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Discrete geometry of a small causal diamond

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We study the discrete causal set geometry of a small causal diamond in a curved spacetime using the average abundance $\langle C_k \rangle$ of k-element chains or total orders in the underlying causal set $\mathbf{C}$. We begin by obtaining the first-order curvature corrections to the flat spacetime expression for $\langle C_k \rangle$ using Riemann normal coordinates. For fixed spacetime dimension this allows us to find a new expression for the discrete scalar curvature of $\mathbf{C}$ as well as the time-time component of its Ricci tensor in terms of the $\langle C_k \rangle$. We also find a new dimension estimator for $\mathbf{C}$ which replaces the flat spacetime Myrheim-Meyer estimator in generic curved spacetimes.

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I. INTRODUCTION

That there is a profound relationship between order and Lorentzian geometry has been evident ever since the work of Malament, Hawking, and others [1,2] where they showed the existence of a bijection between the causal structure, itself a partially ordered set, and the conformal class of the spacetime metric. This is one of the main motivations for the causal set approach to quantum gravity, which assumes that the primitive structure underlying spacetime is a locally finite partially ordered set, or causal set [3]. Instead of considering the spacetime metric as the fundamental dynamical variable in causal set theory (CST), it is the causal structure that one wishes to "quantize." However, to recover the full spacetime geometry from the causal structure, there must be a way to obtain the spacetime volume or equivalently, the conformal factor. This is achieved in CST via the condition of "local finiteness" which implies a fundamental spacetime discreteness: underlying every finite volume region of spacetime is a finite cardinality causal set. Thus, the continuum-discrete correspondence in CST is not exact but approximate, with the continuum being the approximation of the underlying causal set.

In order to maintain the relationship between volume and cardinality in all coordinate systems, a causal set $\mathbf{C}$ approximated by a spacetime (M, g) is obtained via the following random, Poisson discretization of (M, g) [3,4]. Given a fundamental scale ρ^{-1} (which could be the Planck volume), the probability that a spacetime region of volume V contains N-elements of C is given by the Poisson distribution

$$P_V(N) = \exp^{-\rho V} \frac{(\rho V)^N}{N!}, \tag{1}$$

for which

$$\langle N \rangle = \rho V, \tag{2}$$

thus establishing the required number to volume correspondence. To give credence to the existence of a fundamental spacetime discreteness, CST moreover requires the following conjecture. Namely, if a causal set $\mathbf{C}$ approximates to a spacetime (M, g), then (M, g) is unique, up to modifications to it on scales $< \rho^{-1}$. In other words, this conjectures that *all* the meaningful information about the geometry and topology of (M, g) at scales $\gg \rho^{-1}$ is contained in the causal set; continuum information below these scales is irrelevant since the discrete substructure, i.e., the causal set, is fundamental.

From a purely mathematical point of view, this conjecture[1] is very intriguing. While it has been verified in several different cases, a general proof is still not known, though considerable progress has been made in this direction [5–7]. A key question is how to extract continuum topological and geometric properties from $\mathbf{C}$ using purely order theoretic information. Uniqueness of the approximating spacetime with respect to a *given* geometric or topological property then follows, i.e., any two spacetimes which are approximations to $\mathbf{C}$ must share this property on scales $\gg \rho^{-1}$. For example, for a causal set $\mathbf{C}$ that is approximated by flat spacetime, the Myrheim-Meyer dimension gives a good estimate of the spacetime dimension [8,9], while the length of the longest chain or total order between elements in $\mathbf{C}$ gives a good estimate of the timelike distance [10]. An estimator for spatial distance in this case has also been obtained [11]. Additionally, the homology of spatial hypersurfaces can be constructed from the causal set underlying a globally hyperbolic spacetime [12,13]. A very important recent result is the construction of the scalar curvature from which the causal set action is obtained [14].

A natural question to ask of the flat spacetime results of Refs. [8–10] is how they are modified in the presence of curvature. While it is true in the continuum that a sufficiently small neighborhood of a point is approximately flat, the corrections from curvature can in fact be well quantified using Riemann normal coordinates in a convex normal neighbourhood of any point. However, we run into the following issue in the discrete case: there is at present no

[1]It is also often referred to as a "fundamental theorem" of CST.

known purely order theoretic definition of a "small" neighborhood of an element in a causal set which corresponds to a convex normal neighborhood. At best it may be possible to find approximately flat subsets in a causal set but it is still unclear how to do this in a systematic way [15].

This issue however will not be the focus of our current investigation. Instead we consider only those N-element causal sets $\mathbf{C}$ which are approximated by a small causal diamond or Alexandrov interval $\mathbf{I}[p, q]$ in a generic curved spacetime (M, g), where the smallness parameter is given by the proper time T between the events p and q. We expand the metric in Riemann normal coordinates (RNC) about an origin $r = (0, \ldots, 0)$

$$g_{ab}(x) = \eta_{ab}(0) - \frac{1}{3}x^c x^d R_{acbd}(0) + O(x^3), \qquad (3)$$

where the $O(x)$ correction to the metric vanishes since $\Gamma^c_{ab}(0) = 0$. The RNC has been used to calculate the volume of a small interval $\mathbf{I}[p, q]$ [8,16,17] for which the first-order correction to the volume of $\mathbf{I}[p, q]$ due to the effect of curvature occurs at $O(T^{2+n})$.

