One-loop four-graviton amplitudes in $\mathcal{N} = 4$ supergravity models

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We evaluate in great detail one-loop four-graviton field theory amplitudes in pure $\mathcal{N} = 4$ $D = 4$ supergravity. The expressions are obtained by taking the field theory limits of (4,0) and (2,2) space-time supersymmetric string theory models. For each model we extract the contributions of the spin-1 and spin-2 $\mathcal{N} = 4$ supermultiplets running in the loop. We show that all of those constructions lead to the same four-dimensional result for the four-graviton amplitudes in pure supergravity even though they come from different string theory models.

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I. INTRODUCTION

The role of supersymmetry in perturbative supergravity still leaves room for surprises. The construction of candidate counterterms for ultraviolet (UV) divergences in extended four-dimensional supergravity theories does not forbid some particular amplitudes to have an improved UV behavior. For instance, the four-graviton three-loop amplitude in $\mathcal{N} = 4$ supergravity turns out to be UV finite [1,2], despite the construction of a candidate counterterm [3]. (Some early discussion of the three-loop divergence in $\mathcal{N} = 4$ has appeared in Ref. [4], and recent alternative arguments have been given in Ref. [5].)

The UV behavior of extended supergravity theories is constrained in string theory by nonrenormalization theorems that give rise in the field theory limit to supersymmetric protection for potential counterterms. In maximal supergravity, the absence of divergences until six loops in four dimensions [6–8] is indeed a consequence of the supersymmetric protection for $\frac{1}{2}$-, $\frac{1}{4}$- and $\frac{1}{8}$-BPS operators in string [9,10] or field theory [11,12]. In half-maximal supergravity, it was shown recently [2] that the absence of three-loop divergence in the four-graviton amplitude in four dimensions is a consequence of the protection of the $\frac{1}{2}$-BPS R^4 coupling from perturbative quantum corrections beyond one loop in heterotic models. We refer to Refs. [13–15] for a discussion of the nonrenormalization theorems in heterotic string.

Maximal supergravity is unique in any dimension and corresponds to the massless sector of type II string theory compactified on a torus. Duality symmetries relate different phases of the theory and strongly constrain its UV behavior [10,12,16–19].

On the contrary, half-maximal supergravity (coupled to vector multiplets) is not unique and can be obtained in the low-energy limit of (4,0) string theory models—with all the space-time supersymmetries coming from the world-sheet left-moving sector—or (2,2) string theory models—with the space-time supersymmetries originating both from the world-sheet left-moving and right-moving sectors. The two constructions give rise to different low-energy supergravity theories with a different identification of the moduli.

In this work we analyze the properties of the four-graviton amplitude at one loop in pure $\mathcal{N} = 4$ supergravity in four dimensions. We compute the genus one string theory amplitude in different models and extract its field theory limit. This method has been pioneered by Ref. [20]. It has then been developed intensively for gauge theory amplitudes by Refs. [21,22] and then applied to gravity amplitudes in Refs. [23,24]. In this work we will follow more closely the formulation given in Ref. [25].

We consider three classes of four-dimensional string models. The first class, on which was based the analysis in Ref. [2], are heterotic string models. They have (4,0) supersymmetry and $4 \leq n_v \leq 22$ vector multiplets. The models of the second class also carry (4,0) supersymmetry; they are type II asymmetric orbifolds. We will study a model with $n_v = 0$ (the Dabholkar-Harvey construction; see Ref. [26]) and a model with $n_v = 6$. The third class is composed of type II symmetric orbifolds with (2,2) supersymmetry. For a given number of vector multiplets, the (4,0) models are related to one another by strong-weak S duality and related to (2,2) models by U duality [27,28]. Several tests of the duality relations between orbifold models have been given in Ref. [29].

The string theory constructions generically contain matter vector multiplets. By comparing models with $n_v \neq 0$ vector multiplets to a model where $n_v = 0$, we directly check that one can simply subtract these contributions and extract the pure $\mathcal{N} = 4$ supergravity contributions in four dimensions.

We shall show that the four-graviton amplitudes extracted from the (4,0) string models match that obtained in Refs. [24,30–35]. We however note that all of those constructions are based on a (4,0) construction, while our analysis covers both the (4,0) and (2,2) models. The four-graviton amplitudes are expressed in a supersymmetric

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decomposition into $\mathcal{N} = 4s$ spin-s supermultiplets with $s = 1, \frac{3}{2}, 2$, as in Refs. [24,30–35]. The $\mathcal{N} = 8$ and $\mathcal{N} = 6$ supermultiplets have the same integrands in all the models, while the contributions of the $\mathcal{N} = 4$ multiplets have different integrands. Despite the absence of obvious relation between the integrands of the two models, the amplitudes turn out to be equal after integration in all the string theory models. In a nutshell, we find that the four-graviton one-loop field theory amplitudes in the (2,2) construction are identical to the (4,0) ones.

The paper is organized as follows. For each model we evaluate the one-loop four-graviton string theory amplitudes in Sec. II. In Sec. III we compare the expressions that we obtained and check that they are compatible with our expectations from string dualities. We then extract and evaluate the field theory limit in the regime $\alpha' \to 0$ of those string amplitudes in Sec. IV. This gives us the field theory four-graviton one-loop amplitudes for pure $\mathcal{N} = 4$ supergravity. Section V contains our conclusions. Finally, Appendixes A and B contain details about our conventions and the properties of the conformal field theory (CFT) building blocks of our string theory models.

II. ONE-LOOP STRING THEORY AMPLITUDES IN (4,0) AND (2,2) MODELS

In this section, we compute the one-loop four-graviton amplitudes in four-dimensional $\mathcal{N} = 4$ (4,0) and (2,2) string theory models. Their massless spectrum contains an $\mathcal{N} = 4$ supergravity multiplet coupled to n_v $\mathcal{N} = 4$ vector multiplets. Since the heterotic string is made of the tensor product of a left-moving superstring by a right-moving bosonic string, it only gives rise to (4,0) models. However, type II compactifications provide the freedom to build (4,0) and (2,2) models [36].

A. Heterotic CHL models

We evaluate the one-loop four-graviton amplitudes in heterotic string Chaudhuri-Hockney-Lykken (CHL) models in four dimensions [37–39]. Their low-energy limits are (4,0) supergravity models with $4 \le n_v \le 22$ vector supermultiplets matter fields. We first comment on the moduli space of the model, then write the string theory one-loop amplitude and finally compute the CHL partition function. This allows us to extract the massless states contribution to the integrand of the field theory limit.

These models have the following moduli space:

$$\Gamma\backslash SU(1,1)/U(1) \times SO(6,n_v;\mathbb{Z})\backslash SO(6,n_v)/SO(6) \times SO(n_v), \tag{2.1}$$

where n_v is the number of vector multiplets and Γ is a discrete subgroup of $SL(2,\mathbb{Z})$. For instance, $\Gamma = SL(2,\mathbb{Z})$ for $n_v = 22$ and $\Gamma = \Gamma_1(N)$ for the $\mathbb{Z}_N$ CHL (4,0) orbifold. [We refer to Appendix A 3 for a definition of the congruence subgroups of $SL(2,\mathbb{Z})$.] The scalar manifold $SU(1,1)/U(1)$ is parametrized by the axion-dilaton in the $\mathcal{N} = 4$ gravity supermultiplet.

The generic structure of the amplitude has been described in Ref. [2]. We will use the same notations and conventions. The four-graviton amplitude takes the following form[1]:

$$\mathcal{M}^{(n_v)}_{(4,0)\text{het}} = \mathcal{N}\left(\frac{\pi}{2}\right)^4 t_8F^4 \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^{\frac{D-6}{2}}} \int_{\mathcal{T}} \prod_{1\le i<j\le 4} \times \frac{d^2\nu_i}{\tau_2} e^{\mathcal{Q}} \mathcal{Z}^{(n_v)}_{(4,0)\text{het}} \bar{\mathcal{W}}^B, \tag{2.2}$$

where $D = 10 - d$ and $\mathcal{N}$ is the normalization constant of the amplitude. The domains of integration are $\mathcal{F} = \{\tau = \tau_1 + i\tau_2; |\tau_1| \le \frac{1}{2}, |\tau|^2 \ge 1, \tau_2 > 0\}$ and $\mathcal{T} := \{\nu = \nu_1 + i\nu_2; |\nu_1| \le \frac{1}{2}, 0 \le \nu_2 \le \tau_2\}$. Then,

$$\bar{\mathcal{W}}^B := \frac{\langle \prod_{j=1}^4 \tilde{\epsilon}^j \cdot \bar{\partial} X(\nu_j) e^{ik_j\cdot X(\nu_j)}\rangle}{(2\alpha')^4 \langle \prod_{j=1}^4 e^{ik_j\cdot X(\nu_j)}\rangle} \tag{2.3}$$

is the kinematical factor coming from the Wick contractions of the bosonic vertex operators and the plane-wave part is given by $\langle \prod_{j=1}^4 e^{ik_j\cdot X(\nu_j)}\rangle = \exp(\mathcal{Q})$ with

$$\mathcal{Q} = \sum_{1\le i<j\le 4} 2\alpha' k_i \cdot k_j \mathcal{P}(\nu_{ij}), \tag{2.4}$$

where we have made use of the notation $\nu_{ij} := \nu_i - \nu_j$. Using the result of Ref. [41] with our normalizations we explicitly write

$$\begin{aligned}\bar{\mathcal{W}}^B = {}& \prod_{r=1}^4 \tilde{\epsilon}_r \cdot \bar{\mathcal{Q}}_r \\ &+ \frac{1}{2\alpha'}(\tilde{\epsilon}_1 \cdot \bar{\mathcal{Q}}_1 \tilde{\epsilon}_2 \cdot \bar{\mathcal{Q}}_2 \tilde{\epsilon}_3 \cdot \tilde{\epsilon}_4 \bar{\mathcal{T}}(\nu_{34}) + \text{perms}) \\ &+ \frac{1}{4\alpha'^2}(\tilde{\epsilon}_1 \cdot \tilde{\epsilon}_2 \tilde{\epsilon}_3 \cdot \tilde{\epsilon}_4 \bar{\mathcal{T}}(\nu_{12}) \bar{\mathcal{T}}(\nu_{34}) + \text{perms}),\end{aligned} \tag{2.5}$$

where we have introduced

$$\mathcal{Q}_I^\mu := \sum_{r=1}^4 k_r^\mu \partial \mathcal{P}(\nu_{Ir}|\tau); \qquad \mathcal{T}(\nu) := \partial_\nu^2 \mathcal{P}(\nu|\tau), \tag{2.6}$$

with $\mathcal{P}(z)$ the genus one bosonic propagator. We refer to Appendix A 2 for definitions and conventions.

The CHL models studied in this work are asymmetric $\mathbb{Z}_N$ orbifolds of the bosonic sector (in our case the right-moving sector) of the heterotic string compactified on $T^5 \times S^1$. Geometrically, the orbifold rotates N groups of ℓ bosonic fields $\bar{X}^a$ belonging either to the internal T^{16} or to the T^5 and acts as an order N shift on the S^1. More

[1]The t_8 tensor defined in Ref. [40], Appendix 9.A, is given by $t_8F^4 = 4\text{tr}(F^{(1)}F^{(2)}F^{(3)}F^{(4)}) - \text{tr}(F^{(1)}F^{(2)})\text{tr}(F^{(3)}F^{(4)}) + \text{perms}(2,3,4)$, where the traces are taken over the Lorentz indices. Setting the coupling constant to one, $t_8F^4 = stA^{\text{tree}}(1,2,3,4)$, where $A^{\text{tree}}(1,2,3,4)$ is the color stripped ordered tree amplitude between four gluons.

precisely, if we take a boson $\bar{X}^a$ of the $(p+1)$th group $(p = 0, \ldots, N-1)$ of ℓ bosons, we have $a \in \{p\ell, p\ell+1, \ldots, p\ell+(\ell-1)\}$ and for twists $g/2$, $h/2 \in \{0, 1/N, \ldots, (N-1)/N\}$ we get

$$\begin{aligned} \bar{X}^a(z+\tau) &= e^{i\pi g p/N}\bar{X}^a(z), \\ \bar{X}^a(z+1) &= e^{i\pi h p/N}\bar{X}^a(z). \end{aligned} \tag{2.7}$$

We will consider models with $(N, n_v, \ell) \in \{(1,22,16), (2,14,8), (3,10,6), (5,6,4), (7,4,3)\}$. It is in principle possible to build models with $(N, n_v, \ell) = (11,2,2)$ and $(N, n_v, \ell) = (23,0,1)$ and thus decouple totally the matter fields, but it is then required to compactify the theory on a seven- and eight-dimensional torus, respectively. We will not comment about it further, since we have anyway a type II superstring compactification with (4,0) supersymmetry that already has $n_v = 0$ that we discuss in Sec. II B 2. This issue could have been important, but it appears that at one loop in the field theory limit there is no problem to decouple the vector multiplets to obtain pure $\mathcal{N} = 4$ supergravity. The partition function of the right-moving CFT is given by

$$\mathcal{Z}^{(n_v)}_{(4,0)\text{het}}(\tau) = \frac{1}{|G|}\sum_{(g,h)} \mathcal{Z}^{h,g}_{(4,0)\text{het}}(\tau), \tag{2.8}$$

where $|G|$ is the order of the orbifold group i.e., $|G| = N$. The twisted conformal blocks $\mathcal{Z}^{g,h}_{\text{het}}$ are a product of the oscillator and zero mode part

$$\mathcal{Z}^{h,g}_{(4,0)\text{het}} = \mathcal{Z}^{h,g}_{\text{osc}} \times \mathcal{Z}^{h,g}_{\text{latt}}. \tag{2.9}$$

In the field theory limit only the massless states from the $h = 0$ sector will contribute and we are left with

$$\mathcal{Z}^{(n_v)}_{(4,0)\text{het}}(\tau) \to \frac{1}{N}\mathcal{Z}^{0,0}_{(4,0)\text{het}}(\tau) + \frac{1}{N}\sum_{\{g\}} \mathcal{Z}^{0,g}_{(4,0)\text{het}}(\tau). \tag{2.10}$$

The untwisted partition function ($g = h = 0$) with generic diagonal Wilson lines A, as required by modular invariance, is

$$\mathcal{Z}^{0,0}_{(4,0)\text{het}}(\tau) := \frac{\Gamma_{(6,24)}(G,A)}{\bar{\eta}^{24}(\bar{\tau})}, \tag{2.11}$$

where $\Gamma_{(6,24)}(G,A)$ is the lattice sum for the Narain lattice $\Gamma^{(5,5)} \oplus \Gamma^{(1,1)} \oplus \Gamma_{E_8\times E_8}$ with Wilson lines [42]. It drops out in the field theory limit where the radii of compactification $R \sim \sqrt{\alpha'}$ are sent to zero and we are left with the part coming from the oscillators

$$\mathcal{Z}^{0,0}_{(4,0)\text{het}}(\tau) \to \frac{1}{\bar{\eta}^{24}(\bar{\tau})}. \tag{2.12}$$

At a generic point in the moduli space, the 480 gauge bosons of the adjoint representation of $E_8 \times E_8$ get masses due to Wilson lines, and only the ℓ gauge bosons of the $U(1)^\ell$ group left invariant by the orbifold action [43,44] stay in the matter massless spectrum.

