space. The $8n$ coordinates of the reduced phase space can be chosen to be P_μ, w^μ given by Eqs. (3.8) and (2.12). For finding the remaining two coordinates, we should simply resolve the condition (3.14) preserving the Poisson brackets (3.10):

$$I_+ = s\frac{2x}{1+x\bar{x}}, \qquad I_3 = s\frac{1-x\bar{x}}{1+x\bar{x}}, \qquad s\in\mathbb{R}, \qquad x\in\mathbb{C}. \tag{3.16}$$

This yields

$$x = \frac{I_+}{s+I_3}: \{x,\bar{x}\} = \frac{i}{2s}(1+x\bar{x})^2,$$
$$\{P_\mu, x\} = \frac{\bar{w}_\mu x}{1+w\bar{w}} - \imath\frac{x^2\bar{w}_\mu}{1+w\bar{w}},$$
$$\{P_\mu, \bar{x}\} = -\frac{\bar{w}_\mu\bar{x}+\imath w_\mu}{1+w\bar{w}}, \qquad \{w^\mu, x\} = \{w^\mu, \bar{x}\} = 0. \tag{3.17}$$

It is seen that we have reduced the phase space $T^*\mathbb{C}P^{2n+1}$ to the $T^*\mathbb{H}P^n\times S^2$. The latter defines the phase space of the $su(2)$-isospin particle on $\mathbb{H}P^n$ interacting with the BPST instanton field. Its Poisson brackets are defined by the relations (3.11) and (3.17), or, equivalently, by (3.10), (3.11), and (3.12).

The Hamiltonian of the free particle on $\mathbb{C}P^{2n+1}$ given by Eq. (3.3) [or, equivalently, by Eq. (3.6)], results, upon reduction, in the one on $\mathbb{H}P^n$,

$$H_{\mathbb{H}P^n} = (g^{-1})_\mu{}^\nu\bar{P}^\mu P_\nu + s^2. \tag{3.18}$$

So, we reduced the free particle system on $T^*\mathbb{C}P^{2n+1}$ to the isospin particle on $\mathbb{H}P^n$ interacting with the BPST instanton field. The subset of the Noether constants of motion (3.4) commuting with Eq. (3.14) is reduced on $\mathbb{H}P^n$ and forms the $Sp(n+1)$ algebra of the isometries of the reduced system.

These reduced generators are given by the expressions

$$L_\mu{}^\nu = J_\mu{}^\nu + J^\nu{}_\mu = \imath(w^\nu\pi_\mu - \bar{w}_\mu\bar{\pi}^\nu) + \imath(w_\mu\pi^\nu - \bar{w}^\nu\bar{\pi}_\mu), \tag{3.19}$$

$$L_3 = J_{2n+1}{}^{2n+1} = I_3 + \frac{\imath}{2}((\pi w) - (\bar{\pi}\bar{w})),$$
$$L_- = R_{2n+1} = (\bar{\pi}^\mu w_\mu) + I_-, \qquad L_+ = \bar{L}_-, \tag{3.20}$$

$$L^\mu = \imath J_{2n+1}{}^\mu - R^\mu = \bar{\pi}^\mu - ((\bar{\pi}^\nu w_\nu) - I_-)\bar{w}^\mu + ((\pi w) - \imath I_3)w^\mu. \tag{3.21}$$

The $2n(2n+1)/2$ generators $L_{\mu\nu}$ and three generators $L_\pm$, L_3 form the $sp(n)\times sp(1)$ algebra, and these two sets of generators, together with the $4n$ generators L_μ, form the $2n^2+5n+3$-dimensional algebra of the isometries of $\mathbb{H}P^n$, that is $sp(n+1)$:

$$\{L_{\mu\nu}, L_{\rho\sigma}\} = -\imath(\Omega_{\mu\rho}L_{\sigma\nu} + \Omega_{\nu\rho}L_{\sigma\mu} + \Omega_{\nu\sigma}L_{\mu\rho} + \Omega_{\mu\sigma}L_{\nu\rho}), \tag{3.22}$$