The starting point of our analysis is Meyer's work [9] in which a general expression was found for the average abundance $\langle C_k\rangle$ of k-chains or k-element total orders in a causal set $\mathbf{C}_0$ which is approximated by an Alexandroff interval $\mathbf{I}_0[p, q]$ in flat spacetime. Following Myrheim, Meyer used this to find a dimension estimator for the dimension of $\mathbf{I}_0[p, q]$ employing only $\langle C_1\rangle$, the average abundance of elements in $\mathbf{I}_0[p, q]$, and $\langle C_2\rangle$, the average abundance of relations. Using the RNC expansion and with the help of Ref. [17] we extend this analysis to the curved spacetime case. We find that the $\langle C_k\rangle$ in $\mathbf{I}[p, q]$ satisfy a recursion relation and depend on the scalar curvature $R(0)$ and the time-time component of the Ricci tensor $R_{00}(0)$. It is then an easy exercise to invert these relations and find expressions for $R(0)$ and $R_{00}(0)$ in terms of $\langle C_1\rangle$, $\langle C_2\rangle$, $\langle C_3\rangle$ for fixed spacetime dimension. We construct a new dimension estimator for generic curved spacetimes using $\langle C_1\rangle$, $\langle C_2\rangle$, $\langle C_3\rangle$, $\langle C_4\rangle$ and show that it reduces to the Myrheim-Meyer estimator in flat spacetime.

As mentioned earlier, the scalar curvature of an element in a causal set was first calculated by Benincasa and Dowker [14] using a curved spacetime expression for a nonlocal D'Alembertian on a causal set. They showed that $R(q)$ for an element q in the $\mathbf{C}$ can be expressed in terms of the abundance of k-element "inclusive intervals" which are order theoretically very distinct from k-chains. Like k-chains they too have a bottom element e_1 and a top element e_k, but unlike k-chains, every element in the interval $I[e_1, e_k]$ which is the intersection of the future of e_1 and the past of e_k belongs to the inclusive interval, and has precisely $k - 2$ elements satisfying $e_1 \prec e_i \prec e_k$. In contrast, the expression for $R(0)$ that we find depends only on the abundance of k-chains. This may suggest that for manifold-like causal sets there are hidden relations between these seemingly different order theoretic entities.

The plan of our paper is as follows. In Sec. II after first presenting some basic definitions, we reproduce Meyer's results for the $\langle C_k\rangle$ thus setting the notation that we will use in the rest of the paper. In Sec. III we present the main calculation in the paper, where we use the RNC to obtain the lowest order curvature correction to $\langle C_k\rangle$. We show that there is a recursion relation between the coefficients in the expression for the different $\langle C_k\rangle$, but that the form of the dependence on the $R(0)$ and $R_{00}(0)$ is the same for all k. In Sec. IV for a fixed spacetime dimension we find expressions for $R(0)$ and $R_{00}(0)$ which depend only on $\langle C_1\rangle$, $\langle C_2\rangle$, $\langle C_3\rangle$. The coefficients in these expressions again have a simple dependence on dimension. In Sec. V we point out that the Myrheim-Meyer dimension estimator is insufficient in a generic curved spacetime and find a new dimension estimator using $\langle C_1\rangle$, $\langle C_2\rangle$, $\langle C_3\rangle$, $\langle C_4\rangle$. This estimator reduces to the Myrheim-Meyer dimension estimator in the case of flat spacetime. An important question in these calculations is how the error decreases with the sprinkling density ρ (i.e., the inverse of the volume cutoff)—the larger ρ is the closer one comes to the continuum. In Sec. VI using the technique developed in Ref. [9] we show that the error in $\langle C_k\rangle$ grows as $\rho^{\frac{2k-1}{2}}$ which means that the error in $R(0)$, $R_{00}(0)$ and n goes like $\rho^{-1/2}$, thus going to zero in the continuum limit. We discuss the implications of our results in Sec. VII and the questions that need to be addressed in the future. Finally, the Appendix contains explicit calculations of the results that appear in the main body of the paper.

II. THE ABUNDANCE OF k-CHAINS IN FLAT SPACETIME

A k-chain in a causal set $\mathbf{C}$ is a k-element total order, i.e., a set of elements $\{e_1, e_2, \ldots, e_k\}$, $e_i \in \mathbf{C}$ such that $e_i \prec e_{i+1}$ for all i. For any finite element $\mathbf{C}$, the number C_k of k-chains is therefore invariant of the choice of labeling of $\mathbf{C}$. This makes C_k a good observable. Note that the e_i and e_{i+1} need not be "linked," i.e., there could exist an element $e \in \mathbf{C}$ such that $e_i \prec e \prec e_{i+1}$. Moreover, $e_1 \prec e \prec e_k$ does not imply that e belongs to the k-chain. In contrast, a k-inclusive interval is defined as $I_k[e_1, e_k] = \mathrm{Future}(e_1) \cap \mathrm{Past}(e_k)$ [14]. Along with the elements e_1, e_k it also contains precisely k-elements. However, every $e \in \mathbf{C}$ such that $e_1 \prec e \prec e_k$ belongs to $I_k[e_1, e_k]$, which means that the order theoretic structure of a k-chain is very different from that of a k-inclusive interval. The fact that one can express the discrete scalar curvature both in terms of the abundance of the inclusive intervals as shown by Benincasa and Dowker [14] and in terms of the abundance of k-chains as we will show in Sec. IV thus suggests a hidden connection between the two.