The oscillator part is computed to be

$$\mathcal{Z}^{h,g}_{\text{osc}} = \sum_{\{g,h\}}\prod_{p=0}^{N-1}(\mathcal{Z}^{h\times p, g\times p}_X)^\ell, \tag{2.13}$$

where the twisted bosonic chiral blocks $\mathcal{Z}^{h,g}_X$ are given in Appendix A. For $h = 0$, $\mathcal{Z}^{0,g}_{\text{osc}}$ is independent of g when N is prime and it can be computed explicitly. It is the inverse of the unique cusp form $f_k(\tau) = (\eta(\tau)\eta(N\tau))^{k+2}$ for $\Gamma_1(N)$ of modular weight $\ell = k+2 = 24/(N+1)$ with $n_v = 2\ell - 2$ as determined in Refs. [43,44]. Then (2.10) writes

$$\mathcal{Z}_{(4,0)\text{het}} \to \frac{1}{N}\left(\frac{1}{(\bar{\eta}(\bar{\tau}))^{24}} + \frac{N-1}{f_k(\bar{\tau})}\right). \tag{2.14}$$

To conclude this section, we write the part of the integrand of (2.2) that will contribute in the field theory limit. When $\alpha' \to 0$, the region of the fundamental domain of integration $\mathcal{F}$ of interest is the large τ_2 region, such that $t = \alpha'\tau_2$ stays constant. Then, the objects that we have introduced admit an expansion in the variable $q = e^{2i\pi\tau} \to 0$. We find

$$\mathcal{Z}_{(4,0)\text{het}} \to \frac{1}{\bar{q}} + 2 + n_v + o(\bar{q}). \tag{2.15}$$

Putting everything together and using the expansions given in (A16), we find that the integrand in (2.2) is given by

$$\begin{aligned} \mathcal{Z}_{(4,0)\text{het}}\mathcal{W}^B e^{\mathcal{Q}} \to{}& e^{\pi\alpha'\tau_2 Q}((\mathcal{W}^B e^{\mathcal{Q}})|_{\bar{q}} \\ &+ (n_v+2)(\mathcal{W}^B e^{\mathcal{Q}})|_{\bar{q}^0} + o(\bar{q})). \end{aligned} \tag{2.16}$$

Order $\bar{q}$ coefficients are present because of the $1/\bar{q}$ chiral tachyonic pole in the nonsupersymmetric sector of the theory. Since the integral over τ_1 of $\bar{q}^{-1}(\mathcal{W}^B e^{\mathcal{Q}})|_{\bar{q}^0}$ vanishes, as a consequence of the level matching condition, we did not write it. We introduce $\mathcal{A}$, the massless sector contribution to the field theory limit of the amplitude at the leading order in α', for later use in Secs. III and IV:

$$\begin{aligned} \mathcal{A}^{(n_v)}_{(4,0)\text{het}} ={}& \frac{1}{2}\left(\frac{\pi}{2}\right)^4 t_8F^4(\bar{\mathcal{W}}^B|_{\bar{q}}(1+\alpha'\delta\mathcal{Q}) \\ &+ \bar{\mathcal{W}}^B|_{\bar{q}^0}\mathcal{Q}|_{\bar{q}} + (n_v+2)\bar{\mathcal{W}}^B|_{\bar{q}^0}), \end{aligned} \tag{2.17}$$

where we have made use of the notations for the $\bar{q}$ expansion

$$\begin{aligned} \bar{\mathcal{W}}^B &= \bar{\mathcal{W}}^B|_{q^0} + \bar{q}\,\bar{\mathcal{W}}^B|_q + o(\bar{q}^2), \\ \mathcal{Q} &= -\pi\alpha'\tau_2 Q + \alpha'\delta Q + q\mathcal{Q}|_q + \bar{q}\mathcal{Q}|_{\bar{q}} + o(|q|^2). \end{aligned} \tag{2.18}$$

B. Type II asymmetric orbifold

In this section we consider type II string theory on two different kinds of asymmetric orbifolds. They lead to (4,0) models with a moduli space given in (2.1), where the axion-dilaton parametrizes the $SU(1,1)/U(1)$ factor. The first one is a $\mathbb{Z}_2$ orbifold with $n_v = 6$. The others are the Dabholkar-Harvey models [26,45]; they have $n_v = 0$ vector multiplet.

First, we give a general formula for the treatment of those asymmetric orbifolds. We then study in detail the partition function of two particular models and extract the contribution of massless states to the integrand of the field theory limit of their amplitudes. A generic expression for the scattering amplitude of four gravitons at one loop in type IIA and IIB superstring is

$$\mathcal{M}^{(n_v)}_{(4,0)II} = \mathcal{N}\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^{\frac{D-6}{2}}}\int_{\mathcal{T}}\prod_{1\le i<j\le 4}\frac{d^2\nu_i}{\tau_2}e^{\mathcal{Q}}\frac{1}{2}\sum_{a,b=0,1}(-1)^{a+b+ab}\mathcal{Z}_{a,b}\mathcal{W}_{a,b}\frac{1}{2|G|}\sum_{\substack{\bar a,\bar b=0,1\\ g,h}}(-1)^{\bar a+\bar b+\mu\bar a\bar b}(-1)^{C(\bar a,\bar b,g,h)}\bar{\mathcal{Z}}^{h,g}_{\bar a,\bar b}\tilde{\mathcal{W}}_{\bar a,\bar b}, \tag{2.19}$$

where $\mathcal{N}$ is the same normalization factor as for the heterotic string amplitude and $C(\bar a,\bar b,g,h)$ is a model-dependent phase factor determined by modular invariance and discussed below. We have introduced the chiral partition functions in the (a,b)-spin structure

$$\mathcal{Z}_{a,b} = \frac{\theta[^a_b](0|\tau)^4}{\eta(\tau)^{12}};\qquad \mathcal{Z}_{1,1}=0. \tag{2.20}$$

The value of μ determines the chirality of the theory: $\mu=0$ for type IIA and $\mu=1$ for type IIB. The partition function in a twisted sector (h,g) of the orbifold is denoted $\bar{\mathcal{Z}}^{h,g}_{\bar a,\bar b}$. Notice that the four-dimensional fermions are not twisted, so the vanishing of their partition function in the $(a,b)=(1,1)$ sector holds for a (g,h)-twisted sector: $\bar{\mathcal{Z}}^{h,g}_{1,1}=0$. This is fully consistent with the fact that, due to the lack of fermionic zero modes, this amplitude does not receive any contributions from the odd/odd, odd/even or even/odd spin structures. We use the holomorphic factorization of the (0,0)-ghost picture graviton vector operators as

$$V^{(0,0)} = \int d^2z {:}\epsilon^{(i)}\cdot V(z)\tilde\epsilon^{(i)}\cdot\bar V(\bar z)e^{ik\cdot X(z,\bar z)}{:} \tag{2.21}$$

with

$$\begin{aligned}\epsilon^{(i)}\cdot V(z) &= \epsilon^{(i)}\cdot\partial X - i\frac{F^{(i)}_{\mu\nu}}{2}{:}\psi^\mu\psi^\nu{:};\\ \tilde\epsilon^{(i)}\cdot\bar V(\bar z) &= \tilde\epsilon^{(i)}\cdot\bar\partial X + i\frac{\tilde F^{(i)}_{\mu\nu}}{2}{:}\bar\psi^\mu\bar\psi^\nu{:}\end{aligned} \tag{2.22}$$

where we have introduced the field strengths $F^{(i)}_{\mu\nu} = \epsilon^{(i)}_\mu k_{i\nu} - \epsilon^{(i)}_\nu k_{i\mu}$ and $\tilde F^{(i)}_{\mu\nu} = \tilde\epsilon^{(i)}_\mu k_{i\nu} - \tilde\epsilon^{(i)}_\nu k_{i\mu}$.

The correlators of the vertex operators in the (a,b)-spin structure are given by $\mathcal{W}_{a,b}$ and $\bar{\mathcal{W}}_{\bar a,\bar b}$ defined by, respectively,

$$\begin{aligned}\mathcal{W}_{a,b} &= \frac{\langle\prod_{j=1}^4\epsilon^{(j)}\cdot V(z_j)e^{ik_j\cdot X(z_j)}\rangle_{a,b}}{(2\alpha')^4\langle\prod_{j=1}^4 e^{ik_j\cdot X(z_j)}\rangle},\\ \bar{\mathcal{W}}_{\bar a,\bar b} &= \frac{\langle\prod_{j=1}^4\tilde\epsilon^{(j)}\cdot\bar V(\bar z_j)e^{ik_j\cdot X(\bar z_j)}\rangle_{\bar a,\bar b}}{(2\alpha')^4\langle\prod_{j=1}^4 e^{ik_j\cdot X(\bar z_j)}\rangle}.\end{aligned} \tag{2.23}$$

We decompose the $\mathcal{W}_{a,b}$ into one part that depends on the spin structure (a,b), denoted $\mathcal{W}^F_{a,b}$, and another independent of the spin structure $\mathcal{W}^B$:

$$\mathcal{W}_{a,b} = \mathcal{W}^F_{a,b} + \mathcal{W}^B, \tag{2.24}$$

this last term being identical to the one given in (2.3). The spin structure-dependent part is given by the following fermionic Wick's contractions:

$$\mathcal{W}^F_{a,b} = \mathcal{S}_{4;a,b} + \mathcal{S}_{2;a,b}, \tag{2.25}$$

where $\mathcal{S}_{n;a,b}$ arise from Wick contracting n pairs of worldsheet fermions. Note that the contractions involving three pairs of fermion turn out to vanish in all the type II models by symmetry. We introduce the notation $\sum_{\{(i,\dots),(j,\dots)\}=\{1,2,3,4\}}\dots$ for the sum over the ordered partitions of $\{1,2,3,4\}$ into two sets where the partitions $\{(1,2,3),1\}$ and $\{(1,3,2),1\}$ are considered to be independent. In that manner, the two terms in (2.25) can be written explicitly:

$$\begin{aligned}\mathcal{S}_{4;a,b} &= \frac{1}{2^{10}}\sum_{\{(i,j),(k,l)\}=\{1,2,3,4\}} S_{a,b}(z_{ij})S_{a,b}(z_{ji})S_{a,b}(z_{kl})S_{a,b}(z_{lk})\,\mathrm{tr}(F^{(i)}F^{(j)})\,\mathrm{tr}(F^{(k)}F^{(l)})\\ &\quad - \frac{1}{2^8}\sum_{\{(i,j,k,l)\}=\{1,2,3,4\}} S_{a,b}(z_{ij})S_{a,b}(z_{jk})S_{a,b}(z_{kl})S_{a,b}(z_{li})\,\mathrm{tr}(F^{(i)}F^{(j)}F^{(k)}F^{(l)}),\\ \mathcal{S}_{2;a,b} &= -\frac{1}{2^5}\sum_{\{(i,j),(k,l)\}=\{1,2,3,4\}} S_{a,b}(z_{ij})S_{a,b}(z_{ji})\,\mathrm{tr}(F^{(i)}F^{(j)})\Big(\epsilon^{(k)}\cdot\mathcal{Q}_k\epsilon^{(l)}\cdot\mathcal{Q}_l + \frac{1}{2\alpha'}\epsilon^{(k)}\cdot\epsilon^{(l)}\mathcal{T}(z_{kl})\Big).\end{aligned} \tag{2.26}$$

Because the orbifold action only affects the right-moving fermionic zero modes, the left movers are untouched and Riemann's identities imply (see Appendix A 2 for details)

$$\sum_{\substack{a,b=0,1\\ ab=0}} (-1)^{a+b+ab} \mathcal{Z}_{a,b} \mathcal{W}_{a,b} = \left(\frac{\pi}{2}\right)^4 t_8 F^4. \tag{2.27}$$

Notice that a contribution with less than four fermionic contractions vanishes. We now rewrite (2.19):

$$\mathcal{M}^{(6)}_{(4,0)II} = -\mathcal{N} \frac{1}{2}\left(\frac{\pi}{2}\right)^4 t_8 F^4 \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^{\frac{D-6}{2}}} \int_{\mathcal{T}} \prod_{1\le i<j\le 4} \frac{d^2\nu_i}{\tau_2} e^{\mathcal{Q}} \frac{1}{2|G|} \sum_{\substack{\bar a,\bar b=0,1\\ g,h}} (-1)^{\bar a+\bar b+\mu \bar a\bar b} (-1)^{C(\bar a,\bar b,g,h)} \bar{\mathcal{Z}}^{h,g}_{\bar a,\bar b} \bar{\mathcal{W}}_{\bar a,\bar b}. \tag{2.28}$$

For the class of asymmetric $\mathbb{Z}_N$ orbifolds with n_v vector multiplets studied here, the partition function $\mathcal{Z}^{(\mathrm{asym})}_{a,b} = |G|^{-1}\sum_{g,h}(-1)^{C(a,b,g,h)}\mathcal{Z}^{g,h}_{a,b}$ has the following low-energy expansion:

$$\mathcal{Z}^{(\mathrm{asym})}_{0,0} = \frac{1}{\sqrt{q}} + n_v + 2 + o(q); \qquad \mathcal{Z}^{(\mathrm{asym})}_{0,1} = \frac{1}{\sqrt{q}} - (n_v + 2) + o(q); \qquad \mathcal{Z}^{(\mathrm{asym})}_{1,0} = 0 + o(q). \tag{2.29}$$

Because the four-dimensional fermionic zero modes are not saturated we have $\mathcal{Z}^{\mathrm{asym}}_{1,1} = 0$.

Since in those constructions no massless mode arises in the twisted $h \neq 0$ sector, this sector decouples. Hence, at $o(\bar q)$ one has the following relation:

$$\sum_{\bar a,\bar b=0,1} (-1)^{a+b+ab} \bar{\mathcal{Z}}^{(\mathrm{asym})}_{\bar a,\bar b} \bar{\mathcal{W}}_{\bar a,\bar b} \to (\bar{\mathcal{W}}_{0,0} - \bar{\mathcal{W}}_{0,1})|_{\sqrt{\bar q}} + (n_v + 2)(\bar{\mathcal{W}}_{0,0} + \bar{\mathcal{W}}_{0,1})|_{q^0}. \tag{2.30}$$

The contribution of massless states to the field theory amplitude is given by

$$\mathcal{A}^{(n_v)}_{(4,0)II} = \frac{1}{4}\left(\frac{\pi}{2}\right)^4 t_8 F^4 ((\bar{\mathcal{W}}_{0,0} - \bar{\mathcal{W}}_{0,1})|_{\sqrt{\bar q}} + (n_v + 2)(\bar{\mathcal{W}}_{0,0} + \bar{\mathcal{W}}_{0,1})|_{q^0}). \tag{2.31}$$

Using the Riemann identity (2.27) we can rewrite this expression in the following form:

$$\mathcal{A}^{(n_v)}_{(4,0)II} = \frac{1}{4}\left(\frac{\pi}{2}\right)^4 t_8 F^4 \left(\left(\frac{\pi}{2}\right)^4 t_8 \tilde F^4 + (n_v - 6)(\bar{\mathcal{W}}_{0,0} + \bar{\mathcal{W}}_{0,1})|_{q^0} + 16\bar{\mathcal{W}}_{1,0}|_{q^0}\right). \tag{2.32}$$

Higher powers of $\bar q$ in $\mathcal{W}_{a,b}$ or in $\mathcal{Q}$ are suppressed in the field theory limit that we discuss in Sec. IV.

At this level, this expression is not identical to the one derived in the heterotic construction (2.17). The type II and heterotic (4,0) string models with n_v vector multiplets are dual to each other under the transformation $S \to -1/S$, where S is the axion-dilaton scalar in the $\mathcal{N} = 4$ supergravity multiplet. We will see in Sec. III that for the four-graviton amplitudes we obtain the same answer after integrating out the real parts of the positions of the vertex operators.

We now illustrate this analysis on the examples of the asymmetric orbifold with six or zero vector multiplets.

1. Example: A model with six vector multiplets

Let us compute the partition function of the asymmetric orbifold obtained by the action of the right-moving fermion counting operator $(-1)^{F_R}$ and a $\mathbb{Z}_2$ action on the torus T^6 [29,46]. The effect of the $(-1)^{F_R}$ orbifold is to project out the 16 vector multiplets arising from the R/R sector, while preserving supersymmetry on the right-moving sector. The moduli space of the theory is given by (2.1) with $n_v = 6$ and $\Gamma = \Gamma(2)$ (see Ref. [29] for instance).

The partition function for the (4,0) CFT $\mathbb{Z}_2$ asymmetric orbifold model $\mathcal{Z}^{(\mathrm{asym}),(n_v=6)}_{a,b} = \frac{1}{2}\sum_{g,h}\mathcal{Z}^{g,h}_{a,b}$ with

$$\mathcal{Z}^{h,g}_{a,b}(w) := (-1)^{ag+bh+gh} \mathcal{Z}_{a,b} \Gamma_{(4,4)} \Gamma^{w}_{(2,2)}\begin{bmatrix} h \\ g \end{bmatrix}, \tag{2.33}$$

where the shifted lattice sum $\Gamma^{w}_{(2,2)}[^h_g]$ is given in Ref. [29] and recalled in Appendix B. The chiral blocks $\mathcal{Z}_{a,b}$ have been defined in (2.20) and $\Gamma_{(4,4)}$ is the lattice sum of the T^4. Using the fact that $\Gamma^{w}_{(2,2)}[^h_g]$ reduces to 0 for $h = 1$ and to 1 for $h = 0$ and that $\Gamma_{(4,4)} \to 1$ in the field theory limit, we see that the partition function is unchanged in the sectors $(a, b) = (0, 0)$ and (0,1) while for the $(a, b) = (1, 0)$ sector, the $(-1)^{ag}$ in (2.33) cancels the partition function when summing over g. One obtains the following result:

$$\mathcal{Z}^{(\mathrm{asym}),(n_v=6)}_{0,0} = \mathcal{Z}_{0,0}; \qquad \mathcal{Z}^{(\mathrm{asym}),(n_v=6)}_{0,1} = \mathcal{Z}_{0,1}; \qquad \mathcal{Z}^{(\mathrm{asym}),(n_v=6)}_{1,0} = 0. \tag{2.34}$$

Using (A6), one checks directly that it corresponds to (2.29) with $n_v = 6$.

2. Example: Models with zero vector multiplet

Now we consider the type II asymmetric orbifold models with zero vector multiplets constructed in Ref. [26] and discussed in Ref. [45].

Those models are compactifications of the type II superstring on a six-dimensional torus with an appropriate choice for the value of the metric G_{ij} and B-field B_{ij}. The Narain lattice is given by $\Gamma^{DH} = \{p_L, p_R; p_L, p_R \in \Lambda_W(\mathfrak{g}), p_L - p_R \in \Lambda_R(\mathfrak{g})\}$, where $\Lambda_R(\mathfrak{g})$ is the root lattice of a simply laced semisimple Lie algebra $\mathfrak{g}$ and $\Lambda_W(\mathfrak{g})$ is the weight lattice.