$$\{L_+, L_-\} = 2\imath L_3, \qquad \{L_3, L_\pm\} = \pm\imath L_\pm, \qquad \{L_{\mu\nu}, L_\pm\} = \{L_{\mu\nu}, L_3\} = 0, \tag{3.23}$$

$$\{L^\mu, L_-\} = 0, \qquad \{L^\mu, L_+\} = \bar{L}^\mu, \qquad \{L^\mu, L_3\} = \frac{\imath}{2}L^\mu, \qquad \{L^\mu, L_\rho{}^\sigma\} = -\imath\delta^\mu_\rho L^\sigma - \imath L_\rho\Omega^{\mu\sigma}, \tag{3.24}$$

$$\{L^\mu, L^\nu\} = 2\Omega^{\mu\nu}L_-, \qquad \{\bar{L}_\mu, \bar{L}_\nu\} = -2\Omega_{\mu\nu}L_+, \qquad \{L^\mu, \bar{L}_\nu\} = \imath L_\nu{}^\mu + 2\imath\delta^\mu_\nu L_3. \tag{3.25}$$

The Casimir of $sp(n+1)$ is precisely the Hamiltonian on $\mathbb{H}P^n$

$$H_{\mathbb{H}P^n} = \frac{L_\mu{}^\nu L_\nu{}^\mu + 4L^\mu\bar{L}_\mu + 8(L_+L_- + L_3^2)}{4} + I^2. \tag{3.26}$$

Hence, we established a complete correspondence between the $SU(2)$ (classical) Landau problem on $\mathbb{H}P^n$ and the free particle system on $\mathbb{C}P^{2n+2}$.

IV. OSCILLATOR

In the previous section, we considered the free particle on the n-dimensional quaternionic projective space $\mathbb{H}P^n$ and demonstrated that the inclusion of the $SU(2)$ instanton preserves its whole symmetry algebra $sp(n+1)$. In fact, $\mathbb{H}P^n$ seems to be a natural candidate for the role of configuration spaces of the (super)integrable systems interacting with the BPST instanton field. At least, on these spaces there should exist the proper generalizations of the systems on $\mathbb{R}^4$ respecting the inclusion of the BPST instanton. The simplest system of this sort, besides the free particle, is the $4n$-dimensional isotropic oscillator. How can we construct its appropriate analog on $\mathbb{H}P^n$?

Let us consider a more complicated integrable system on $\mathbb{H}P^n$, that is, the generalization of the oscillator, given by the following expression:

$$H_{\rm osc} = H_{\mathbb{H}P^n} + \omega_0^2 w\bar{w}, \tag{4.1}$$

where the first term is simply the free particle Hamiltonian on $\mathbb{H}P^n$ given by Eq. (3.18).

This potential has been suggested in Refs. [9,10] in analogy with the earlier constructed oscillator potential on $\mathbb{C}P^n$. In our opinion, it is deductive to present here the speculations which lead to the suggestion of the above system. Namely, in Ref. [8] two of the authors constructed the model of the oscillator on $\mathbb{C}P^n$ requiring that it should have the hidden symmetries, resulting, in the flat limit, in the ordinary oscillator on $\mathbb{C}^n$. Such a model was found to be unique. On $\mathbb{C}P^1$ it was found to be the well-known Higgs oscillator, while for $n > 1$ it was defined by the potential

$$V_{\mathbb{C}P^n} = \omega_0^2 z\bar{z}, \tag{4.2}$$

with z^i being the inhomogeneous coordinates on $\mathbb{C}P^n$. Besides the $u(n)$ Nöther constants of motion defined by the second expression in Eq. (3.4), this system was found to have hidden constants of motion (for $n > 1$) given by the expression