We now reproduce Meyer's results in n-dimensional flat spacetime (M_0, η) using notation that we will find convenient in the curved spacetime generalization. Let $p,\ q \in M_0$ such that $p = (-T/2, 0, \ldots, 0)$ and $q = (T/2, 0, \ldots, 0)$. For a causal set $\mathbf{C}_0$ which is approximated by $\mathbf{I}_0[p, q]$ for a given sprinkling density ρ, the average abundance of elements, or 1-chains $\langle C_1\rangle_\eta$ is given by

$$\begin{aligned}\langle C_1\rangle_\eta &= \rho \mathbf{V_0} = \rho \int_{\mathbf{I}_0[p,q]} dx_1 \\ &= 2\rho \int_0^{\frac{T}{2}} dt \int_0^{\frac{T}{2}-t} dr r^{n-2} \int d\Omega_{n-2} \\ &= \rho \frac{2A_{n-2}}{n(n-1)}\left(\frac{T}{2}\right)^n = \rho \zeta_0 T^n, \end{aligned} \qquad (4)$$

where A_{n-2} is the volume of the unit $(n-2)$ sphere S^{n-2}. Next, the average number of 2-chains or relations in $\mathbf{I}_0[p, q]$ is given by the probability of there being a pair of elements $x_1, x_2 \in \mathbf{I}_0[p, q]$ such that $x_1 \prec x_2$, or

$$\langle C_2\rangle_\eta = \rho^2 \int_{\mathbf{I}_0[p,q]} dx_1 \int_{J^+(x_1)\cap \mathbf{I}_0[p,q]} dx_2. \qquad (5)$$

Recognizing that the integral over dx_2 is simply the volume of the smaller interval $\mathbf{I}_0[x_1, q]$ and using Eq. (4)

$$\begin{aligned}\langle C_2\rangle_\eta &= \rho^2 \frac{2A_{n-2}}{2^n n(n-1)} \int_{\mathbf{I}_0[p,q]} dx_1 T_1^n \\ &= \rho^2 {\mathbf{V_0}}^2 \frac{\Gamma(n+1)\Gamma(\frac{n}{2})}{4\Gamma(\frac{3n}{2})}, \end{aligned} \qquad (6)$$

Meyer was able to similarly use the nested integral expression for $\langle C_k\rangle_\eta$

$$\begin{aligned}\langle C_k\rangle_\eta &= \rho^k \int_{\mathbf{I}_0[p,q]} dx_1 \\ &\quad \times \int_{J^+(x_1)\cap \mathbf{I}_0[p,q]} dx_2 \ldots \int_{J^+(x_{k-1})\cap \mathbf{I}_0[p,q]} dx_k \\ &= \rho^k \int_{\mathbf{I}_0[p,q]} dx_1 \langle C_{k-1}(x_1)\rangle_\eta, \end{aligned} \qquad (7)$$

to find by induction the general form

$$\langle C_k\rangle_\eta = \rho^k \chi_k {\mathbf{V_0}}^k = \rho^k \zeta_k T^{kn}, \qquad (8)$$

where

$$\begin{aligned}\chi_k &\equiv \frac{1}{k}\left(\frac{\Gamma(n+1)}{2}\right)^{k-1} \frac{\Gamma(\frac{n}{2})\Gamma(n)}{\Gamma(\frac{kn}{2})\Gamma(\frac{(k+1)n}{2})}, \\ \zeta_k &\equiv \left(\frac{2A_{n-2}}{2^n n(n-1)}\right)^k \chi_k = {\zeta_0}^k \chi_k, \end{aligned} \qquad (9)$$

with ζ_0 defined as in Eq. (4). Note, in particular, that $\chi_1 = 1$. We will find it useful to express $\langle C_k\rangle_\eta$ as

$$\langle C_k\rangle_\eta = \rho^k \zeta_{k-1} \int_{\mathbf{I}_0[p,q]} dx_1 T_1^{(k-1)n} = \rho^k \zeta_{k-1} \mathrm{I}_1((k-1)n), \qquad (10)$$

where $\mathrm{I}_1(m)$ is evaluated in Eq. (A4) of the Appendix. As discussed above, the average number of chains in a finite element causal set $\mathbf{C}$ is itself a covariant observable. In particular, the distribution of the abundance of k-chains as a function of k in a finite element causal set can be compared with the distribution of $\langle C_k\rangle_\eta$; if the two distributions agree, it is an indication that the $\mathbf{C}$ may be well approximated by flat spacetime and is therefore manifold-like. A similar comparison using k-inclusive intervals was found to be useful in determining flat spacetime behavior in a model of two-dimensional causal set quantum gravity [18]. It is therefore important to find a generalization of $\langle C_k\rangle_\eta$ to curved spacetime.