The asymmetric orbifold action is given by $|p_L, p_R\rangle \to e^{2i\pi p_L\cdot v_L}|p_L, g_R p_R\rangle$, where g_R is an element of the Weyl group of $\mathfrak{g}$ and v_L is a shift vector appropriately chosen to avoid any massless states in the twisted sector [26,45]. With such a choice of shift vector and because the asymmetric orbifold action leaves p_L invariant, we have a (4,0) model of four-dimensional supergravity with no vector multiplets.

The partition function is given by

$$Z^{\rm asym}_{\bar a,\bar b} = \frac{\theta[^{\bar a}_{\bar b}]}{(\eta(\bar\tau))^3}\frac{1}{|G|}\sum_{\{g_j,h_j\}}\prod_{i=1}^{3} Z^{h_j,g_j}_{\bar a,\bar b}, \tag{2.35}$$

where the sum runs over the sectors of the orbifold. For instance, in the $\mathbb{Z}_9$ model of Dabholkar and Harvey, one has $g_j \in j\times\{\frac{2}{9},\frac{4}{9},\frac{8}{9}\}$ with $j = 0,\ldots,8$ and the same for h_j. The twisted conformal blocks are

$$Z^{h,g}_{\bar a,\bar b} = \begin{cases} \left(\frac{\theta[^{\bar a}_{\bar b}]}{\eta(\bar\tau)}\right)^3 \times \left(\frac{1}{\eta(\bar\tau)}\right)^6 & \text{if } (g,h) = (0,0) \mod 2, \\ e^{i\frac{\pi}{2}a(g-b)} 2\sin\left(\frac{\pi g}{2}\right)\frac{\theta[^{a+h}_{b+g}]}{\theta[^{1+h}_{1+g}]} & \forall\ (g,h) \neq (0,0) \mod 2. \end{cases} \tag{2.36}$$

The phase in (2.19) is determined by modular invariance to be $C(\bar a, \bar b, g_R, h_R) = \sum_{i=1}^3(\bar a g^i_R + \bar b h^i_R + g^i_R h^i_R)$.

In the field theory limit, we perform the low-energy expansion of this partition function and we find that it takes the following form for all of the models in Refs. [26,45]:

$$\begin{aligned} Z^{({\rm asym}),(n_v=0)}_{0,0} &= \frac{1}{\sqrt{q}} + 2 + o(q); \\ Z^{({\rm asym}),(n_v=0)}_{0,1} &= \frac{1}{\sqrt{q}} - 2 + o(q); \\ Z^{({\rm asym}),(n_v=0)}_{1,0} &= 0 + o(q), \end{aligned} \tag{2.37}$$

which is (2.29) with $n_v = 0$ as expected.

C. Type II symmetric orbifold

In this section we consider (2,2) models of four-dimensional $\mathcal{N} = 4$ supergravity. These models can be obtained from the compactification of type II string theory on symmetric orbifolds of $K_3 \times T^2$. The difference with the heterotic models considered in Sec. II A is that the scalar parametrizing the coset space $SU(1,1)/U(1)$ that used to be the axiodilaton S is now the Kähler modulus of the two-torus T^2 for the type IIA case or complex structure modulus for the type IIB case. The nonperturbative duality relation between these two models is discussed in detail in Refs. [29,39].

Models with $n_v \geq 2$ have been constructed in Ref. [36]. The model with $n_v = 22$ is a $T^4/\mathbb{Z}_2 \times T^2$ orbifold, and the following models with $n_v \in \{14, 10\}$ are successive $\mathbb{Z}_2$ orbifolds of the first one. The model with $n_v = 6$ is a freely acting $\mathbb{Z}_2$ orbifold of the $T^4/\mathbb{Z}_2 \times T^2$ theory that simply projects out the 16 vector multiplets of the R/R sector. The four-graviton amplitude can be effectively written in terms of the (g, h) sectors of the first $\mathbb{Z}_2$ orbifold of the T^4, and writes

$$\begin{aligned}\mathcal{M}^{(n_v)}_{(2,2)} = \mathcal{N}\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^{\frac{D-6}{2}}}\int_{\mathcal{T}}\prod_{1\le i<j\le 4}\frac{d^2\nu_i}{\tau_2}e^{\mathcal{Q}}\frac{1}{4|G|} \\ \times\sum_{h,g=0}^{1}\sum_{\substack{a,b=0,1\\ \bar a,\bar b=0,1}}(-1)^{a+b+ab}(-1)^{\bar a+\bar b+\bar a\bar b}Z^{h,g,(n_v)}_{a,b} \\ \times \bar Z^{h,g,(n_v)}_{\bar a,\bar b}(\mathcal{W}_{a,b}\bar{\mathcal{W}}_{\bar a,\bar b} + \mathcal{W}_{a,b;\bar a,\bar b}),\end{aligned} \tag{2.38}$$

where $\mathcal{N}$ is the same overall normalization as for the previous amplitudes and $Z^{h,g,(n_v)}_{a,b}$ is defined in Appendix B. The term $\mathcal{W}_{a,b;\bar a,\bar b}$ is a mixed term made of contractions between holomorphic and antiholomorphic fields. It does not appear in the (4,0) constructions since the left/right contractions vanish due to the totally unbroken supersymmetry in the left-moving sector.

Two types of contributions arise from the mixed correlators

$$\begin{aligned}\mathcal{W}^1_{a,b;\bar a,\bar b} &= \frac{\langle :\epsilon^{(i)}\cdot\partial X\tilde\epsilon^{(i)}\cdot\bar\partial X::\epsilon^{(j)}\cdot\partial X\tilde\epsilon^{(j)}\cdot\bar\partial X: \prod_{r=1}^4 e^{ik_r\cdot X(z_r)}\rangle_{a,b;\bar a,\bar b}}{(2\alpha')^4\langle\prod_{j=1}^4 e^{ik_j\cdot X(z_j)}\rangle}; \\ \mathcal{W}^2_{a,b;\bar a,\bar b} &= \frac{\langle\epsilon^{(i)}\cdot\partial X\epsilon^{(j)}\cdot\partial X\tilde\epsilon^{(k)}\cdot\bar\partial X\tilde\epsilon^{(l)}\cdot\bar\partial X\prod_{r=1}^4 e^{ik_r\cdot X(z_r)}\rangle_{a,b;\bar a,\bar b}}{(2\alpha')^4\langle\prod_{j=1}^4 e^{ik_j\cdot X(z_j)}\rangle}, \qquad (ij)\neq(kl),\end{aligned} \tag{2.39}$$

with at least one operator product expansion between a holomorphic and an antiholomorphic operator. Explicitly, we find

$$\begin{aligned}\mathcal{W}_{a,b;\bar a;\bar b} = &\sum_{\substack{\{i,j,k,l\}\in\{1,2,3,4\}\\(i,j)\neq(k,l)}} (S_{a,b}(\nu_{ij}))^2(\bar S_{\bar a,\bar b}(\bar\nu_{kl}))^2\times \mathrm{tr}(F^{(i)}F^{(j)})\,\mathrm{tr}(F^{(k)}F^{(l)})\times(\epsilon^{(k)}\cdot\tilde\epsilon^{(i)}\hat{\mathcal{T}}(k,i)(\epsilon^{(l)}\cdot\tilde\epsilon^{(j)}\hat{\mathcal{T}}(l,j)+\epsilon^{(l)}\cdot\mathcal{Q}_l\tilde\epsilon^{(j)}\cdot\bar{\mathcal{Q}}_j)\\ &+(i\leftrightarrow j)+\epsilon^{(k)}\cdot\mathcal{Q}_k(\epsilon^{(l)}\cdot\tilde\epsilon^{(i)}\tilde\epsilon^{(j)}\cdot\bar{\mathcal{Q}}_j\hat{\mathcal{T}}(l,i)+\epsilon^{(l)}\cdot\tilde\epsilon^{(j)}\hat{\mathcal{T}}(l,j)\tilde\epsilon^{(i)}\cdot\bar{\mathcal{Q}}_i))\\ &+\sum_{\{i,j,k,l\}\in\{1,2,3,4\}}|S_{a,b}(\nu_{ij})|^4\times(\mathrm{tr}(F^{(i)}F^{(j)}))^2(\epsilon^{(k)}\cdot\tilde\epsilon^{(l)}\hat{\mathcal{T}}(k,l)(\epsilon^{(l)}\cdot\tilde\epsilon^{(k)}\hat{\mathcal{T}}(l,k)+\epsilon^{(l)}\cdot\mathcal{Q}_l\tilde\epsilon^{(k)}\cdot\bar{\mathcal{Q}}_k)+(k\leftrightarrow l)),\end{aligned}\tag{2.40}$$

where

$$\begin{aligned}\hat{\mathcal{T}}(i,j) &:= \partial_{\nu_i}\bar\partial_{\bar\nu_j}\mathcal{P}(\nu_i-\nu_j|\tau)\\ &= \frac{\pi}{4}\Big(\frac{1}{\tau_2}-\delta^{(2)}(\nu_i-\nu_j)\Big).\end{aligned}\tag{2.41}$$

Forgetting about the lattice sum, which at any rate is equal to one in the field theory limit,

$$\begin{aligned}\mathcal{Z}^{h,g}_{a,b} &= c_h\frac{\Big(\theta\big[{a\atop b}\big](0|\tau)\Big)^2\theta\big[{a+h\atop b+g}\big]\Big(0|\tau)\theta\big[{a-h\atop b-g}\big](0|\tau)}{(\eta(\tau))^6\Big(\theta\big[{1+h\atop 1+g}\big](0|\tau)\Big)^2}\\ &= c_h(-1)^{(a+h)g}\left(\frac{\theta\big[{a\atop b}\big]\Big(0|\tau)\theta\big[{a+h\atop b+g}\big](0|\tau)}{(\eta(\tau))^3\theta\big[{1+h\atop 1+g}\big](0\tau)}\right)^2,\end{aligned}\tag{2.42}$$

where c_h is an effective number whose value depends on h in the following way: $c_0=1$ and $c_1=\sqrt{n_v-6}$. This number represents the successive halving of the number of twisted R/R states. We refer to Appendix B for details.

The sum over the spin structures in the untwisted sector $(g,h)=(0,0)$ is once again performed using Riemann's identities:

$$\sum_{\substack{a,b=0,1\\ab=0}}(-1)^{a+b+ab}\mathcal{Z}^{0,0}_{a,b}\mathcal{W}_{a,b}=\Big(\frac{\pi}{2}\Big)^4 t_8F^4.\tag{2.43}$$

In the twisted sectors $(h,g)\neq(0,0)$ we remark that $\mathcal{Z}^{0,1}_{0,1}=\mathcal{Z}^{0,1}_{0,0}$, $\mathcal{Z}^{1,0}_{1,0}=\mathcal{Z}^{1,0}_{0,0}$, $\mathcal{Z}^{1,1}_{1,0}=\mathcal{Z}^{1,1}_{0,1}$, and $\mathcal{Z}^{1,0}_{0,1}=\mathcal{Z}^{0,1}_{1,0}=\mathcal{Z}^{1,1}_{0,0}=0$, which gives for the chiral blocks in (2.38)

$$\begin{aligned}\sum_{\substack{a,b=0,1\\ab=0}}(-1)^{a+b+ab}\mathcal{Z}^{0,1}_{a,b}\mathcal{W}_{a,b}&=\mathcal{Z}^{0,1}_{0,0}(\mathcal{W}_{0,0}-\mathcal{W}_{0,1}),\\ \sum_{\substack{a,b=0,1\\ab=0}}(-1)^{a+b+ab}\mathcal{Z}^{1,0}_{a,b}\mathcal{W}_{a,b}&=\mathcal{Z}^{1,0}_{0,0}(\mathcal{W}_{0,0}-\mathcal{W}_{1,0}),\\ \sum_{\substack{a,b=0,1\\ab=0}}(-1)^{a+b+ab}\mathcal{Z}^{1,1}_{a,b}\mathcal{W}_{a,b}&=\mathcal{Z}^{1,1}_{0,1}(\mathcal{W}_{0,1}-\mathcal{W}_{1,0}).\end{aligned}\tag{2.44}$$

Therefore the factorized terms in the correlator take the simplified form

$$\begin{aligned}\frac{1}{4|G|}\sum_{g,h}\sum_{\substack{a,b=0,1\\\bar a,\bar b=0,1}}&(-1)^{a+b+ab}(-1)^{\bar a+\bar b+\bar a\bar b}\mathcal{Z}^{h,g}_{a,b}\bar{\mathcal{Z}}^{h,g}_{\bar a,\bar b}\mathcal{W}_{a,b}\bar{\mathcal{W}}_{\bar a,\bar b}\\ &=\frac{1}{8}\Big(\frac{\pi}{2}\Big)^8 t_8t_8R^4+\frac{1}{8}|\mathcal{Z}^{0,1}_{0,0}(\mathcal{W}_{0,0}-\mathcal{W}_{0,1})|^2\\ &\quad+\frac{1}{8}|\mathcal{Z}^{1,0}_{0,0}(\mathcal{W}_{0,0}-\mathcal{W}_{1,0})|^2\\ &\quad+\frac{1}{8}|\mathcal{Z}^{1,1}_{0,1}(\mathcal{W}_{0,1}-\mathcal{W}_{1,0})|^2,\end{aligned}\tag{2.45}$$

where $t_8t_8R^4$ is the Lorentz scalar built from four powers of the Riemann tensor arising at the linearized level as the product $t_8t_8R^4=t_8F^4t_8\tilde F^4$.[2]

The mixed terms can be treated in the same way with the result

$$\begin{aligned}\frac{1}{4|G|}\sum_{g,h}\sum_{\substack{a,b=0,1\\\bar a,\bar b=0,1}}&(-1)^{a+b+ab}(-1)^{\bar a+\bar b+\bar a\bar b}\mathcal{Z}^{h,g}_{a,b}\bar{\mathcal{Z}}^{h,g}_{\bar a,\bar b}\mathcal{W}_{a,b;\bar a,\bar b}\\ &=\frac{1}{8}|\mathcal{Z}^{0,1}_{0,0}|^2(\mathcal{W}_{0,0;0,0}-\mathcal{W}_{0,1;0;1})\\ &\quad+\frac{1}{8}|\mathcal{Z}^{1,0}_{0,0}|^2(\mathcal{W}_{0,0;0,0}-\mathcal{W}_{1,0;1,0})\\ &\quad+\frac{1}{8}|\mathcal{Z}^{1,1}_{0,1}|^2(\mathcal{W}_{0,1;0,1}-\mathcal{W}_{1,0;1,0}).\end{aligned}\tag{2.46}$$

Since the conformal blocks $\mathcal{Z}^{h,g}_{a,b}$ have the q expansion [see (A3)]

$$\mathcal{Z}^{0,1}_{0,0}=\frac{1}{\sqrt{q}}+4\sqrt{q}+o(q);\qquad \mathcal{Z}^{1,0}_{0,0}=4\sqrt{n_v-6}+o(q);$$
$$\mathcal{Z}^{1,1}_{0,1}=4\sqrt{n_v-6}+o(q),\tag{2.47}$$

the massless contribution to the integrand of (2.45) is given by

[2]This Lorentz scalar is the one obtained from the four-graviton tree amplitude $t_8t_8R^4=stuM^{\mathrm{tree}}(1,2,3,4)$ setting Newton's constant to one.

$$A^{(n_v)}_{(2,2)} = \frac{1}{8}\left(\frac{\pi}{2}\right)^8 t_8 t_8 R^4 + \frac{1}{8}|\mathcal{W}_{0,0}|_{\sqrt{q}} - \mathcal{W}_{0,1}|_{\sqrt{q}}|^2 + \frac{1}{8}(\mathcal{W}_{0,0;0,0}|_{\sqrt{q}} - \mathcal{W}_{0,1;0,1}|_{\sqrt{q}}) + 2(n_v - 6) \times (|\mathcal{W}_{0,0}|_{q^0} - \mathcal{W}_{1,0}|_{q^0}|^2 + |\mathcal{W}_{0,1}|_{q^0} - \mathcal{W}_{1,0}|_{q^0}|^2) + 2(n_v - 6)(\mathcal{W}_{0,0;0,0}|_{q^0;\bar{q}^0} + \mathcal{W}_{0,1;0,1}|_{q^0;\bar{q}^0} - 2\mathcal{W}_{1,0;1,0}|_{q^0;\bar{q}^0}). \tag{2.48}$$

We notice that the bosonic piece $\mathcal{W}^B$ in $\mathcal{W}_{a,b}$ in (2.24) cancels in each term of the previous expression, due to the minus sign between the $\mathcal{W}_{a,b}$'s in the squares.

The integrand of the four-graviton amplitude takes a different form in the (2,2) construction compared with the expression for the (4,0) constructions heterotic in (2.17) and asymmetric type II models in (2.32). We will show that after taking the field theory limit and performing the integrals the amplitudes will turn out to be the same.

As mentioned above, for a given number of vector multiplets the type II (2,2) models are only nonperturbatively equivalent (U duality) to the (4,0) models. However, we will see that this nonperturbative duality does not affect the perturbative one-loop multigraviton amplitudes. Nevertheless, we expect that both α' corrections to those amplitudes and amplitudes with external scalars and vectors should be model dependent.

In the next section, we analyze the relationships between the string theory models.