$$I^i_j = \bar{R}^i R_j + \omega_0^2 z^i \bar{z}_j, \tag{4.3}$$

where R_i are the translation generators defined by the first expression in Eq. (3.4). Surprisingly, it was found that the inclusion of a constant magnetic field preserves all symmetries (and, respectively, superintegrability) of the system. Moreover, it was found that even on $\mathbb{C}P^1$, this potential is a distinguished one. Namely, though the system is not superintegrable in this case, it is exactly solvable and preserves the exact solvability property after inclusion of the constant magnetic field, while the Higgs oscillator on $S^2 = \mathbb{C}P^1$, being a superintegrable system, looses the superintegrability (and even the exact solvability) property upon inclusion of a constant magnetic field. This allowed the authors to call that system "$\mathbb{C}P^n$ oscillator" for any n. It was further studied in Ref. [11].

Keeping in mind that the potential of the Higgs oscillator on the n-dimensional sphere (to be more precise, on the real projective space) reads, in inhomogeneous coordinates,

$$V_{\mathbb{R}P^n} = \frac{\omega_0^2 y^2}{2}, \qquad y^i = \frac{u^i}{u^0}, \tag{4.4}$$

with u^i, u^0 being coordinates of the ambient $\mathbb{R}^n$ space ($(u^i)^2 + (u^0)^2 = 1$), the authors of Refs. [9,10] claimed that the oscillator potential on $\mathbb{H}P^n$ should be given by the same expression as in the case of $\mathbb{C}P^n$, with the replacement of inhomogeneous complex coordinates with quaternionic ones. And this system has to respect the inclusion of the BPST instanton field. They have checked this in the simplest case of $\mathbb{H}P^n = S^4$ and found that it is indeed the case. However, in contrast with the Higgs oscillator on $\mathbb{R}P^1 = S^1$, and with the $\mathbb{C}P^1(= S^2)$ oscillator, the spectrum of the $\mathbb{H}P^1$ oscillator (and of its hyperbolic analog [10]) system was found to be degenerate, which is a precise indication of the existence of hidden symmetries. Unfortunately, no explanation of these symmetries was offered there. Moreover, this claim has never been checked for nontrivial (higher-dimensional) cases.

Now, let us show that the Hamiltonian (4.1), together with Poisson brackets (3.10), (3.11), and (3.12), defines a well-defined oscillator system on $\mathbb{H}P^n$.

It is clear that the added oscillator potential does not commute with the coset generators L^μ, $\bar{L}_\mu$, while the rest of the $sp(n) \times sp(1)$ generators $L_\mu{}^\nu$, $L_{\pm,3}$ remain as symmetries of the system. However, the system possesses a set of hidden symmetries:

$$I_\nu{}^\mu = L^\mu \bar{L}_\nu - \bar{L}^\mu L_\nu + \omega_0^2 (w^\mu \bar{w}_\nu - \bar{w}^\mu w_\nu), \tag{4.5}$$

which are constructed by analogy with the corresponding integrals for the $\mathbb{C}P^n$ oscillator (4.3). These quantities commute with $L_{\pm,3}$ and transform linearly with respect to $L_{\mu\nu}$:

$$\{I_{\mu\nu}, L_{\rho\sigma}\} = \imath(\Omega_{\mu\sigma} I_{\nu\rho} + \Omega_{\rho\mu} I_{\sigma\nu} - \Omega_{\nu\sigma} I_{\mu\rho} - \Omega_{\nu\rho} I_{\sigma\mu}),$$
$$\{L_{\mu\nu}, L_{\pm,3}\} = \{I_{\mu\nu}, L_{\pm,3}\} = 0. \tag{4.6}$$