Meyer obtained a dimension estimator from $\langle C_k\rangle_\eta$ by observing that the ratio

$$\mathsf{f}_0(n) \equiv \frac{\langle C_2\rangle_\eta}{\langle C_1\rangle_\eta^2} = \frac{\Gamma(n+1)\Gamma(\frac{n}{2})}{4\Gamma(\frac{3n}{2})} \qquad (11)$$

is only a function of n. Thus, one has an expression for the dimension which depends only on order-theoretic information in the causal set. Indeed, $\mathsf{f}_0(n)$ is one-half of Myrheim's ordering fraction

$$\mathbf{f}(\mathbf{C}) \equiv R\binom{N}{2}^{-1} \approx \frac{2R}{N^2}, \qquad (12)$$

where $R = \langle C_2\rangle$ is the number of relations and $N = \langle C_1\rangle$. In two spacetime dimensions, for example, $\mathbf{f}(2) = 1/2$, i.e., the inverse of the spacetime dimension. In Sec. V we will show that Eq. (11) does not suffice in curved spacetime and there is need for a new dimension estimator.

III. THE ABUNDANCE OF k-CHAINS IN A SMALL CAUSAL DIAMOND IN CURVED SPACETIME

The RNC expansion to order T^2 gives an expression for $\langle C_1\rangle$ in $\mathbf{I}[p, q]$ [8,16]

$$\begin{aligned}\langle C_1\rangle &= \rho \mathbf{V} = \rho \int_{\mathbf{I}[p,q]} \sqrt{-g_1} dx_1 \\ &= \rho \mathbf{V_0}(1 + \alpha_1 R(0) T^2 + \beta_1 R_{00}(0) T^2), \end{aligned} \qquad (13)$$

where

$$\alpha_1 = -\frac{n}{24(n+1)(n+2)}, \qquad \beta_1 = \frac{n}{24(n+1)}, \qquad (14)$$

and uses the RNC expansion

$$\sqrt{-g_1} = 1 - \frac{1}{6} x_1^\mu x_1^\nu R_{\mu\nu}(0) + O(x^3). \qquad (15)$$

Now, the average number of 2-chains or relations is given by the similar generalization

$$\begin{aligned}\langle C_2\rangle &= \rho^2 \int_{\mathbf{I}[p,q]} \sqrt{-g_1} dx_1 \int_{J^+(x_1)\cap \mathbf{I}[p,q]} \sqrt{-g_2} dx_2 \\ &= \rho^2 \int_{\mathbf{I}[p,q]} \sqrt{-g_1} dx_1 \mathbf{V}_1, \end{aligned} \qquad (16)$$

where $\mathbf{V}_1$ denotes the volume of the region $J^+(x_1) \cap \mathbf{I}[p, q]$. Using the covariant form of Eq. (13) we see that

$$\langle C_2\rangle = \rho^2\zeta_1 \int_{\mathbf{I}[p,q]} dx_1 T_1^n\Big(1-\frac{1}{6}x_1^\mu x_1^\nu R_{\mu\nu}(0)+\alpha_1 T_1^2 R(y_1) + \beta_1 T_1^\mu T_1^\nu R_{\mu\nu}(y_1)\Big) + O(T^{n+3})$$
$$= \rho^2\zeta_1\Big[\int_{\mathbf{I}[p,q]} dx_1 T_1^n + \int_{\mathbf{I}_0[p,q]} dx_1 T_1^n\Big(-\frac{1}{6}x_1^\mu x_1^\nu R_{\mu\nu}(0) + \alpha_1 T_1^2 R(y_1)+\beta_1 T_1^\mu T_1^\nu R_{\mu\nu}(y_1)\Big)\Big] + O(T^{n+3}), \quad (17)$$

where we have split the integral in the manner of Ref. [8]: the first is a *flat spacetime* integral over the curved spacetime interval $\mathbf{I}[p,q]$, whereas the second is the contribution from the curvature terms over the flat spacetime interval $\mathbf{I}_0[p,q]$. y_1 is the midpoint of the interval $\mathbf{I}[x_1,q]$ as shown in Fig 1. Using the light-cone coordinates $u=t-r$, $v=t+r$ we see from Fig. 1 that

$$T_1^\mu = \frac{1}{2}T^\mu - x_1^\mu, \qquad T_1 = \sqrt{\Big(\frac{T}{2}-u_1\Big)\Big(\frac{T}{2}-v_1\Big)}. \quad (18)$$

Thus,

$$T_1^\mu T_1^\nu R_{\mu\nu}(y_1) = \frac{T^2}{4}R_{00}(y_1) - x_1^\mu T R_{0\mu}(y_1) + x_1^\mu x_1^\nu R_{\mu\nu}(y_1). \quad (19)$$

The first integral was evaluated in Ref. [17] and shown to be of the form

$$\int_{\mathbf{I}[p,q]} dx_1 T_1^m = \int_{\mathbf{I}_0[p,q]} dx_1 T_1^m\Big(1+\frac{T^2}{24}R_{00}(0)\Big) \quad (20)$$

for any non-negative integer m. Moreover, as can be readily seen, to order T^2, $R_{\mu\nu}(y_1)$ can be replaced with $R_{\mu\nu}(0)$.