III. COMPARISON OF THE STRING MODELS

A. Massless spectrum

The spectrum of the type II superstring in ten dimensions is given by the following sectors in the Gliozzi-Scherk-Olive projection the graviton G_{MN}, the B-field B_{MN}, and the dilaton Φ come from the NS/NS sector, the gravitini ψ^M, and the dilatini λ come from the R/NS and NS/R sectors, while the one-form C_M and three-form C_{MNP} come from the R/R sector in the type IIA string. The dimensional reduction of type II string on a torus preserves all of the 32 supercharges and leads to the $\mathcal{N} = 8$ supergravity multiplet in four dimensions.

The reduction to $\mathcal{N} = 4$ supersymmetry preserves 16 supercharges and leads to the following content. The NS/NS sector contributes to the $\mathcal{N} = 4$ supergravity multiplet and to six vector multiplets. The R/R sector contributes to the $\mathcal{N} = 4$ spin-$\frac{3}{2}$ multiplet and to the vector multiplets with a multiplicity depending on the model.

In the partition function, the first Riemann vanishing identity

$$\sum_{a,b=0,1} (-1)^{a+b+ab} \mathcal{Z}_{a,b} = 0, \tag{3.1}$$

reflects the action of the $\mathcal{N} = 4$ supersymmetry inside the one-loop amplitudes in the following manner. The q expansion of this identity gives

$$\left(\frac{1}{\sqrt{q}} + 8 + o(\sqrt{q})\right) - \left(\frac{1}{\sqrt{q}} - 8 + o(\sqrt{q})\right) - (16 + o(q)) = 0. \tag{3.2}$$

The first two terms are the expansion of $\mathcal{Z}_{0,0}$ and $\mathcal{Z}_{0,1}$, and the last one is the expansion of $\mathcal{Z}_{1,0}$. The cancellation of the $1/\sqrt{q}$ terms shows that the Gliozzi-Scherk-Olive projection eliminates the tachyon from the spectrum, and at the order q^0 the cancellation results in the matching between the bosonic and fermionic degrees of freedom.

In the amplitudes, chiral $\mathcal{N} = 4$ supersymmetry implies the famous Riemann identities, stating that for $0 \leq n \leq 3$ external legs, the one-loop n-point amplitude vanishes [see Eq. (A25)]. At four points it gives

$$\sum_{a,b=0,1} (-1)^{a+b+ab} \mathcal{Z}_{a,b} \mathcal{W}_{a,b} = \left(\frac{\pi}{2}\right)^4 t_8 F^4. \tag{3.3}$$

In $\mathcal{W}_{a,b}$ [see (2.25)] the term independent of the spin structure $\mathcal{W}^B$ and the terms with less than four fermionic contractions $\mathcal{S}_{2;a,b}$ cancel in the previous identity. The cancellation of the tachyon yields at the first order in the q expansion of (3.3)

$$\mathcal{W}_{0,0}|_{q^0} - \mathcal{W}_{0,1}|_{q^0} = 0. \tag{3.4}$$

The next term in the expansion gives an identity describing the propagation of the $\mathcal{N} = 4$ super-Yang-Mills multiplet in the loop

$$8(\mathcal{W}_{0,0}|_{q^0} + \mathcal{W}_{0,1}|_{q^0} - 2\mathcal{W}_{1,0}|_{q^0}) + (\mathcal{W}_{0,0}|_{\sqrt{q}} - \mathcal{W}_{0,1}|_{\sqrt{q}}) = \left(\frac{\pi}{2}\right)^4 t_8 F^4. \tag{3.5}$$

In this equation, one should have vectors, spinors and scalars propagating according to the sector of the theory. In $\mathcal{W}_{a,b}$, $a = 0$ is the NS sector, and $a = 1$ is the Ramond sector. The scalars have already been identified in (3.4) and correspond to $\mathcal{W}_{0,0}|_{q^0} + \mathcal{W}_{0,1}|_{q^0}$. The vector, being a massless bosonic degree of freedom, should then correspond to $\mathcal{W}_{0,0}|_{\sqrt{q}} - \mathcal{W}_{0,1}|_{\sqrt{q}}$. Finally, the fermions correspond to $\mathcal{W}_{1,0}|_{q^0}$. The factor of 8 in front of the first term is the number of degrees of freedom of a vector in ten dimensions; one can check that the number of bosonic degrees of freedom matches the number of fermionic degrees of freedom.

B. Amplitudes and supersymmetry

In this section we discuss the relationships between the four-graviton amplitudes in the various $\mathcal{N} = 4$ supergravity models in the field theory limit. We apply the logic of the previous section about the spectrum of left or right movers to the tensor product spectrum and see that we can precisely identify the contributions to the amplitude, both in the (4,0) and (2,2) models. The complete evaluation of the amplitudes will be performed in Sec. IV.

As mentioned above, the field theory limit is obtained by considering the large τ_2 region, and the integrand of the field theory amplitude is given by

$$A_X^{(n_v)} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \prod_{i=1}^{4} d\nu_i^1 \mathcal{A}_X^{(n_v)}, \tag{3.6}$$

where $X \in \{(4,0)\mathrm{het}, (4,0)II, (2,2)\}$ indicates the model, as in (2.17) and (2.32) or (2.48), respectively.

At one loop this quantity is the sum of the contribution from n_v $\mathcal{N} = 4$ vector (spin-1) supermultiplets running in the loop and the $\mathcal{N} = 4$ spin-2 supermultiplet

$$A_X^{(n_v)} = A_X^{\text{spin } 2} + n_v A_X^{\text{spin } 1}. \tag{3.7}$$

For the case of the type II asymmetric orbifold models with n_v vector multiplets we deduce from (2.32)

$$A_{(4,0)II}^{\text{spin } 1} = \frac{1}{4}\left(\frac{\pi}{2}\right)^4 t_8 F^4 \int_{-\frac{1}{2}}^{\frac{1}{2}} \prod_{i=1}^{4} d\nu_i^1 (\mathcal{W}_{0,0}|_{q^0} + \mathcal{W}_{0,1}|_{q^0}). \tag{3.8}$$

Since $t_8 F^4$ is the supersymmetric left-moving sector contribution [recall the supersymmetry identity in (3.3)], it corresponds to an $\mathcal{N} = 4$ vector multiplet and we recognize in (3.8) the product of this multiplet with the scalar from the right-moving sector:

$$(1_{\mathbf{1}}, 1/2_{\mathbf{4}}, 0_{\mathbf{6}})_{\mathcal{N}=4} = (1_{\mathbf{1}}, 1/2_{\mathbf{4}}, 0_{\mathbf{6}})_{\mathcal{N}=4} \otimes (0_{\mathbf{1}})_{\mathcal{N}=0}. \tag{3.9}$$

This agrees with the identification made in the previous subsection where $\mathcal{W}_{0,0}|_{q^0} + \mathcal{W}_{0,1}|_{q^0}$ was argued to be a scalar contribution.

The contribution from the $\mathcal{N} = 4$ supergravity multiplet running in the loop is given by

$$A_{(4,0)II}^{\text{spin } 2} = \frac{1}{4}\left(\frac{\pi}{2}\right)^4 t_8 F^4 \int_{-\frac{1}{2}}^{\frac{1}{2}} \prod_{i=1}^{4} d\nu_i^1 [2(\mathcal{W}_{0,0}|_{q^0} + \mathcal{W}_{0,1}|_{q^0}) + (\mathcal{W}_{0,0}|_{\sqrt{q}} - \mathcal{W}_{0,1}|_{\sqrt{q}})]. \tag{3.10}$$

The factor of 2 is the number of degrees of freedom of a vector in four dimensions. Since $\mathcal{Z}_{1,0}^{\text{asym}} = 0 + o(q)$ for the (4,0) model asymmetric orbifold construction, the integrand of the four-graviton amplitude in (2.29) does not receive any contribution from the right-moving R sector. Stated differently, the absence of $\mathcal{W}_{1,0}$ implies that both R/R and NS/R sectors are projected out, leaving only the contribution from the NS/NS and R/NS. Thus, the four $\mathcal{N} = 4$ spin-$\frac{3}{2}$ supermultiplets and 16 $\mathcal{N} = 4$ spin-1 supermultiplets are projected out, leaving at most six vector multiplets. This number is further reduced to zero in the Dabholkar-Harvey construction [26].

From (3.10) we recognize that the $\mathcal{N} = 4$ supergravity multiplet is obtained by the following tensor product:

$$(2_{\mathbf{1}}, 3/2_{\mathbf{4}}, 1_{\mathbf{6}}, 1/2_{\mathbf{4}}, 0_{\mathbf{2}})_{\mathcal{N}=4} = (1_{\mathbf{1}}, 1/2_{\mathbf{4}}, 0_{\mathbf{6}})_{\mathcal{N}=4} \otimes (1_{\mathbf{1}})_{\mathcal{N}=0}. \tag{3.11}$$

The two real scalars arise from the trace part and the antisymmetric part (after dualization in four dimensions) of the tensorial product of the two vectors. Using the identification of $\mathcal{W}_{0,0}|_{q^0} + \mathcal{W}_{0,1}|_{q^0}$ with a scalar contribution and Eq. (A31) we can now identify $\mathcal{W}_{0,0}|_{\sqrt{q}} - \mathcal{W}_{0,0}|_{\sqrt{q}}$ with the contribution of a vector and two scalars. This confirms the identification of $\mathcal{W}_{1,0}|_{q^0}$ with a spin-$\frac{1}{2}$ contribution in the end of Sec. III A.

Since

$$(3/2_{\mathbf{1}}, 1_{\mathbf{4}}, 1/2_{\mathbf{6}+\mathbf{1}}, 0_{\mathbf{4}+\bar{\mathbf{4}}})_{\mathcal{N}=4} = (1_{\mathbf{1}}, 1/2_{\mathbf{4}}, 0_{\mathbf{6}})_{\mathcal{N}=4} \otimes (1/2)_{\mathcal{N}=0}, \tag{3.12}$$

we see that removing the four spin $\frac{1}{2}$ (that is, the term $\mathcal{W}_{1,0}|_{q^0}$) of the right-moving massless spectrum of the string theory construction in asymmetric type II models removes the contribution from the massless spin $\frac{3}{2}$ to the amplitudes. For the asymmetric type II model, using (3.5), we can present the contribution from the $\mathcal{N} = 4$ supergravity multiplet in a form that reflects the decomposition of the $\mathcal{N} = 8$ supergravity multiplet into $\mathcal{N} = 4$ supermultiplets:

$$\begin{aligned}(2_{\mathbf{1}}, 3/2_{\mathbf{8}}, 1_{\mathbf{28}}, 1/2_{\mathbf{56}}, 0_{\mathbf{70}})_{\mathcal{N}=8} &= (2_{\mathbf{1}}, 3/2_{\mathbf{4}}, 1_{\mathbf{6}}, 1/2_{\mathbf{4}}, 0_{\mathbf{1}+\mathbf{1}})_{\mathcal{N}=4} \\ &\oplus \mathbf{4}(3/2_{\mathbf{1}}, 1_{\mathbf{4}}, 1/2_{\mathbf{6}+\mathbf{1}}, 0_{\mathbf{4}+\bar{\mathbf{4}}})_{\mathcal{N}=4} \\ &\oplus \mathbf{6}(1_{\mathbf{1}}, 1/2_{\mathbf{4}}, 0_{\mathbf{6}})_{\mathcal{N}=4},\end{aligned} \tag{3.13}$$

as

$$A_{(4,0)II}^{\text{spin } 2} = A_{\mathcal{N}=8}^{\text{spin } 2} - 6A_{(4,0)II}^{\text{spin } 1} - 4A_{(4,0)II}^{\text{spin } \frac{3}{2}}, \tag{3.14}$$

where we have introduced the $\mathcal{N} = 8$ spin-2 supergravity contribution

$$A_{\mathcal{N}=8}^{\text{spin } 2} = \frac{1}{4}\left(\frac{\pi}{2}\right)^8 t_8 t_8 R^4, \tag{3.15}$$

and the $\mathcal{N} = 4$ spin-$\frac{3}{2}$ supergravity contribution

$$A_{(4,0)II}^{\text{spin } \frac{3}{2}} = -\left(\frac{\pi}{2}\right)^4 t_8 F^4 \int_{-\frac{1}{2}}^{\frac{1}{2}} \prod_{i=1}^{4} d\nu_i^1 \mathcal{W}_{1,0}|_{q^0}. \tag{3.16}$$

For the (2,2) models the contribution of the massless states to the amplitude is given in (2.48). The contribution from a vector multiplet is

$$\begin{aligned}A_{(2,2)}^{\text{spin } 1} = {}& 2\int_{-\frac{1}{2}}^{\frac{1}{2}} \prod_{i=1}^{4} d\nu_i^1 (|\mathcal{W}_{0,0}|_{q^0} - \mathcal{W}_{1,0}|_{q^0}|^2 \\ &+ |\mathcal{W}_{0,1}|_{q^0} - \mathcal{W}_{1,0}|_{q^0}|^2) \\ &+ 2\int_{-\frac{1}{2}}^{\frac{1}{2}} \prod_{i=1}^{4} d\nu_i^1 (\mathcal{W}_{0,0;0,0}|_{q^0;\bar{q}^0} \\ &+ \mathcal{W}_{0,1;0,1}|_{q^0;\bar{q}^0} - 2\mathcal{W}_{1,0;1,0}|_{q^0;\bar{q}^0}).\end{aligned} \tag{3.17}$$

Using that $|\mathcal{W}_{0,0}|_{q^0} - \mathcal{W}_{1,0}|_{q^0}|^2 = |\mathcal{W}_{0,1}|_{q^0} - \mathcal{W}_{1,0}|_{q^0}|^2 = \frac{1}{4}|\mathcal{W}_{0,0}|_{q^0} + \mathcal{W}_{0,1}|_{q^0} - 2\mathcal{W}_{1,0}|_{q^0}|^2$ as a consequence of (3.4), we can rewrite this as

$$A^{\text{spin 1}}_{(2,2)} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \prod_{1\le i<j\le 4} d\nu_i^1 |\mathcal{W}_{0,0}|_{q^0} + \mathcal{W}_{0,1}|_{q^0} - 2\mathcal{W}_{1,0}|_{q^0}|^2 + 2\int_{-\frac{1}{2}}^{\frac{1}{2}} \prod_{i=1}^{4} d\nu_i^1 (\mathcal{W}_{0,0;0,0}|_{q^0;\bar q^0} + \mathcal{W}_{0,1;0,1}|_{q^0;\bar q^0} - 2\mathcal{W}_{1,0;1,0}|_{q^0;\bar q^0}), \tag{3.18}$$

showing that this spin-1 contribution in the (2,2) models arises as the product of two $\mathcal{N}=2$ hypermultiplets $Q = (2\times 1/2_{\mathbf{1}}, 2\times 0_{\mathbf{2}})_{\mathcal{N}=2}$:

$$2\times(1_{\mathbf{1}}, 1/2_{\mathbf{4}}, 0_{\mathbf{6}})_{\mathcal{N}=4} = (2\times 1/2_{\mathbf{1}}, 2\times 0_{\mathbf{2}})_{\mathcal{N}=2} \otimes (2\times 1/2_{\mathbf{1}}, 2\times 0_{\bar{\mathbf{2}}})_{\mathcal{N}=2}. \tag{3.19}$$

The contribution from the $\mathcal{N}=4$ supergravity multiplet running in the loop [obtained from (2.48) by setting $n_v = 0$] can be presented in a form reflecting the decomposition in (3.13):

$$A^{\text{spin 2}}_{(2,2)} = A^{\text{spin 2}}_{\mathcal{N}=8} - 6A^{\text{spin 1}}_{(2,2)} - 4A^{\text{spin}\frac{3}{2}}_{(2,2)}, \tag{3.20}$$

where $A^{\text{spin}\frac{3}{2}}_{(2,2)}$ is given by

$$A^{\text{spin}\frac{3}{2}}_{(2,2)} = -\frac{1}{8}A^{\text{spin 2}}_{\mathcal{N}=8} - \frac{1}{32}\int_{-\frac{1}{2}}^{\frac{1}{2}} \prod_{1\le i<j\le 4} d\nu_i^1 |\mathcal{W}_{0,0}|_{\sqrt{q}} - \mathcal{W}_{0,1}|_{\sqrt{q}}|^2. \tag{3.21}$$

C. Comparing of the string models

The integrands of the amplitudes in the two (4,0) models in (2.17) and (2.32) and the (2,2) models in (2.48) take a different form. In this section we show first the equality between the integrands of the (4,0) models and then that any difference with the (2,2) models can be attributed to the contribution of the vector multiplets.

The comparison is done in the field theory limit where $\tau_2 \to +\infty$ and $\alpha' \to 0$ with $t = \alpha'\tau_2$ held fixed. The real parts of the ν_i variables are integrated over the range $-\frac{1}{2} \le \nu_i^1 \le \frac{1}{2}$. In this limit the position of the vertex operators scale as $\nu_i = \nu_i^1 + i\tau_2\omega_i$. The positions of the external legs on the loop are then denoted by $0 \le \omega_i \le 1$ and are ordered according to the kinematical region under consideration. In this section we discuss the integration over the ν_i^1's only; the integration over the ω_i's will be performed in Sec. IV.