However, in contrast to the case of the $\mathbb{C}P^n$ oscillator, where the symmetries of the system form a quadratic algebra, in the case of the $\mathbb{H}P^n$ oscillator, the Poisson brackets between the hidden symmetry generators cannot be expressed through the combination of L and I and give us a new set of integrals of motion, which are, already, cubic in momenta:

$$\begin{aligned}\{I_{\mu\nu}, I_{\rho\sigma}\} = {}& \imath(I_{\mu\rho} L_{\nu\sigma} + I_{\nu\sigma} L_{\mu\rho} - I_{\mu\sigma} L_{\nu\rho} - I_{\nu\rho} L_{\mu\sigma}) \\ & + \Omega_{\mu\rho} S_{\nu\sigma} + \Omega_{\nu\sigma} S_{\mu\rho} - \Omega_{\mu\sigma} S_{\nu\rho} - \Omega_{\nu\rho} S_{\mu\sigma},\end{aligned} \tag{4.7}$$

where

$$\begin{aligned}S_{\mu\nu} = {}& 2L_\mu L_\nu L_- + 2\bar{L}_\mu \bar{L}_\nu L_+ + 2\imath L_3 (L_\mu \bar{L}_\nu + L_\nu \bar{L}_\mu) \\ & - \omega_0^2 (L_{\mu\nu} + 2(w_\mu w_\nu L_+ + \bar{w}_\mu \bar{w}_\nu L_- \\ & - \imath L_3 (w_\mu \bar{w}_\nu + w_\nu \bar{w}_\mu)))\end{aligned} \tag{4.8}$$

defines the new set of cubic constants of motion. Their Poisson brackets yield an additional, last set of constants of motion,

$$\begin{aligned}T_{\mu\nu} = {}& I_-(\bar{\pi}_\mu w_\nu - \bar{\pi}_\nu w_\mu) + I_+(\pi_\mu \bar{w}_\nu - \pi_\nu \bar{w}_\mu) \\ & - \imath I_3 (\pi_\mu w_\nu - \pi_\nu w_\mu + \bar{\pi}_\mu \bar{w}_\nu - \bar{\pi}_\nu \bar{w}_\mu).\end{aligned} \tag{4.9}$$

Let us notice that the constants of motion (4.8) and (4.9) have no analogs neither in Higgs nor in $\mathbb{C}P^n$ oscillator models. It seems that precisely these constants of motion are responsible for the degeneracy of the spectrum of the $\mathbb{H}P^1$ oscillator observed in Ref. [10].

Finally, let us notice that, repeating the speculations given for the $\mathbb{C}P^n$ oscillator, we can define the singular version of the $\mathbb{H}P^n$ oscillator respecting the inclusion of the BPST instanton field,

$$V = \frac{\alpha^2}{w\bar{w}} + \omega_0^2 w\bar{w}. \tag{4.10}$$

The classical and semiclassical analysis of its $\mathbb{C}P^1$ analog has been carried out in Ref. [12].

V. CONCLUSION

In this paper we constructed two basic one-particle integrable systems on quaternionic projective spaces, which are the quaternionic Landau problem on $\mathbb{H}P^n$, i.e., the particle moving in the presence of an instanton field, and the $\mathbb{H}P^n$ oscillator (interacting with the instanton field). Both systems are superintegrable; the first one possesses the $Sp(n+1)$ symmetry algebra, while the symmetry algebra of the second one is highly nonlinear, and it still needs to be calculated. Note that both systems can be easily lifted to the ones on the complex projective space, where the instanton field is "absorbed" in the spatial coordinates. The obvious next step is to consider the respective quantum mechanical systems for $n > 1$ (for the $n = 1$ case, it was considered in earlier works). With the quantum mechanics at hand, one can consider, e.g., the "quantum Hall effect on $\mathbb{H}P^n$", in analogy with the "quantum Hall effect on $\mathbb{C}P^n$", considered by Karabali and Nair (see Refs. [4,5]). A more detailed description of the "singular $\mathbb{H}P^n$ oscillator" defined by the potential (4.10) and the construction and proper generalization of the Coulomb system are also in order. Supersymmetric extensions of these systems, which are similar to those on complex projective spaces, are of special importance [13]. However, the natural desire to obtain them by the Hamiltonian reduction from the complex projective space seems to be technically irrelevant [14], so that one should try to do it in a less obvious way.

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