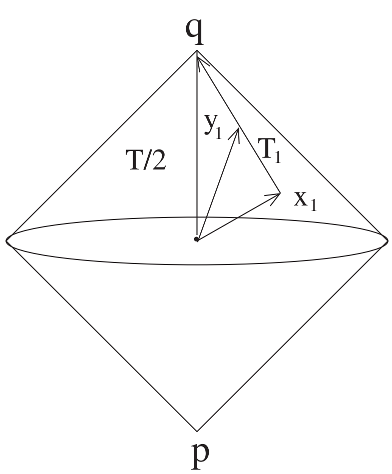

FIG. 1. An Alexandroff interval $\mathbf{I}[p,q]$ in flat spacetime. T_1 is the proper time between the events x_1 and q and y_1 is the midpoint of $\mathbf{I}[x_1,q]$.

We can moreover simplify the expressions in $\langle C_2\rangle$ substantially by using the symmetries of $\mathbf{I}_0[p,q]$. Expanding the term

$$\int_{\mathbf{I}_0[p,q]} dx_1 x_1^\mu x_1^\nu R_{\mu\nu}(0)T_1^m = \int_{\mathbf{I}_0[p,q]} dx_1 t_1^2 R_{00}(0)T_1^m + 2\int_{\mathbf{I}_0[p,q]} dx_1 t x_1^i R_{0i}(0)T_1^m + \int_{\mathbf{I}_0[p,q]} dx_1 x_1^i x_1^j R_{ij}(0)T_1^m, \quad (21)$$

for m a positive integer, we see that the cross terms do not contribute, so that we are left with

$$\int_{\mathbf{I}_0[p,q]} dx_1 x_1^\mu x_1^\nu R_{\mu\nu}(0)T_1^m = R_{00}(0)\int_{\mathbf{I}_0[p,q]} dx_1 t_1^2 T_1^m + \sum_{i=1}^{n-1} R_{ii}(0)\int_{\mathbf{I}_0[p,q]} dx_1 (x_1^i)^2 T_1^m, \quad (22)$$

since the last integral is independent of the spatial direction i, due to the symmetry of $\mathbf{I}_0[p,q]$.

Gathering the coefficients of $R(0)$ and $R_{00}(0)$

$$\langle C_2\rangle = \rho^2\zeta_1\Big[\mathrm{I}_1(n) + R(0)\Big[\Big(\beta_1-\frac{1}{6}\Big)\mathrm{I}_2(n)+\alpha_1\mathrm{I}_1(n+2)\Big] + R_{00}(0)\Big[\frac{T^2}{24}\mathrm{I}_1(n)+\Big(\beta_1-\frac{1}{6}\Big)\mathrm{I}_2(n) - \frac{1}{6}\mathrm{I}_3(n)+\beta_1\mathrm{I}_4(n)\Big]\Big], \quad (23)$$

where we have used $\sum_{i=1}^{n-1} R_{ii}(0) = R_{00}(0)+R(0)$ and we define the general class of integrals

$$\begin{aligned} \mathrm{I}_1(m) &= \int_{\mathbf{I}_0[p,q]} dx_1 T_1^m, \\ \mathrm{I}_2(m) &= \int_{\mathbf{I}_0[p,q]} dx_1 T_1^m r_1^2\cos^2\theta_1, \\ \mathrm{I}_3(m) &= \int_{\mathbf{I}_0[p,q]} dx_1 t_1^2 T_1^m, \\ \mathrm{I}_4(m) &= \int_{\mathbf{I}_0[p,q]} dx_1 \Big(\frac{T}{2}-t_1\Big)^2 T_1^m \end{aligned} \quad (24)$$

for non-negative integers m. These integrals have been evaluated in the Appendix. Using the terminology defined therein, we find it useful to reexpress the last three integrals in terms of the first

$$\mathrm{I}_2(m) = f_2(m)\mathrm{I}_1(m)T^2,$$
$$\mathrm{I}_3(m) = f_3(m)\mathrm{I}_1(m)T^2,$$
$$\mathrm{I}_4(m) = f_4(m)\mathrm{I}_1(m)T^2,$$

and

$$\mathrm{I}_1(m+2) = g_1(m)\mathrm{I}_1(m)T^2,$$

where $g_1(m)$, $f_2(m)$, $f_3(m)$, and $f_4(m)$ are defined in Eq. (A16) of the Appendix. Given Eq. (10) we find that

$$\langle C_2\rangle = \langle C_2\rangle_\eta[1 + T^2\alpha_2 R(0) + T^2\beta_2 R_{00}(0)], \quad (25)$$

where

$$\begin{aligned}\alpha_2 &= \left(\beta_1 - \frac{1}{6}\right)f_2(n) + \alpha_1 g_1(n),\\ \beta_2 &= \frac{1}{24} + \left(\beta_1 - \frac{1}{6}\right)f_2(n) - \frac{1}{6}f_3(n) + \beta_1 f_4(n).\end{aligned} \quad (26)$$

We find that

$$\alpha_2 = -\frac{4n}{24(2n+2)(3n+2)}, \qquad \beta_2 = \frac{4n}{24(3n+2)}. \quad (27)$$