1. Comparing the (4,0) models

In the heterotic string amplitude (2.17), we can identify two distinct contributions: n_v vector multiplets and one $\mathcal{N}=4$ supergravity multiplet running in the loop. At the leading order in α', the contribution of the vector multiplets is given by

$$A^{\text{spin1}}_{(4,0)\text{het}} = \frac{1}{2}\left(\frac{\pi}{2}\right)^4 t_8F^4 \int_{-\frac{1}{2}}^{\frac{1}{2}} \prod_{i=1}^{4} d\nu_i^1 \bar{\mathcal{W}}^B|_{\bar q^0}, \tag{3.22}$$

and the one of the supergravity multiplet by

$$A^{\text{spin2}}_{(4,0)\text{het}} = \frac{1}{2}\left(\frac{\pi}{2}\right)^4 t_8F^4 \int_{-\frac{1}{2}}^{\frac{1}{2}} \prod_{i=1}^{4} d\nu_i^1 ((\bar{\mathcal{W}}^B|_{\bar q}(1+\alpha'\delta Q) + \bar{\mathcal{W}}^B|_{\bar q^0} Q|_{\bar q} + 2\bar{\mathcal{W}}^B|_{\bar q^0}). \tag{3.23}$$

The vector multiplet contributions take different forms in the heterotic construction in (3.22) and the type II models in (3.8). However using the expansion of the fermionic propagators given in Appendix A 2, it is not difficult to perform the integration over ν_i^1 in (3.8). We see that

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \prod_{1\le i<j\le 4} d\nu_i^1 (\mathcal{W}^F_{0,0}|_{\bar q^0} + \mathcal{W}^F_{0,1}|_{\bar q^0}) = 0. \tag{3.24}$$

Thus there only remains the bosonic part of $\mathcal{W}_{a,b}$, and we find that the contribution of the vector multiplet is the same in the heterotic and asymmetric orbifold constructions:

$$A^{\text{spin 1}}_{(4,0)\text{het}} = A^{\text{spin 1}}_{(4,0)II}. \tag{3.25}$$

The case of the $\mathcal{N}=4$ supergraviton is a little more involved. In order to simplify the argument we make the following choice of helicity to deal with more manageable expressions: $(1^{++}, 2^{++}, 3^{--}, 4^{--})$. We set as well the reference momenta q_i's for graviton $i = 1,\ldots,4$ as follows: $q_1 = q_2 = k_3$ and $q_3 = q_4 = k_1$. At four points in supersymmetric theories, amplitudes with more $+$ or $-$ helicity states vanish. In that manner the covariant quantities t_8F^4 and $t_8t_8R^4$ are written in the spinor helicity formalism[3] $2t_8F^4 = \langle k_1k_2\rangle^2[k_3k_4]^2$ and $4t_8t_8R^4 = \langle k_1k_2\rangle^4[k_3k_4]^4$, respectively. With this choice of gauge $\epsilon^{(1)}\cdot\epsilon^{(k)} = 0$ for $k = 2, 3, 4$, $\epsilon^{(3)}\cdot\epsilon^{(l)} = 0$ with $l = 2, 4$ and only $\epsilon^{(2)}\cdot\epsilon^{(4)} \neq 0$. The same relationships hold for the scalar product between the right-moving $\tilde\epsilon$ polarizations and the left- and right-moving polarizations. We can now simplify the various kinematical factors $\mathcal{W}^B$ for the heterotic string and the $\mathcal{W}_{a,b}$'s for the type II models. We find $\bar{\mathcal{W}}^B = \frac{1}{2}t_8\tilde F^4\tilde{\mathcal{W}}^B$, where

$$\tilde{\mathcal{W}}^B = \tilde{\mathcal{W}}^B_1 + \frac{1}{\alpha' u}\tilde{\mathcal{W}}^B_2, \tag{3.26}$$

with

[3] A null vector $k^2 = 0$ is parametrized as $k_{\alpha\dot\alpha} = k_\alpha\bar k_{\dot\alpha}$, where $\alpha, \dot\alpha = 1, 2$ are $SL(2,\mathbb{C})$ two-dimensional spinor indices. The positive and negative helicity polarization vectors are given by $\epsilon^+(k,q)_{\alpha\dot\alpha} := \frac{q_\alpha\bar k_{\dot\alpha}}{\sqrt{2}\langle qk\rangle}$ and $\epsilon^-(k,q)_{\alpha\dot\alpha} := -\frac{k_\alpha\bar q_{\dot\alpha}}{\sqrt{2}[qk]}$, respectively, where q is a massless reference momentum. The self-dual and anti-self-dual field strengths read $F^-_{\alpha\beta} := \sigma^{mn}_{\alpha\beta}F_{mn} = \frac{k_\alpha k_\beta}{\sqrt{2}}$ and $F^+_{\dot\alpha\dot\beta} := \bar\sigma^{mn}_{\dot\alpha\dot\beta}F_{mn} = -\frac{\bar k_{\dot\alpha}\bar k_{\dot\beta}}{\sqrt{2}}$, respectively.

$$\tilde{\mathcal{W}}_1^B = (\bar\partial\mathcal{P}(12) - \bar\partial\mathcal{P}(14))(\bar\partial\mathcal{P}(21) - \bar\partial\mathcal{P}(24))(\bar\partial\mathcal{P}(32) - \bar\partial\mathcal{P}(34))(\bar\partial\mathcal{P}(42) - \bar\partial\mathcal{P}(43)),$$
$$\tilde{\mathcal{W}}_2^B = \bar\partial^2\mathcal{P}(24)(\bar\partial\mathcal{P}(12) - \bar\partial\mathcal{P}(14))(\bar\partial\mathcal{P}(32) - \bar\partial\mathcal{P}(34)). \quad (3.27)$$

In these equations it is understood that $\mathcal{P}(ij)$ stands for $\mathcal{P}(\nu_i - \nu_j)$. We find as well that $\mathcal{W}_{a,b}^F = \frac{1}{2}t_8F^4\tilde{\mathcal{W}}_{a,b}^F$ with $\tilde{\mathcal{W}}_{a,b}^F = \tilde{\mathcal{S}}_{4;a,b} + \tilde{\mathcal{S}}_{2;a,b}$, where

$$\tilde{\mathcal{S}}_{4;a,b} = \frac{1}{2^8}(S_{a,b}(12)^2S_{a,b}(34)^2 - S_{a,b}(1234) - S_{a,b}(1243) - S_{a,b}(1423)),$$
$$\tilde{\mathcal{S}}_{2;a,b} = \frac{1}{2^4}(\partial\mathcal{P}(12) - \partial\mathcal{P}(14))(\partial\mathcal{P}(21) - \partial\mathcal{P}(24)) \times (S_{a,b}(34))^2 + \frac{1}{2^4}(\partial\mathcal{P}(32) - \partial\mathcal{P}(34))(\partial\mathcal{P}(42) - \partial\mathcal{P}(43))(S_{a,b}(12))^2, \quad (3.28)$$

where we have used a shorthand notation: $S_{a,b}(ij)$ stands for $S_{a,b}(z_i - z_j)$ while $S_{a,b}(ijkl)$ stands for $S_{a,b}(z_i - z_j)S_{a,b}(z_j - z_k)S_{a,b}(z_k - z_l)S_{a,b}(z_l - z_i)$. With that choice of helicity, we can immediately give a simplified expression for the contribution of a spin-1 supermultiplet in the (4,0) models. We introduce the field theory limit of $\tilde{\mathcal{W}}^B$:

$$W^B := \lim_{\tau_2\to\infty}\left(\frac{2}{\pi}\right)^4\int_{-\frac12}^{\frac12}\prod_{i=1}^4 d\nu_i^1\tilde{\mathcal{W}}^B|_{\bar q^0}. \quad (3.29)$$

In this limit, this quantity is given by $W^B = W_1 + W_2$ with

$$W_1 = \frac{1}{16}(\partial P(12) - \partial P(14))(\partial P(21) - \partial P(24))(\partial P(32) - \partial P(34))(\partial P(42) - \partial P(43)),$$
$$W_2 = \frac{1}{4\pi}\frac{1}{\alpha'\tau_2 u}\partial^2 P(24)(\partial P(12) - \partial P(14))(\partial P(32) - \partial P(34)), \quad (3.30)$$

where $\partial^n P(\omega)$ is the nth derivative of the field theory propagator (3.35) and where $\alpha'\tau_2$ is the proper time of the field one-loop amplitude. We can now rewrite (3.22) and find

$$A_{(4,0)\text{het}}^{\text{spin }1} = \frac{1}{4}\left(\frac{\pi}{2}\right)^8 t_8t_8R^4W^B. \quad (3.31)$$

Let us come back to the comparison of the $\mathcal{N} = 4$ spin-2 multiplet contributions in the type II asymmetric orbifold model given in (3.10) and the heterotic one given in (3.23). We consider the following part of (3.23):

$$\int_{-\frac12}^{\frac12}\prod_{i=1}^4 d\nu_i^1(\bar{\mathcal{W}}^B|_{\bar q}(1 + \alpha'\delta Q) + \bar{\mathcal{W}}^B|_{\bar q^0}\mathcal{Q}|_{\bar q}), \quad (3.32)$$

defined in the field theory limit for large τ_2.

The integral over the ν_i^1 will kill any term that have a nonzero phase $e^{i\pi(a\nu_1^1 + b\nu_2^1 + c\nu_3^1 + d\nu_4^1)}$, where a, b, c, d are non-all-vanishing integers. In $\tilde W_1^B$ we have terms of the form $\partial\delta P(ij)\times\partial\mathcal{P}(ji)|_{\bar q}\times(\partial P(kl) - \partial P(k'l'))\times(\partial P(rs) - \partial P(r's'))$. Using the definition of $\delta\mathcal{P}(ij)$ given in (A15) and the order $\bar q$ coefficient of the propagator in (A14), we find that

$$\partial\delta P(ij)\times\partial\mathcal{P}(ji)|_{\bar q} = -\frac{i\pi^2}{2}\sin(2\pi\nu_{ij}) \times \operatorname{sgn}(\omega_{ij})e^{2i\pi\operatorname{sgn}(\omega_{ij})\nu_{ij}}, \quad (3.33)$$

which integrates to $-\pi^2/4$. All such terms with $(ij) = (12)$ and $(ij) = (34)$ contribute in total to

$$\frac{1}{2}\left(\frac{\pi}{2}\right)^4[(\partial P(12) - \partial P(14))(\partial P(21) - \partial P(24)) + (\partial P(32) - \partial P(34))(\partial P(42) - \partial P(43))], \quad (3.34)$$

where $\partial P(ij)$ is for the derivative of the propagator in the field theory $\partial P(\omega_i - \omega_j)$ given by

$$\partial P(\omega) = 2\omega - \operatorname{sgn}(\omega). \quad (3.35)$$

The last contraction in $\tilde W_1^B$ for $(ij) = (24)$ leads to the same kind of contribution. However, they will actually be cancelled by terms coming from similar contractions in $\tilde{\mathcal{W}}_2^B|_{\bar q}$. More precisely, the nonzero contractions involved yield

$$(\partial^2\mathcal{P}(24)e^{\mathcal{Q}})|_{\bar q} = -\frac{\alpha'\pi^2}{2}(\cos(2\pi\nu_{24})e^{2i\pi\operatorname{sgn}(\omega_{24})\nu_{24}} - 2e^{2i\pi\operatorname{sgn}(\omega_{24})\nu_{24}}\sin^2(\pi\nu_{24})), \quad (3.36)$$

which integrates to $-\alpha'\pi^2/2$. The α' compensates the $1/\alpha'$ factor in (3.26) and this contribution precisely cancels the one from (3.33) with $(ij) = (24)$. Other types of terms with more phase factors from the propagator turn out to vanish after summation. In all, we get $-\pi^4W_3/4$, where

$$W_3 = -\frac{1}{8}((\partial P(12) - \partial P(14))(\partial P(21) - \partial P(24)) + (\partial P(32) - \partial P(34))(\partial P(42) - \partial P(43))). \quad (3.37)$$

Finally, let us look at the totally contracted terms of the form $\partial\delta P(ik)\partial\delta P(kl)\partial\delta P(lj)\times\partial\mathcal{P}(ij)|_{\bar q}$ that come from $\tilde{\mathcal{W}}_1^B|\bar q$. Those are the only terms of that type that survive the ν^1 integrations since they form closed chains in their arguments. They give the following terms:

$$i\frac{\pi^4}{8}\sin(\pi\nu_{ij})\mathrm{sgn}(\omega_{ik})\mathrm{sgn}(\omega_{kl})\mathrm{sgn}(\omega_{lj}) \times e^{2i\pi(\mathrm{sgn}(\omega_{ik})\nu_{ik}+\mathrm{sgn}(\omega_{kl})\nu_{kl}+\mathrm{sgn}(\omega_{lj})\nu_{lj})}. \tag{3.38}$$

They integrate to $\pi^4/16$ if the vertex operators are ordered according to $0\leq\omega_i<\omega_k<\omega_l<\omega_j\leq 1$ or in the reversed ordering. Hence, from $\tilde{W}_1^B$ we will get one of the orderings we want in our polarization choice, namely the region (s,t). From $\tilde{W}_2 e^{\mathcal{Q}}$, a similar computation yields the two other kinematical regions (s,u) and (t,u). In all we have a total integrated contribution of $\pi^4/16$. We collect all the different contributions that we have obtained, and (3.23) writes

$$A^{\mathrm{spin2}}_{(4,0)\mathrm{het}}=\frac{1}{4}\left(\frac{\pi}{2}\right)^8 t_8t_8R^4(1-4W_3+2W^B), \tag{3.39}$$

where we used that $t_8t_8R^4=t_8F^4t_8\tilde{F}^4$ and (3.29) and (3.30).

We now turn to the spin-2 contribution in the type II asymmetric orbifold models given in (3.10). Using the q-expansion detailed in Appendix A 2 c, we find that

$$\begin{aligned}\int_{-\frac{1}{2}}^{\frac{1}{2}}\prod_{i=1}^4 d\nu_i^1(\tilde{W}_{0,0}|_{\sqrt{q}}-\tilde{W}_{0,1}|_{\sqrt{q}}) &= \int_{-\frac{1}{2}}^{\frac{1}{2}}\prod_{i=1}^4 d\nu_i^1(\tilde{W}^F_{0,0}|_{\sqrt{q}}-\tilde{W}^F_{0,1}|_{\sqrt{q}})\\ &= 2\int_{-\frac{1}{2}}^{\frac{1}{2}}\prod_{i=1}^4 d\nu_i^1\tilde{W}^F_{0,0}|_{\sqrt{q}}.\end{aligned} \tag{3.40}$$

We have then terms of the form $\tilde{\mathcal{S}}_{2;0,0}$ and $\tilde{\mathcal{S}}_{4;0,0}$. Their structure is similar to the terms in the heterotic case with, respectively, two and four bosonic propagators contracted. The bosonic propagators do not have a $\sqrt{q}$ piece and since $\tilde{\mathcal{S}}_{0,0}(12)^2|_{\sqrt{q}}=\tilde{\mathcal{S}}_{0,0}(34)^2|_{\sqrt{q}}=4\pi^2$ we find that the terms in $\mathcal{S}_{2;0,0}$ give

$$2\int_{-\frac{1}{2}}^{\frac{1}{2}}\prod_{i=1}^4 d\nu_i^1\tilde{\mathcal{S}}_{2;0,0}|_{\sqrt{q}}=-4\left(\frac{\pi}{2}\right)^4 W_3, \tag{3.41}$$

including the $1/2^4$ present in (2.26). The $\tilde{\mathcal{S}}_{4;0,0}$ terms have two different kind of contributions: double trace and single trace [see, respectively, first and second lines in (2.26)]. In the spin structure (0,0) the double trace always vanishes in $\tilde{\mathcal{S}}_{4;0,0}|_{\sqrt{q}}$ since

$$\int_{-\frac{1}{2}}^{\frac{1}{2}}\prod_{i=1}^4 d\nu_i^1\frac{\sin(\pi\nu_{ij})}{\sin^2(\pi\nu_{kl})\sin(\pi\nu_{ij})}=\int_{-\frac{1}{2}}^{\frac{1}{2}}\prod_{i=1}^4 d\nu_i^1\frac{1}{\sin^2(\pi\nu_{kl})} = 0. \tag{3.42}$$

However the single trace terms are treated in the same spirit as for the heterotic string. Only closed chains of sines contribute and are nonzero only for specific ordering of the vertex operators. For instance,

$$-4\pi^4\frac{\sin(\pi\nu_{ij})}{\sin(\pi\nu_{jk})\sin(\pi\nu_{kl})\sin(\pi\nu_{li})}\sim_{\tau_2\to\infty}-(2\pi)^4, \tag{3.43}$$

for the ordering $0\leq\omega_j<\omega_l<\omega_k<\omega_i\leq 1$. Summing all of the contributions from $\tilde{\mathcal{S}}_{4;0,0}$ gives a total factor of $-\pi^4/16$, including the normalization in (2.26). We can now collect all the terms to get

$$A^{\mathrm{spin}\,2}_{(4,0)II}=\frac{1}{4}\left(\frac{\pi}{2}\right)^8 t_8t_8R^4(1-4W_3+2W^B), \tag{3.44}$$

showing the equality with the heterotic expression

$$A^{\mathrm{spin}\,2}_{(4,0)\mathrm{het}}=A^{\mathrm{spin}\,2}_{(4,0)II}. \tag{3.45}$$

We remark that the same computations give the contribution of the spin-$\frac{3}{2}$ multiplets in the two models, which are equal as well and write

$$A^{\mathrm{spin}\frac{3}{2}}_{(4,0)\mathrm{het}}=A^{\mathrm{spin}\frac{3}{2}}_{(4,0)II}=\frac{1}{4}\left(\frac{\pi}{2}\right)^8 t_8t_8R^4(W_3-2W^B). \tag{3.46}$$

Thanks to those equalities for the spin 2, spin $\frac{3}{2}$ and spin 1 in (3.25), from now we will use the notation $A^{\mathrm{spin}\,s}_{(4,0)}$ with $s=1,\frac{3}{2},2$.