We can go one step further and calculate

$$\begin{aligned}\langle C_3\rangle &= \rho\int dx_1\sqrt{-g_1}\langle C_2(x_1)\rangle\\ &= \rho^3\zeta_2\Bigg[\mathrm{I}_1(2n) + R(0)\Bigg(\left(\beta_2 - \frac{1}{6}\right)\mathrm{I}_2(2n)\\ &\quad + \alpha_2\mathrm{I}_1(2n+2)\Bigg) + R_{00}(0)\Bigg(\frac{T^2}{24}\mathrm{I}_1(2n)\\ &\quad + \left(\beta_2 - \frac{1}{6}\right)\mathrm{I}_2(2n) - \frac{1}{6}\mathrm{I}_3(2n) + \beta_2\mathrm{I}_4(2n)\Bigg)\Bigg]\\ &= \langle C_3\rangle_\eta\left[1 + T^2\alpha_3 R(0) + T^2\beta_3 R_{00}(0)\right],\end{aligned} \quad (28)$$

where again we have used Eq. (10) and

$$\begin{aligned}\alpha_3 &= \left(\beta_2 - \frac{1}{6}\right)f_2(2n) + \alpha_2 g_1(2n)\\ &= -\frac{6n}{24(3n+2)(4n+2)},\\ \beta_3 &= \frac{1}{24} + \left(\beta_2 - \frac{1}{6}\right)f_2(2n) - \frac{1}{6}f_3(2n) + \beta_2 f_4(2n)\\ &= \frac{6n}{24(4n+2)}.\end{aligned} \quad (29)$$

This suggests an iterative formula

$$\begin{aligned}\alpha_{k+1} &= \left(\beta_k - \frac{1}{6}\right)f_2(kn) + \alpha_k g_1(kn),\\ \beta_{k+1} &= \frac{1}{24} + \left(\beta_k - \frac{1}{6}\right)f_2(kn) - \frac{1}{6}f_3(kn) + \beta_k f_4(kn),\end{aligned} \quad (30)$$

with

$$\begin{aligned}\alpha_k &= -\frac{nk}{12(kn+2)((k+1)n+2)},\\ \beta_k &= \frac{nk}{12((k+1)n+2)}.\end{aligned} \quad (31)$$

Lemma 1. To the lowest order correction in the flat spacetime expression, the average number of k-element chains in a small causal diamond is

$$\langle C_k\rangle = \langle C_k\rangle_\eta[1 + T^2\alpha_k R(0) + T^2\beta_k R_{00}(0)] + O(T^{kn+3}), \quad (32)$$

where α_k and β_k are given by Eq. (31).

Proof: We will prove this inductively. We have already shown it for $k = 2$. Now, let us assume $\langle C_k\rangle$ is of the form Eq. (32). Just as in the flat spacetime case, one has nested integrals so that

$$\begin{aligned}\langle C_{k+1}\rangle &= \rho\int_{\mathbf{I}[p,q]} dx_1\sqrt{-g_1}\langle C_k(x_1)\rangle\\ &= \rho^{k+1}\zeta_k\int_{\mathbf{I}_0[p,q]} dx_1 T_1^{kn}\Bigg(1 + \frac{T^2}{24}R_{00}(0)\\ &\quad - \frac{1}{6}x_1^\mu x_1^\nu R_{\mu\nu}(0) + \alpha_k T_1^2 R(0)\\ &\quad + \beta_k T_1^\mu T_1^\nu R_{\mu\nu}(0)\Bigg) + O(T^{kn+3}),\end{aligned} \quad (33)$$

where we have used Eq. (20) to reduce the integral over $\mathbf{I}[p,q]$ to one over $\mathbf{I}_0[p,q]$ to order $O(T^{kn+2})$. Using the integrals $\mathrm{I}_{1,2,3,4}(m)$ from the Appendix, we can reduce this to

$$\begin{aligned}\langle C_{k+1}\rangle &= \rho^{k+1}\zeta_k\Big[\mathrm{I}_1(kn) + R(0)\Big(\Big(\beta_k - \frac{1}{6}\Big)\mathrm{I}_2(kn) + \alpha_k\mathrm{I}_1(kn+2)\Big)\\ &\quad + R_{00}(0)\Big(\frac{T^2}{24}\mathrm{I}_1(kn) + \Big(\beta_k - \frac{1}{6}\Big)\mathrm{I}_2(kn) - \frac{1}{6}\mathrm{I}_3(kn) + \beta_k\mathrm{I}_4(kn)\Big)\Big],\\ &= \langle C_{k+1}\rangle_\eta\Big[1 + T^2R(0)\Big(\Big(\beta_k - \frac{1}{6}\Big)f_2(kn) + \alpha_k g_1(kn+2)\Big)\\ &\quad + T^2R_{00}(0)\Big(\frac{1}{24} + \Big(\beta_k - \frac{1}{6}\Big)f_2(kn) - \frac{1}{6}f_3(kn) + \beta_k f_4(kn)\Big)\Big].\end{aligned} \tag{34}$$

Writing it in the form of Eq. (32), with

$$\begin{aligned}\alpha_{k+1} &= \Big(\beta_k - \frac{1}{6}\Big)f_2(kn) + \alpha_k g_1(kn+2),\\ \beta_{k+1} &= \frac{1}{24} + \Big(\beta_k - \frac{1}{6}\Big)f_2(kn) - \frac{1}{6}f_3(kn) + \beta_k f_4(kn),\end{aligned} \tag{35}$$

we find the desired form for $\langle C_{k+1}\rangle$. It then follows from the expressions for $g_1(m)$, $f_2(m)$, $f_3(m)$, and $f_4(m)$ [Eqs. (A5), (A7), (A8), and (A15)] that

$$\begin{aligned}\alpha_{k+1} &= -\frac{n(k+1)}{12((k+1)n+2)((k+2)n+2)},\\ \beta_{k+1} &= \frac{n(k+1)}{12((k+2)n+2)}.\end{aligned} \tag{36}$$

□

IV. SCALAR CURVATURE FROM THE ABUNDANCE OF *k*-CHAINS

For a fixed n $\langle C_k\rangle$ contains three unknowns, T, $R(0)$, and $R_{00}(0)$. Thus, we need at least three values of k in order to determine $R(0)$. For each $\langle C_k\rangle$ the lowest order correction due to curvature is $O(T^{kn+2})$. Hence, as in the flat spacetime calculation of the Myrheim-Meyer dimension Eq. (11) we must take appropriate powers of $\langle C_1\rangle$, $\langle C_2\rangle$, and $\langle C_3\rangle$ to be able to compare their lowest order corrections. Defining

$$Q_k \equiv \Big(\frac{\langle C_k\rangle}{\rho^k\zeta_k}\Big)^{3/k} = \frac{1}{\zeta_0^3}\Big(\frac{\langle C_k\rangle}{\rho^k\chi_k}\Big)^{3/k} \tag{37}$$

for $k = 1, 2, 3$

$$Q_1 = T^{3n}(1 + 3\alpha_1R(0)T^2 + 3\beta_1R_{00}(0)T^2) + O(T^{3n+2}), \tag{38}$$

$$Q_2 = T^{3n}\Big(1 + \frac{3}{2}\alpha_2R(0)T^2 + \frac{3}{2}\beta_2R_{00}(0)T^2\Big) + O(T^{3n+2}), \tag{39}$$

$$Q_3 = T^{3n}(1 + \alpha_3R(0)T^2 + \beta_3R_{00}(0)T^2) + O(T^{3n+2}). \tag{40}$$

Thus, the Q_k are independent of the sprinkling density ρ and hence can be used to construct continuum geometric parameters. It is useful to gather a few identities and definitions before we proceed:

$$\begin{aligned}\beta_k\alpha_k^{-1} &= -(kn+2),\\ \Psi_k &\equiv \alpha_k\beta_{k+1} - \alpha_{k+1}\beta_k = -n\alpha_k\alpha_{k+1},\\ \Phi_k &\equiv \frac{k}{k+1}\beta_{k+1} - \beta_k,\\ \Theta_k &= \frac{k}{k+1}\alpha_{k+1} - \alpha_k,\\ K_k &\equiv ((k+1)n+2)Q_k,\\ J_k &\equiv (kn+2)K_k.\end{aligned} \tag{41}$$

We eliminate the $R_{00}(0)$ term from Eqs. (38) and (39) and subsequently from Eqs. (39) and (40) to get the pair of equations

$$\Big(\frac{\beta_2}{2}Q_1 - \beta_1Q_2\Big)T^{-3n} = \Phi_1 + \frac{3}{2}\Psi_1T^2R(0), \tag{42}$$

$$\Big(\frac{2\beta_3}{3}Q_2 - \beta_2Q_3\Big)T^{-3n} = \Phi_2 + \Psi_2T^2R(0). \tag{43}$$

Since both $R(0)$ and T are unknowns, we first eliminate the $R(0)T^2$ term:

$$\begin{aligned}&\Big(\frac{\beta_2}{3}\Psi_2Q_1 - \frac{2}{3}(\beta_1\Psi_2 - \beta_3\Psi_1)Q_2 + \beta_2\Psi_1Q_3\Big)T^{-3n}\\ &\quad = \frac{2}{3}\Psi_2\Phi_1 - \Psi_1\Phi_2.\end{aligned} \tag{44}$$

We find after some algebraic manipulation that

$$T^{3n} = \frac{1}{2n^2}(J_1 - 2J_2 + J_3), \tag{45}$$

with the J_i's given by Eq. (41). We thus obtain the expression for the scalar curvature

$$R(0) = -\frac{2(n+2)(2n+2)(3n+2)}{n^3T^{3n+2}}(K_1 - 2K_2 + K_3) \tag{46}$$

or more explicitly

$$R(0) = -2(n+2)(2n+2)(3n+2)2^{\frac{3n+2}{3n}}n^{\frac{4}{3n}-1} \times \frac{(K_1 - 2K_2 + K_3)}{(J_1 - 2J_2 + J_3)^{\frac{3n+2}{3n}}}. \quad (47)$$

We may additionally solve for $R_{00}(0)$ by eliminating $R(0)$ from Eqs. (38)–(40):

$$\left(\frac{\alpha_2}{2}Q_1 - \alpha_1 Q_2\right)T^{-3n} = \Theta_1 - \frac{3}{2}\Psi_1 T^2 R(0), \quad (48)$$

$$\left(\frac{2\alpha_3}{3}Q_2 - \alpha_2 Q_3\right)T^{-3n} = \Theta_2 - \Psi_2 T^2 R(0). \quad (49)$$