The perturbative equality between these two (4,0) models is not surprising. For a given number of vector multiplets n_v the heterotic and asymmetric type II construction lead to two string theory (4,0) models related by S duality, $S\to -1/S$, where S is the axion-dilaton complex scalar in the $\mathcal{N}=4$ supergravity multiplet. The perturbative expansion in these two models is defined around different points in the $SU(1,1)/U(1)$ moduli space. The action of $\mathcal{N}=4$ supersymmetry implies that the one-loop amplitudes between gravitons, which are neutral under the $U(1)$ R symmetry, are the same in the strong and weak coupling regimes.

2. Comparing the (4,0) and (2,2) models

In the case of the (2,2) models, the contribution from the vector multiplets is given in (3.18). The string theory integrand is different from the one in (3.8) for the (4,0) as it can be seen using the supersymmetric Riemann identity in (3.5). Let us first write the spin-1 contribution in the (2,2) models. Performing the ν_i^1 integrations and the same kind of manipulations that we have done in the previous section, we can show that it is given by

$$A^{\mathrm{spin}\,1}_{(2,2)}=\frac{1}{4}\left(\frac{\pi}{2}\right)^8 t_8t_8R^4\left((W_3)^2+\frac{1}{2}W_2\right). \tag{3.47}$$

This is to be compared with (3.31). The expressions are clearly different but will lead to the same amplitude. In the same manner, we find for the spin $\frac{3}{2}$

$$A^{\mathrm{spin}\,\frac{3}{2}}_{(2,2)}=\frac{1}{4}\left(\frac{\pi}{2}\right)^8 t_8t_8R^4\left(W_3-2\left((W_3)^2+\frac{1}{2}W_2\right)\right). \tag{3.48}$$

This differs from (3.46) by a factor coming solely from the vector multiplets.

We now compare the spin-2 contributions in the (4,0) model in (3.16) and the (2,2) model in (3.21). Again, a similar computation to the one we have done gives the contribution of the spin-2 multiplet running in the loop for the (2,2) model:

$$A^{\text{spin } 2}_{(2,2)} = \frac{1}{4}\left(\frac{\pi}{2}\right)^8 t_8 t_8 R^4 \left(1 - 4W_3 + 2\left((W_3)^2 + \frac{1}{2}W_2\right)\right). \tag{3.49}$$

We compare this with (3.39) and (3.44) that we rewrite in the following form:

$$A^{\text{spin } 2}_{(4,0)II} = \frac{1}{4}\left(\frac{\pi}{2}\right)^8 t_8 t_8 R^4 (1 - 4W_3 + 2(W_1 + W_2)). \tag{3.50}$$

The difference between the two expressions originates again solely from the vector multiplet sector. Considering that the same relation holds for the contribution of the $\mathcal{N} = 4$ spin-$\frac{3}{2}$ multiplets, we deduce that this is coherent with the supersymmetric decomposition (3.13) that gives

$$A^{\text{spin } 2}_{(2,2)} = A^{\text{spin } 2}_{(4,0)} + 2(A^{\text{spin } 1}_{(2,2)} - A^{\text{spin } 1}_{(4,0)}). \tag{3.51}$$

The difference between the spin-2 amplitudes in the two models is completely accounted for by the different vector multiplet contributions. The string theory models are related by a U duality exchanging the axion-dilaton scalar S of the gravity multiplet with a geometric modulus [27,28,36]. This transformation affects the coupling of the multiplet running in the loop, thus explaining the difference between the two string theory models. However at the supergravity level, the four graviton amplitudes that we compute are not sensitive to this fact and are equal in all models, as we will see now.

IV. FIELD THEORY ONE-LOOP AMPLITUDES IN $\mathcal{N} = 4$ SUPERGRAVITY

In this section we shall extract and compute the field theory limit $\alpha' \to 0$ of the one-loop string theory amplitudes studied in previous sections. We show some relations between loop momentum power counting and the spin or supersymmetry of the multiplet running in the loop.

As mentioned above, the region of the fundamental domain integration corresponding to the field theory amplitude is $\tau_2 \to \infty$, such that $t = \alpha' \tau_2$ is fixed. We then obtain a world-line integral of total proper time t. The method for extracting one-loop field theory amplitudes from string theory was pioneered in Ref. [20]. The general method that we apply consists in extracting the $o(q)^0$ terms in the integrand and taking the field theory limit and was developed extensively in Refs. [23,24,47]. Our approach will follow the formulation given in Ref. [25].

The generic form of the field theory four-graviton one-loop amplitude for $\mathcal{N} = 4$ supergravity with a spin-s ($s = 1, \frac{3}{2}, 2$) $\mathcal{N} = 4$ supermultiplet running in the loop is given by

$$M^{\text{spin } s}_X = \left(\frac{4}{\pi}\right)^4 \frac{\mu^{2\epsilon}}{\pi^{\frac{D}{2}}} \int_0^\infty \frac{dt}{t^{\frac{D-6}{2}}} \times \int_{\Delta_\omega} \prod_{i=1}^{3} d\omega_i e^{-\pi t Q(\omega)} \times A^{\text{spin } s}_X, \tag{4.1}$$

where $D = 4 - 2\epsilon$ and X stands for the model, $X = (4,0)$het, $X = (4,0)II$ or $X = (2,2)$, while the respective amplitudes $A^{\text{spin } s}_X$ are given in Secs. III B and III C. We have set the overall normalization to unity.

The domain of integration $\Delta_\omega = [0,1]^3$ is decomposed into three regions $\Delta_w = \Delta_{(s,t)} \cup \Delta_{(s,u)} \cup \Delta_{(t,u)}$ given by the union of the (s,t), (s,u) and (t,u) domains. In the $\Delta_{(s,t)}$ domain the integration is performed over $0 \le \omega_1 \le \omega_2 \le \omega_3 \le 1$, where $Q(\omega) = -s\omega_1(\omega_3 - \omega_2) - t(\omega_2 - \omega_1) \times (1 - \omega_3)$ with equivalent formulas obtained by permuting the external legs labels in the (t,u) and (s,u) regions (see Ref. [48] for details). We used that $s = -(k_1 + k_2)^2$, $t = -(k_1 + k_4)^2$ and $u = -(k_1 + k_3)^2$ with our convention for the metric $(- + \cdots +)$.

We now turn to the evaluation of the amplitudes. The main properties of the bosonic and fermionic propagators are provided in Appendix A 2. We work with the helicity configuration detailed in the previous section. This choice of polarization makes the intermediate steps easier as the expressions are explicitly gauge invariant.

A. Supersymmetry in the loop

Before evaluating the amplitudes we discuss the action of supersymmetry on the structure of the one-loop amplitudes. An n-graviton amplitude in dimensional regularization with $D = 4 - 2\epsilon$ can generically be written in the following way:

$$M_{n;1} = \mu^{2\epsilon} \int \frac{d^D \ell}{(2\pi)^D} \frac{\mathfrak{N}(\epsilon_i, k_i; \ell)}{\ell^2 (\ell - k_1)^2 \ldots (\ell - \sum_{i=1}^{n-1} k_i)^2}, \tag{4.2}$$

where the numerator is a polynomial in the loop momentum ℓ with coefficients depending on the external momenta k_i and polarization of the gravitons ϵ_i. For ℓ large this numerator behaves as $\mathfrak{N}(\epsilon_i, k_i; \ell) \sim \ell^{2n}$ in nonsupersymmetric theories. In an $\mathcal{N}$ extended supergravity theory, supersymmetric cancellations improve this behavior, which becomes $\ell^{2n-\mathcal{N}}$, where $\mathcal{N}$ is the number of four-dimensional supercharges:

$$\mathfrak{N}^{\mathcal{N}}(\epsilon_i, k_i; \ell) \sim \ell^{2n-\mathcal{N}} \quad \text{for } |\ell| \to \infty. \tag{4.3}$$

The dictionary between the Feynman integral presentation given in (4.2) and the structure of the field theory limit of the string theory amplitude states that the first derivative of a bosonic propagator counts as one power of loop momentum $\partial\mathcal{P} \sim \ell$, $\partial^2\mathcal{P} \sim \ell^2$ while fermionic propagators count for zero power of loop momentum $S_{a,b} \sim 1$. This dictionary was first established in Ref. [47] for gauge theory computation and then applied to supergravity amplitudes computations in Ref. [24] and more recently in Ref. [25].

With this dictionary we find that in the (4,0) model the integrand of the amplitudes have the following behavior:

$$A^{\text{spin } 1}_{(4,0)} \sim \ell^4, \qquad A^{\text{spin } \frac{3}{2}}_{(4,0)} \sim \ell^2 + \ell^4,$$
$$A^{\text{spin } 2}_{(4,0)} \sim 1 + \ell^2 + \ell^4. \tag{4.4}$$

The spin-1 contribution to the four-graviton amplitude has four powers of loop momentum as required for an $\mathcal{N} = 4$ amplitude according (4.3). The $\mathcal{N} = 4$ spin-$\frac{3}{2}$ supermultiplet contribution can be decomposed into an $\mathcal{N} = 6$ spin-$\frac{3}{2}$ supermultiplet term with two powers of loop momentum and an $\mathcal{N} = 4$ spin-1 supermultiplet contribution with four powers of loop momentum. The spin-2 contribution has an $\mathcal{N} = 8$ spin-2 piece with no powers of loop momentum, an $\mathcal{N} = 6$ spin-$\frac{3}{2}$ piece with two powers of loop momentum and an $\mathcal{N} = 4$ spin-1 piece with four powers of loop momentum.

For the (2,2) construction we have the following behavior:

$$A^{\text{spin } 1}_{(2,2)} \sim (\ell^2)^2, \qquad A^{\text{spin } \frac{3}{2}}_{(2,2)} \sim \ell^2 + (\ell^2)^2,$$
$$A^{\text{spin } 2}_{(2,2)} \sim 1 + \ell^2 + (\ell^2)^2. \tag{4.5}$$

Although the superficial counting of the number of loop momenta is the same for each spin $s = 1, \frac{3}{2}, 2$ in the two models, the precise dependence on the loop momentum differs in the two models, as indicated by the symbolic notation ℓ^4 and $(\ell^2)^2$. This is a manifestation of the model dependence for the vector multiplet contributions. As we have seen in the previous section, the order 4 terms in the loop momentum in the spin-$\frac{3}{2}$ and spin-2 parts are due to the spin-1 part.

At the level of the string amplitude, the multiplets running in the loop (spin 2 and spin 1) are naturally decomposed under the $\mathcal{N} = 4$ supersymmetry group. However, at the level of the amplitudes in field theory it is convenient to group the various blocks according to the number of powers of loop momentum in the numerator:

$$A^{\text{spin } s}_{\mathcal{N}=4s} \sim \ell^{4(2-s)}, \qquad s = 1, \frac{3}{2}, 2, \tag{4.6}$$

which is the same as organizing the terms according to the supersymmetry of the corresponding $\mathcal{N} = 4s$ spin-$s = 1, \frac{3}{2}, 2$ supermultiplet. In this decomposition it is understood that for the two $\mathcal{N} = 4$ models the dependence in the loop momenta is not identical.

From these blocks, one can reconstruct the contribution of the spin-2 $\mathcal{N} = 4$ multiplet that we are concerned with using the following relations:

$$M^{\text{spin } 2}_{X} = M^{\text{spin } 2}_{\mathcal{N}=8} - 4M^{\text{spin } \frac{3}{2}}_{\mathcal{N}=6} + 2M^{\text{spin } 1}_{X}, \tag{4.7}$$

where the index X refers to the type of model, (4,0) or (2,2).

This supersymmetric decomposition of the one-loop amplitude reproduces the one given in Refs. [24,30–35].

We shall come now to the evaluation of those integrals. We will see that even though the spin-1 amplitudes have different integrands, i.e., different loop momentum dependence in the numerator of the Feynman integrals, they are equal after integration.

B. Model-dependent part: $\mathcal{N} = 4$ vector multiplet contribution

In this section we first compute the field theory amplitude with an $\mathcal{N} = 4$ vector multiplet running in the loop for the two models. This part of the amplitude is model dependent as far as concerns the integrands. However, the value of the integrals is the same in the different models. Then we provide an analysis of the IR and UV behavior of these amplitudes.

1. Evaluation of the field theory amplitude

The contribution from the $\mathcal{N} = 4$ spin-1 vector supermultiplets in the (4,0) models is

$$M^{\text{spin } 1}_{(4,0)} = \left(\frac{4}{\pi}\right)^4 \frac{\mu^{2\epsilon}}{\pi^{\frac{D}{2}}} \int_0^\infty \frac{dt}{t^{\frac{D-6}{2}}} \int_{\Delta_\omega} d^3\omega e^{-\pi t Q(\omega)} \times A^{\text{spin1}}_{(4,0)}, \tag{4.8}$$

where $A^{\text{spin } 1}_{(4,0)}$ is given in (3.31) for instance and Q defined in (A17). Integrating over the proper time t and setting $D = 4 - 2\epsilon$, the amplitude reads

$$M^{\text{spin } 1}_{(4,0)} = t_8 t_8 R^4 \int_{\Delta_\omega} d^3\omega [\Gamma(1+\epsilon) Q^{-1-\epsilon} W_2 + \Gamma(2+\epsilon) Q^{-2-\epsilon} W_1]. \tag{4.9}$$

The quantities W_1 and W_2 are given in (3.30); they have the following form in terms of the variables ω_i:

$$W_1 = \frac{1}{8}(\omega_2 - \omega_3)(\text{sgn}(\omega_1 - \omega_2) + 2\omega_2 - 1)(\text{sgn}(\omega_2 - \omega_1) + 2\omega_1 - 1)(\text{sgn}(\omega_3 - \omega_2) + 2\omega_2 - 1),$$
$$W_2 = -\frac{1}{4}\frac{1}{u}(2\omega_2 - 1 + \text{sgn}(\omega_3 - \omega_2)) \times (2\omega_2 - 1 + \text{sgn}(\omega_1 - \omega_2))(1 - \delta(\omega_{24})). \tag{4.10}$$

Using the dictionary between the world-line propagators and the Feynman integral from the string-based rules [24,25,47], we recognize in the first term in (4.9) a six-dimensional scalar box integral and in the second term four-dimensional scalar bubble integrals.[4] Evaluating the integrals with standard techniques, we find[5]

$$M^{\rm spin\,1}_{(4,0)} = \frac{t_8 t_8 R^4}{2s^4}\left(s^2 - s(u-t)\log\left(\frac{-t}{-u}\right) - tu\left(\log^2\left(\frac{-t}{-u}\right) + \pi^2\right)\right). \tag{4.11}$$

The crossing symmetry of the amplitude has been broken by our choice of helicity configuration. However, it is still invariant under the exchange of the legs $1 \leftrightarrow 2$ and $3 \leftrightarrow 4$ which amounts to exchanging t and u. The same comment applies to all the field theory amplitudes evaluated in this paper. This result matches the one derived in Refs. [24,30–34] and in particular Eq. (3.20) in Ref. [35].

Now we turn to the amplitude in the (2,2) models:

$$M^{\rm spin\,1}_{(2,2)} = \left(\frac{4}{\pi}\right)^4 \frac{\mu^{2\epsilon}}{\pi^{\frac{D}{2}}} \int_0^\infty \frac{dt}{t^{\frac{D-6}{2}}} \int_{\Delta_\omega} d^3\omega\, e^{-\pi t Q(\omega)} \times A^{\rm spin\,1}_{(2,2)}, \tag{4.12}$$

where $A^{\rm spin\,1}_{(2,2)}$ is defined in (3.18). After integrating over the proper time t, one gets

$$M^{\rm spin\,1}_{(2,2)} = t_8 t_8 R^4 \int_{\Delta_\omega} d^3\omega\left[\Gamma(2+\epsilon)Q^{-2-\epsilon}(W_3)^2 + \frac{1}{2}\Gamma(1+\epsilon)Q^{-1-\epsilon}W_2\right], \tag{4.13}$$

where W_3, defined in (3.37), is given in terms of the ω_i variables by

$$W_3 = -\frac{1}{8}(\mathrm{sgn}(\omega_1-\omega_2) + 2\omega_2 - 1) \times (\mathrm{sgn}(\omega_2-\omega_1) + 2\omega_1 - 1) + \frac{1}{4}(\mathrm{sgn}(\omega_3-\omega_2) + 2\omega_2 - 1)(\omega_3-\omega_2). \tag{4.14}$$

There is no obvious relation between the integrand of this amplitude with the one for (4,0) model in (4.9). Expanding the square one can decompose this integral in three pieces that are seen to be proportional to the (4,0) vector multiplet contribution in (4.11). A first contribution is

$$\frac{t_8 t_8 R^4}{2}\int_{\Delta_\omega} d^3\omega[\Gamma(2+\epsilon)Q^{-2-\epsilon}W_1 + \Gamma(1+\epsilon)Q^{-1-\epsilon}W_2] = \frac{1}{2}M^{\rm spin\,1}_{(4,0)}, \tag{4.15}$$

and we have the additional contributions

$$\frac{t_8 t_8 R^4}{64}\int_{\Delta_\omega} d^3\omega \frac{\Gamma(2+\epsilon)}{Q^{2+\epsilon}}((\mathrm{sgn}(\omega_1-\omega_2) + 2\omega_2 - 1) \times (\mathrm{sgn}(\omega_2-\omega_1) + 2\omega_1 - 1))^2 = \frac{1}{4}M^{\rm spin\,1}_{(4,0)}, \tag{4.16}$$

and

$$\frac{t_8 t_8 R^4}{64}\int_{\Delta_\omega} d^3\omega \frac{\Gamma(2+\epsilon)}{Q^{2+\epsilon}} \times ((\mathrm{sgn}(\omega_3-\omega_2) + 2\omega_2 - 1)(\omega_3-\omega_2))^2 = \frac{1}{4}M^{\rm spin\,1}_{(4,0)}. \tag{4.17}$$

Performing all the integrations leads to

$$M^{\rm spin\,1}_{(2,2)} = M^{\rm spin\,1}_{(4,0)}. \tag{4.18}$$

It is now clear that the vector multiplet contributions to the amplitude are equal in the two theories, (4,0) and (2,2). It would be interesting to see if this expression could be derived with the double-copy construction of Ref. [35].

In this one-loop amplitude there is no interaction between the vector multiplets. Since the coupling of individual vector multiplet to gravity is universal [see for instance the $\mathcal{N}=4$ Lagrangian given in Eq. (4.18) in Ref. [50]], the four-graviton one-loop amplitude in pure $\mathcal{N}=4$ supergravity has to be independent of the model it comes from.

2. IR and UV behavior

The graviton amplitudes with vector multiplets running in the loop in (4.11) and (4.18) are free of UV and IR divergences. The absence of IR divergence is expected, since no spin-2 state is running in the loop. The IR divergence occurs only when a graviton is exchanged between two soft graviton legs (see Fig. 1). This fact has already been noticed in Ref. [30].

This behavior is easily understood by considering the soft graviton limit of the coupling between the graviton and a spin-$s \neq 2$ state. It occurs through the stress-energy tensor $V^{\mu\nu}(k,p) = T^{\mu\nu}(p-k,p)$, where k and p are, respectively, the momentum of the graviton and of the exchanged state. In the soft graviton limit the vertex behaves as $V^{\mu\nu}(p-k,p) \sim -k^\mu p^\nu$ for $p^\mu \sim 0$, and the amplitude behaves in the soft limit as

$$\int_{\ell\sim 0}\frac{d^4\ell}{\ell^2(\ell\cdot k_1)(\ell\cdot k_2)} T_{\mu\nu}(\ell-k_1,\ell)T^{\mu\nu}(\ell,\ell+k_2) \sim (k_1\cdot k_2)\int_{\ell\sim 0}\frac{d^4\ell}{\ell^2(\ell\cdot k_1)(\ell\cdot k_2)}\ell^2, \tag{4.19}$$

[4]In Refs. [25,49] it was wrongly claimed that $\mathcal{N}=4$ amplitudes do not have rational pieces. The argument in Ref. [25] was based on a naive application of the reduction formulas for $\mathcal{N}=8$ supergravity amplitudes to $\mathcal{N}=4$ amplitudes where boundary terms do not cancel anymore.

[5]The analytic continuation in the complex energy plane corresponds to the $+i\varepsilon$ prescription for the Feynman propagators $1/(\ell^2 - m^2 + i\varepsilon)$. We are using the notation that $\log(-s) = \log(-s-i\varepsilon)$ and that $\log(-s/-t) := \log((-s-i\varepsilon)/(-t-i\varepsilon))$.

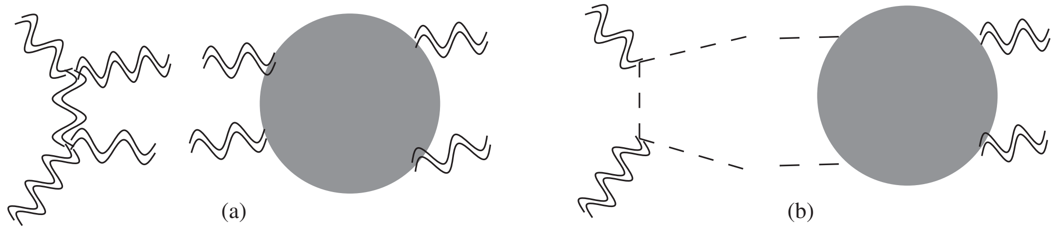

FIG. 1. Contribution to the IR divergences when two external gravitons (double wavy lines) become soft. If a graviton is exchanged as in (a), the amplitude presents an IR divergence. No IR divergences are found when another massless state of spin different from two is exchanged as in (b).

which is finite for small values of the loop momentum $\ell \sim 0$. In the soft graviton limit, the three-graviton vertex behaves as $V^{\mu\nu}(k,p) \sim k^\mu k^\nu$ and the amplitude has a logarithmic divergence at $\ell \sim 0$:

$$(k_1 \cdot k_2)^2 \int_{\ell\sim 0} \frac{d^4\ell}{\ell^2(\ell\cdot k_1)(\ell\cdot k_2)} = \infty. \tag{4.20}$$

The absence of UV divergence is due to the fact that the R^2 one-loop counterterm is the Gauss-Bonnet term. It vanishes in the four-point amplitude since it is a total derivative [51].

C. Model-independent part

In this section we compute the field theory amplitudes with an $\mathcal{N}=8$ supergraviton and an $\mathcal{N}=6$ spin-$\frac{3}{2}$ supermultiplet running in the loop. These quantities are model independent in the sense that their integrands are the same in the different models.

1. The $\mathcal{N}=6$ spin-$\frac{3}{2}$ supermultiplet contribution

The integrand for the $\mathcal{N}=4$ spin-$\frac{3}{2}$ supermultiplet contribution is different in the two (4,0) and (2,2) constructions of the $\mathcal{N}=4$ supergravity models. As shown in Eqs. (3.46) and (3.48), this is accounted for by the contribution of the vector multiplets. However, we exhibit an $\mathcal{N}=6$ spin-$\frac{3}{2}$ supermultiplet model-independent piece by adding two $\mathcal{N}=4$ vector multiplet contributions to the one of an $\mathcal{N}=4$ spin-$\frac{3}{2}$ supermultiplet:

$$M^{\text{spin}\,\frac{3}{2}}_{\mathcal{N}=6} = M^{\text{spin}\,\frac{3}{2}}_{X} + 2M^{\text{spin}\,1}_{X}. \tag{4.21}$$

The amplitude with an $\mathcal{N}=6$ spin-$\frac{3}{2}$ multiplet running in the loop is

$$M^{\text{spin}\,\frac{3}{2}}_{\mathcal{N}=6} = -\frac{t_8t_8R^4}{8}\int_{\Delta_\omega} d^3\omega\Gamma(2+\epsilon)W_3Q^{-2-\epsilon}, \tag{4.22}$$

where W_3 is given in (4.14). The integral is equal to the six-dimensional scalar box integral given in Eq. (3.16) in Ref. [35] up to $o(\epsilon)$ terms. We evaluate it and get

$$M^{\text{spin}\,\frac{3}{2}}_{\mathcal{N}=6} = -\frac{t_8t_8R^4}{2s^2}\left(\log^2\left(\frac{-t}{-u}\right)+\pi^2\right). \tag{4.23}$$

This result is UV finite as expected from the superficial power counting of loop momentum in the numerator of the amplitude given in (4.4). It is free of IR divergences because no graviton state is running in the loop (see the previous section). It matches the one derived in Refs. [24,30–34] and in particular Eq. (3.17) in Ref. [35].

2. The $\mathcal{N}=8$ spin-2 supermultiplet contribution

We now turn to the $\mathcal{N}=8$ spin-2 supermultiplet contribution in (4.7). It has already been evaluated in Refs. [20,52] and can be written as

$$M^{\text{spin}\,2}_{\mathcal{N}=8} = \frac{t_8t_8R^4}{4}\int_{\Delta_\omega} d^3\omega\Gamma(2+\epsilon)Q^{-2-\epsilon}. \tag{4.24}$$

Performing the integrations we have

$$M^{\text{spin}\,2}_{\mathcal{N}=8} = \frac{t_8t_8R^4}{4}\Bigg[\frac{2}{\epsilon}\left(\frac{\log(\frac{-t}{\mu^2})}{su}+\frac{\log(\frac{-s}{\mu^2})}{tu}+\frac{\log(\frac{-u}{\mu^2})}{st}\right) + 2\left(\frac{\log(\frac{-s}{\mu^2})\log(\frac{-t}{\mu^2})}{st}+\frac{\log(\frac{-t}{\mu^2})\log(\frac{-u}{\mu^2})}{tu} + \frac{\log(\frac{-u}{\mu^2})\log(\frac{-s}{\mu^2})}{us}\right)\Bigg], \tag{4.25}$$

where μ^2 is an IR mass scale. This amplitudes carries an ϵ pole signaling the IR divergence due to the graviton running in the loop.

Now we have all the blocks entering the expression for the $\mathcal{N}=4$ pure gravity amplitude in (4.7).

V. CONCLUSION

In this work we have evaluated the four-graviton amplitude at one loop in $\mathcal{N}=4$ supergravity in four dimensions from the field theory limit of string theory constructions. The string theory approach includes (4,0) models where all of the supersymmetry come from the left-moving sector of the theory and (2,2) models where the supersymmetry is split between the left- and right-moving sectors of the theory.

For each model the four-graviton one-loop amplitude is linearly dependent on the number of vector multiplets n_v. Thus we define the pure $\mathcal{N}=4$ supergravity amplitude by subtraction of these contributions. This matches the result obtained in the Dabholkar-Harvey construction of string theory models with no vector multiplets. We have seen that, except when gravitons are running in the loop, the one-loop amplitudes are free of IR divergences. In addition, all the amplitudes are UV finite because the R^2 candidate counterterm vanishes for these amplitudes. Amplitudes with external vector states are expected to be UV divergent [53].

Our results reproduce the ones obtained with the string-based rules in Refs. [24,30] unitarity-based method in Refs. [31–34] and the double-copy approach of Ref. [35]. The structure of the string theory amplitudes of the (4,0) and (2,2) models takes a very different form. There could have been differences at the supergravity level due to the different nature of the couplings of the vector multiplet in the two theories as indicated by the relation between the two amplitudes in (3.51). However, the coupling to gravity is universal. The difference between the various $\mathcal{N} = 4$ supergravity models are visible once interactions between vectors and scalars occur, as can be seen on structure of the $\mathcal{N} = 4$ Lagrangian in Ref. [50], which is not the case in our amplitudes since they involve only external gravitons. Our computation provides a direct check of this fact.

The supergravity amplitudes studied in this paper are naturally organized as a sum of $\mathcal{N} = 4s$ spin-$s = 1, \frac{3}{2}, 2$ contributions, with a simple power counting dependence on the loop momentum $\ell^{4(2-s)}$. Such a decomposition has been already used in the string-based approach to supergravity amplitudes in Ref. [24]. Our analysis reproduces these results and shows that the $\mathcal{N} = 4$ part of the four-graviton amplitude does not depend on whether one starts from (4,0) or (2,2) construction. We expect amplitudes with external scalars or vectors to take a different form in the two constructions.

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APPENDIX A: WORLD-SHEET CFT: CHIRAL BLOCKS, PROPAGATORS

In this Appendix we collect various results about the conformal blocks, fermionic and bosonic propagators at genus one, and their q expansions.

1. Bosonic and fermionic chiral blocks

The genus one theta functions are defined to be

$$\theta\begin{bmatrix} a \\ b \end{bmatrix}(z|\tau) = \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}(n+\frac{a}{2})^2} e^{2i\pi(z+\frac{b}{2})(n+\frac{a}{2})}, \qquad \text{(A1)}$$

and Dedekind eta function

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n), \qquad \text{(A2)}$$

where $q = \exp(2i\pi\tau)$. Those functions have the following $q \to 0$ behavior:

$$\theta\begin{bmatrix} 1 \\ 1 \end{bmatrix}(0,\tau) = 0; \quad \theta\begin{bmatrix} 1 \\ 0 \end{bmatrix}(0,\tau) = -2q^{1/8} + o(q);$$
$$\theta\begin{bmatrix} 0 \\ 0 \end{bmatrix}(0,\tau) = 1 + 2\sqrt{q} + o(q);$$
$$\theta\begin{bmatrix} 0 \\ 1 \end{bmatrix}(0,\tau) = 1 - 2\sqrt{q} + o(q); \quad \eta(\tau) = q^{1/24} + o(q). \qquad \text{(A3)}$$

The partition function of eight world-sheet fermions in the (a,b)-spin structure, $\Psi(z+1) = -(-1)^{2a}\Psi(z)$ and $\Psi(z+\tau) = -(-1)^{2b}\Psi(z)$, and eight chiral bosons is

$$Z_{a,b}(\tau) \equiv \frac{\theta[^a_b](0|\tau)^4}{\eta^{12}(\tau)}; \qquad \text{(A4)}$$

it has the following behavior for $q \to 0$:

$$Z_{1,1} = 0, \qquad Z_{1,0} = 16 + 16^2 q + o(q^2),$$
$$Z_{0,0} = \frac{1}{\sqrt{q}} + 8 + o(\sqrt{q}), \qquad Z_{0,1} = \frac{1}{\sqrt{q}} - 8 + o(\sqrt{q}). \qquad \text{(A5)}$$

The partition function of the twisted (X, Ψ) system in the (a,b)-spin structure is

$$\begin{aligned} X(z+1) &= (-1)^{2h}X(z); \\ \Psi(z+1) &= -(-1)^{2a+2h}\Psi(z), \\ X(z+\tau) &= (-1)^{2g}X(z); \\ \Psi(z+\tau) &= -(-1)^{2b+2g}\Psi(z). \end{aligned} \qquad \text{(A6)}$$

The twisted chiral blocks for a real boson are

$$Z^{h,g}[X] = \left(ie^{-i\pi g}q^{-h^2/2}\frac{\eta(\tau)}{\theta\begin{bmatrix} 1+h \\ 1+g \end{bmatrix}}\right)^{1/2}. \qquad \text{(A7)}$$

The twisted chiral blocks for a Majorana or Weyl fermion are

$$Z^{h,g}_{a,b}[\Psi] = \left(e^{-i\pi a(g+b)/2}q^{h^2/2}\frac{\theta\begin{bmatrix} a+h \\ b+g \end{bmatrix}}{\eta(\tau)}\right)^{1/2}. \qquad \text{(A8)}$$

The total partition function is given by

$$\begin{aligned} Z^{h,g}_{a,b}[(X,\Psi)] &= Z^{h,g}[X]Z^{h,g}_{a,b}[\Psi] \\ &= e^{i\frac{\pi}{4}(1+2g+a(g+b))}\sqrt{\frac{\theta\begin{bmatrix} a+h \\ b+g \end{bmatrix}}{\theta\begin{bmatrix} 1+h \\ 1+g \end{bmatrix}}}. \end{aligned} \qquad \text{(A9)}$$

2. Bosonic and fermionic propagators

a. Bosonic propagators

Our convention for the bosonic propagator is

$$\langle x^\mu(\nu)x^\nu(0)\rangle_{\text{one-loop}} = 2\alpha'\eta^{\mu\nu}\mathcal{P}(\nu|\tau), \tag{A10}$$

with

$$\begin{aligned}\mathcal{P}(\nu|\tau) &= -\frac{1}{4}\ln\left|\frac{\theta[^1_1](\nu|\tau)}{\partial_\nu\theta[^1_1](0|\tau)}\right|^2 + \frac{\pi\nu_2^2}{2\tau_2} + C(\tau)\\ &= \frac{\pi\nu_2^2}{2\tau_2} - \frac{1}{4}\ln\left|\frac{\sin(\pi\nu)}{\pi}\right|^2\\ &\quad - \sum_{m\geq 1}\left(\frac{q^m}{1-q^m}\frac{\sin^2(m\pi\nu)}{m} + \text{c.c.}\right) + C(\tau),\end{aligned} \tag{A11}$$

where $C(\tau)$ is a contribution of the zero modes (see e.g., Ref. [48]) that anyway drops out of the string amplitude because of momentum conservation so we will forget it in the following.

We have as well the expansions

$$\begin{aligned}\partial_\nu\mathcal{P}(\nu|\tau) &= \frac{\pi}{2i}\frac{\nu_2}{\tau_2} - \frac{\pi}{4}\frac{1}{\tan(\pi\nu)} - \pi q\sin(2\pi\nu) + o(q),\\ \partial_\nu^2\mathcal{P}(\nu|\tau) &= -\frac{\pi}{4\tau_2} + \frac{\pi^2}{4}\frac{1}{\sin^2(\pi\nu)} - 2\pi^2 q\cos(2\pi\nu) + o(q),\\ \partial_\nu\bar\partial_{\bar\nu}\mathcal{P}(\nu|\tau) &= \frac{\pi}{4}\left(\frac{1}{\tau_2} - \delta^{(2)}(\nu)\right),\end{aligned} \tag{A12}$$

leading to the following Fourier expansion with respect to ν_1:

$$\begin{aligned}\partial_\nu\mathcal{P}(\nu|\tau) &= \frac{\pi}{4i}\left(\frac{2\nu_2}{\tau_2} - \text{sgn}(\nu_2)\right) + i\frac{\pi}{4}\text{sgn}(\nu_2)\sum_{m\neq 0}e^{2i\pi m\,\text{sgn}(\nu_2)\nu}\\ &\quad - \pi q\sin(2\pi\nu) + o(q),\\ \partial_\nu^2\mathcal{P}(\nu|\tau) &= \frac{\pi}{4\tau_2}(\tau_2\delta(\nu_2) - 1) - \pi^2\sum_{m\geq 1}me^{2i\pi m\,\text{sgn}(\nu_2)\nu}\\ &\quad - 2\pi^2 q\cos(2\pi\nu) + o(q).\end{aligned} \tag{A13}$$

Setting $\nu = \nu_1 + i\tau_2\omega$ we can rewrite these expressions in a form relevant for the field theory limit $\tau_2 \to \infty$ with $t = \alpha'\tau_2$ kept fixed. The bosonic propagator can be decomposed in an asymptotic value for $\tau_2 \to \infty$ (the field theory limit) and corrections originating from massive string modes:

$$\begin{aligned}\mathcal{P}(\nu|\tau) &= -\frac{\pi t}{2\alpha'}P(\omega) + \delta P(\nu) - q\sin^2(\pi\nu)\\ &\quad - \bar q\sin^2(\pi\bar\nu) + o(q^2)\end{aligned} \tag{A14}$$

and

$$\begin{aligned}P(\omega) &= \omega^2 - |\omega|;\\ \delta P(\nu) &= \sum_{m\neq 0}\frac{1}{4|m|}e^{2i\pi m\nu_1 - 2\pi|m\nu_2|}.\end{aligned} \tag{A15}$$

The contribution δP corresponds to the effect of massive string states propagating between two external massless states. The quantity $\mathcal{Q}$ defined in (2.4) writes in this limit

$$\begin{aligned}\mathcal{Q} &= -t\pi Q(\omega) + \alpha'\delta Q - 2\pi\alpha'\sum_{1\leq i<j\leq 4}k_i\cdot k_j(q\sin^2(\pi\nu_{ij})\\ &\quad + \bar q\sin^2(\pi\bar\nu_{ij})) + o(q^2),\end{aligned} \tag{A16}$$

where

$$\begin{aligned}Q(\omega) &= \sum_{1\leq i<j\leq 4}k_i\cdot k_j P(\omega_{ij}),\\ \delta Q &= 2\sum_{1\leq i<j\leq 4}k_i\cdot k_j\delta P(\nu_{ij}).\end{aligned} \tag{A17}$$

b. Fermionic propagators

Our normalization for the fermionic propagators in the (a, b)-spin structure is given by

$$\langle\psi^\mu(z)\psi^\nu(0)\rangle_{\text{one-loop}} = \frac{\alpha'}{2}S_{a,b}(z|\tau). \tag{A18}$$

In the even spin structure fermionic propagators are

$$S_{a,b}(z|\tau) = \frac{\theta[^a_b](z|\tau)}{\theta[^a_b](0|\tau)}\frac{\partial_z\theta[^1_1](0|\tau)}{\theta[^1_1](z|\tau)}. \tag{A19}$$

The odd spin structure propagator is

$$S_{1,1}(z|\tau) = \frac{\partial_z\theta[^1_1](z|\tau)}{\theta[^1_1](z|\tau)}, \tag{A20}$$

and the fermionic propagator orthogonal to the zero modes is

$$\tilde S_{1,1}(z|\tau) = S_{1,1}(z|\tau) - 2i\pi\frac{z_2}{\tau_2} = -4\partial_z\mathcal{P}(z|\tau). \tag{A21}$$

The fermionic propagators have the following q-expansion representation [54]:

$$\begin{aligned}S_{1,1}(z|\tau) &= \frac{\pi}{\tan(\pi z)} + 4\pi\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}\sin(2n\pi z),\\ S_{1,0}(z|\tau) &= \frac{\pi}{\tan(\pi z)} - 4\pi\sum_{n=1}^{\infty}\frac{q^n}{1+q^n}\sin(2n\pi z),\\ S_{0,0}(z|\tau) &= \frac{\pi}{\sin(\pi z)} - 4\pi\sum_{n=1}^{\infty}\frac{q^{n-\frac12}}{1+q^{n-\frac12}}\sin((2n-1)\pi z),\\ S_{0,1}(z|\tau) &= \frac{\pi}{\sin(\pi z)} + 4\pi\sum_{n=1}^{\infty}\frac{q^{n-\frac12}}{1-q^{n-\frac12}}\sin((2n-1)\pi z).\end{aligned} \tag{A22}$$

Riemann supersymmetric identities written in the text (2.27) derive from the following Riemann relation relation:

$$\sum_{a,b=0,1}(-1)^{a+b+ab}\prod_{i=1}^{4}\theta\begin{bmatrix}a\\b\end{bmatrix}(v_i) = -2\prod_{i=1}^{4}\theta_1(v_i'), \quad (A23)$$

with $v_1'=\frac{1}{2}(-v_1+v_2+v_3+v_4)$, $v_2'=\frac{1}{2}(v_1-v_2+v_3+v_4)$, $v_3'=\frac{1}{2}(v_1+v_2-v_3+v_4)$, and $v_4'=\frac{1}{2}(v_1+v_2+v_3-v_4)$. This identity can be written, in the form used in the main text, as vanishing identities

$$\sum_{\substack{a,b=0,1\\ab=0}}(-1)^{a+b+ab}Z_{a,b}(\tau)=0, \quad (A24)$$

$$\sum_{\substack{a,b=0,1\\ab=0}}(-1)^{a+b+ab}Z_{a,b}(\tau)\prod_{r=1}^{n}S_{a,b}(z)=0 \quad 1\le n\le 3, \quad (A25)$$

and the first nonvanishing one

$$\sum_{\substack{a,b=0,1\\ab=0}}(-1)^{a+b+ab}Z_{a,b}(\tau)\prod_{i=1}^{4}S_{a,b}(z_i|\tau)=-(2\pi)^4 \quad (A26)$$

with $z_1+\cdots+z_4=0$ and where we used that $\partial_z\theta[^1_1](0|\tau)=\pi\theta[^0_0](0|\tau)\theta[^1_0](0|\tau)\theta[^0_1](0|\tau)=2\pi\eta^3(\tau)$.

Two identities consequences of the Riemann relation in (A23) are

$$\begin{aligned}S_{0,0}^2(z)-S_{1,0}^2(z)&=\pi^2\left(\theta\begin{bmatrix}0\\1\end{bmatrix}(0|\tau))^4\frac{(\partial_z\theta[^1_1](z|\tau)}{\partial\theta[^1_1](0|\tau)}\right)^2,\\ S_{0,1}^2(z)-S_{1,0}^2(z)&=\pi^2\left(\theta\begin{bmatrix}0\\0\end{bmatrix}(0|\tau))^4\frac{(\partial_z\theta[^1_1](z|\tau)}{\partial\theta[^1_1](0|\tau)}\right)^2.\end{aligned} \quad (A27)$$

c. q expansion

The q expansions of the fermionic propagators in the even spin structure are given by

$$\begin{aligned}S_{1,0}(z|\tau)&=\frac{\pi}{\tan(\pi z)}-4\pi q\sin(2\pi z)+o(q^2),\\ S_{0,0}(z|\tau)&=\frac{\pi}{\sin(\pi z)}-4\pi\sqrt{q}\sin(\pi z)+o(q),\\ S_{0,1}(z|\tau)&=\frac{\pi}{\sin(\pi z)}+4\pi\sqrt{q}\sin(\pi z)+o(q).\end{aligned} \quad (A28)$$

Setting $\mathcal{S}^n_{a,b}=\prod_{i=1}^n S_{a,b}(z_i|\tau)$ we have the following expansion:

$$\begin{aligned}\mathcal{S}^n_{1,0}&=\prod_{i=1}^n\pi\cot(\pi z_i)\left(1-8q\sum_{i=1}^n\sin^2(\pi z_i)\right)+o(q^2),\\ \mathcal{S}^n_{0,0}&=\prod_{i=1}^n\pi(\sin(\pi z_i))^{-1}\left(1-4q\sum_{i=1}^n\sin^2(\pi z_i)\right)+o(q^2),\\ \mathcal{S}^n_{0,1}&=\prod_{i=1}^n\pi(\sin(\pi z_i))^{-1}\left(1+4q\sum_{i=1}^n\sin^2(\pi z_i)\right)+o(q^2).\end{aligned} \quad (A29)$$

Applying these identities with $n=2$ and $n=4$ we derive the following relations between the correlators $\mathcal{W}^F_{a,b}$ defined in (2.25):

$$\mathcal{W}^F_{0,0}|_{q^0}=\mathcal{W}^F_{0,1}|_{q^0}; \qquad \mathcal{W}^F_{0,0}|_{\sqrt{q}}=-\mathcal{W}^F_{0,1}|_{\sqrt{q}}. \quad (A30)$$

Using the q expansion of the bosonic propagator, it is not difficult to realize that $\mathcal{W}^B|_{\sqrt{q}}=0$, and we can promote the previous relation to the full correlator $\mathcal{W}_{a,b}$ defined in (2.23) [using the identities in (A27)]:

$$\mathcal{W}_{0,0}|_{q^0}=\mathcal{W}_{0,1}|_{q^0}; \qquad \mathcal{W}_{0,0}|_{\sqrt{q}}=-\mathcal{W}_{0,1}|_{\sqrt{q}}. \quad (A31)$$

Other useful relations are between the q expansion of the derivative bosonic propagator $\partial\mathcal{P}$ and the fermionic propagator $S_{1,0}$:

$$\begin{aligned}\partial_\nu\mathcal{P}(\nu|\tau)|_{q^0}-\frac{\pi\nu_2}{2i\tau_2}&=-\frac{1}{4}S_{1,0}(\nu|\tau)|_{q^0},\\ \partial_\nu\mathcal{P}(\nu|\tau)|_{q^1}&=+\frac{1}{4}S_{1,0}(\nu|\tau)|_{q}.\end{aligned} \quad (A32)$$

3. Congruence subgroups of $SL(2,\mathbb{Z})$

We denote by $SL(2,\mathbb{Z})$ the group of 2×2 matrix with integers entries of determinant 1. For any N integers we have the following subgroups of $SL(2,\mathbb{Z})$:

$$\begin{aligned}\Gamma_0(N)&=\left\{\begin{pmatrix}a&b\\c&d\end{pmatrix}\in SL(2,\mathbb{Z})|\begin{pmatrix}a&b\\c&d\end{pmatrix}=\begin{pmatrix}*&*\\0&*\end{pmatrix}\ \mathrm{mod}\,N\right\},\\ \Gamma_1(N)&=\left\{\begin{pmatrix}a&b\\c&d\end{pmatrix}\in SL(2,\mathbb{Z})|\begin{pmatrix}a&b\\c&d\end{pmatrix}=\begin{pmatrix}1&*\\0&1\end{pmatrix}\ \mathrm{mod}\,N\right\},\\ \Gamma(N)&=\left\{\begin{pmatrix}a&b\\c&d\end{pmatrix}\in SL(2,\mathbb{Z})|\begin{pmatrix}a&b\\c&d\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}\ \mathrm{mod}\,N\right\}.\end{aligned} \quad (A33)$$

They satisfy the properties that $\Gamma(N)\subset\Gamma_1(N)\subset\Gamma_0(N)\subset SL(2,\mathbb{Z})$.

APPENDIX B: CHIRAL BLOCKS FOR THE TYPE II ORBIFOLDS

We recall some essential facts from the construction of Ref. [29]. The shifted T^2 lattice sum writes

$$\Gamma^w_{(2,2)}\begin{bmatrix}h\\g\end{bmatrix}:=\sum_{P_L,p_R\in\Gamma_{(2,2)}+w\frac{h}{2}}e^{i\pi g l\cdot w}q^{\frac{P_L^2}{2}}\bar{q}^{\frac{P_R^2}{2}}, \quad (B1)$$

where $\ell\cdot w=m_Ib^I+a_In^I$ where the shift vector $w=(a_I,b^I)$ is such that $w^2=2a\cdot b=0$ and

$$P_L^2 = \frac{|U(m_1 + a_1\frac{h}{2}) - (m_2 + a_2\frac{h}{2}) + T(n^1 + b^1\frac{h}{2}) + TU(n^2 + b^2\frac{h}{2})|^2}{2T_2U_2}, \quad P_L^2 - P_R^2 = 2\Big(m_I + a_I\frac{h}{2}\Big)\Big(n^I + b^I\frac{h}{2}\Big). \tag{B2}$$

T and U are the moduli of the T^2. We recall the full expressions for the orbifold blocks:

$$Z_{a,b}^{(22);h,g} := \begin{cases} Z_{a,b} = (2.20) & (h,g) = (0,0) \\ 4(-1)^{(a+h)g}\Big(\frac{\theta[^a_b](0|\tau)\theta[^{a+h}_{b+g}](0|\tau)}{\eta(\tau)^3\theta[^{1+h}_{1+g}]}\Big)^2 \times \Gamma_{(2,2)}(T,U) & (h,g) \neq (0,0) \end{cases}, \tag{B3}$$

$$Z_{a,b}^{(14);h,g} = \frac{1}{2}\sum_{h',g'=0}^{1} Z_{a,b}^{h,g}\begin{bmatrix} h' \\ g' \end{bmatrix}\Gamma^{w}_{(2,2)}\begin{bmatrix} h' \\ g' \end{bmatrix}, \tag{B4}$$

$$Z_{a,b}^{(10);h,g} = \frac{1}{2}\sum_{h_1,g_1=0}^{1}\frac{1}{2}\sum_{h_2,g_2=0}^{1} Z_{a,b}^{h,g}\begin{bmatrix} h; h_1, h_2 \\ g; g_1, g_2 \end{bmatrix}\Gamma^{w_1,w_2}_{(2,2)}\begin{bmatrix} h_1, h_2 \\ g_1, g_2 \end{bmatrix}, \quad \forall\ h, g. \tag{B5}$$

For the $n_v = 6$ model, the orbifold acts differently and we get

$$Z_{a,b}^{(6);h,g} = \frac{1}{2}\sum_{h',g'=0}^{1}(-1)^{hg'+gh'} Z_{a,b}^{h,g}\Gamma^{w}_{(2,2)}\begin{bmatrix} h' \\ g' \end{bmatrix}. \tag{B6}$$

In the previous expressions, the crucial point is that the shifted lattice sums $\Gamma^w_{(2,2)}[^{h'}_{g'}]$ act as projectors on their untwisted $h' = 0$ sector, while the g' sector is left free. We recall now the diagonal properties of the orbifold action (see Ref. [29] again) on the lattice sums:

$$\Gamma^{w_1,w_2}_{(2,2)}\begin{bmatrix} h, 0 \\ g, 0 \end{bmatrix} = \Gamma^{w_1}_{(2,2)}\begin{bmatrix} h \\ g \end{bmatrix}, \qquad \Gamma^{w_1,w_2}_{(2,2)}\begin{bmatrix} 0, h \\ 0, g \end{bmatrix} = \Gamma^{w_2}_{(2,2)}\begin{bmatrix} h \\ g \end{bmatrix}, \qquad \Gamma^{w_1,w_2}_{(2,2)}\begin{bmatrix} h, h \\ g, g \end{bmatrix} = \Gamma^{w_{12}}_{(2,2)}\begin{bmatrix} h \\ g \end{bmatrix}. \tag{B7}$$

The four-dimensional blocks $Z_{a,b}^{h,g}$ have the following properties : $Z_{a,b}^{h,g}[^0_0] = Z_{a,b}^{h,g}[^h_g] = Z_{a,b}^{h,g}$ (ordinary twist); $Z_{a,b}^{0,0}[^h_g]$ is a (4,4) lattice sum with one shifted momentum and thus projects out the $h = 0$ sector. Equivalent properties stand as well for the $n_v = 10$ model.

One has then in the field theory limit

$$Z_{a,b}^{(14);h,g} \in \Big\{Z_{a,b}^{0,0}, Z_{a,b}^{0,1}, \frac{1}{2}Z_{a,b}^{1,0}, \frac{1}{2}Z_{a,b}^{1,1}\Big\}, \qquad Z_{a,b}^{(10);h,g} \in \Big\{Z_{a,b}^{0,0}, Z_{a,b}^{0,1}, \frac{1}{4}Z_{a,b}^{1,0}, \frac{1}{4}Z_{a,b}^{1,1}\Big\}, \qquad Z_{a,b}^{(6);h,g} \in \{Z_{a,b}^{0,0}, Z_{a,b}^{0,1}, 0, 0\}, \tag{B8}$$

from where we easily deduce the effective definition given in (2.42) and the number c_h.

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