Either equation along with Eq. (45) gives

$$R_{00}(0) = -\frac{4(2n+2)(3n+2)}{n^3 T^{3n+2}}((n+2)Q_1 - (5n+4)Q_2 + (4n+2)Q_3). \quad (50)$$

As a check, let us consider the case $R_{00}(0) = 0$ so that from Eqs. (38)–(40) we see that

$$\begin{aligned} R(0) > 0 &\Rightarrow Q_1 < Q_2 < Q_3 \\ R(0) < 0 &\Rightarrow Q_1 > Q_2 > Q_3 \\ R(0) = 0 &\Rightarrow Q_1 = Q_2 = Q_3 = T^{3n}. \end{aligned} \quad (51)$$

Moreover,

$$K_1 - 2K_2 + K_3 = -\frac{n^3}{2(n+2)(2n+2)(3n+2)}R(0)T^{3n+2}, \quad (52)$$

which therefore has the opposite sign to $R(0)$ and is zero when $R(0) = 0$.

As a further check, we note that if both $R_{00}(0) = 0$ and $R(0) = 0$, $Q_1 = Q_2 = Q_3 = T^{3n}$ so that not only is $K_1 - 2K_2 + K_3 = (2n+2)Q_1 - 2(3n+2)Q_2 + (4n+2)Q_3 = 0$ but also $(n+2)Q_1 - (5n+4)Q_2 + (4n+2)Q_3 = 0$ which appears in Eq. (50).

V. A NEW DIMENSION ESTIMATOR FOR CURVED SPACETIME

As one can guess by now, the ordering fraction or equivalently the function $\mathsf{f}(n)$ in curved spacetime, clearly involves the curvature contribution nontrivially. Expanding to order T^2 the curved spacetime version of the Myrheim-Meyer dimension estimator is

$$\begin{aligned} \mathsf{f}(n) = \frac{\langle C_2\rangle}{\langle C_1\rangle^2} &= \mathsf{f}_0(n)(1 + T^2(\alpha_2 - 2\alpha_1)R(0) + T^2(\beta_2 - 2\beta_1)R_{00}(0)) + O(T^3) \\ &= \mathsf{f}_0(n)\left(1 + \frac{n^2T^2}{12(n+1)(3n+2)}\left(\frac{2}{(n+2)}R(0) - R_{00}(0)\right)\right) + O(T^3). \end{aligned} \quad (53)$$

In the special case that $R(0) = R_{00}(0) = 0$, $\mathsf{f}(n) \approx \mathsf{f}_0(n)$ up to order T^2. For a generic spacetime $\mathsf{f}(n)$ is, however, insufficient as a dimension estimator and we must find a replacement. Given that along with n there are 4 unknowns to be solved in terms of the $\langle C_k\rangle$, the simplest way to do so is to include $\langle C_4\rangle$ in our analysis.

We define

$$S_k = (\langle C_k\rangle/\rho^k\zeta_k)^{4/k} \quad (54)$$

analogous to the Q_i in the previous section, with $k = 1, 2, 3, 4$ which again is independent of the sprinkling density. To the lowest order correction we have the four equations

$$\begin{aligned} S_1 &= T^{4n}(1 + 4\alpha_1 R(0)T^2 + 4\beta_1 R_{00}(0)T^2), \\ S_2 &= T^{4n}(1 + 2\alpha_2 R(0)T^2 + 2\beta_2 R_{00}(0)T^2), \\ S_3 &= T^{4n}\left(1 + \frac{4}{3}\alpha_3 R(0)T^2 + \frac{4}{3}\beta_3 R_{00}(0)T^2\right), \\ S_4 &= T^{4n}(1 + \alpha_4 R(0)T^2 + \beta_4 R_{00}(0)T^2). \end{aligned} \quad (55)$$

Eliminating $R(0)T^2$ from the above we get

$$\begin{aligned} \left(\frac{1}{2}\alpha_2 S_1 - \alpha_1 S_2\right) &= T^{4n}(\Theta_1 - 2\Psi_1 R_{00}T^2), \\ \left(\frac{2}{3}\alpha_3 S_2 - \alpha_2 S_3\right) &= T^{4n}\left(\Theta_2 - \frac{4}{3}\Psi_2 R_{00}T^2\right), \\ \left(\frac{3}{4}\alpha_4 S_3 - \alpha_3 S_4\right) &= T^{4n}(\Theta_3 - \Psi_3 R_{00}T^2), \end{aligned} \quad (56)$$

from which we may eliminate $R_{00}(0)T^2$ to get

$$\begin{aligned} &\frac{4}{3}\left(\frac{1}{2}\alpha_2 S_1 - \alpha_1 S_2\right)\Psi_2 - 2\left(\frac{2}{3}\alpha_3 S_2 - \alpha_2 S_3\right)\Psi_1 \\ &\quad = T^{4n}\left(\frac{4}{3}\Theta_1\Psi_2 - 2\Theta_2\Psi_1\right), \\ &\frac{3}{4}\left(\frac{2}{3}\alpha_3 S_2 - \alpha_2 S_3\right)\Psi_3 - \left(\frac{3}{4}\alpha_4 S_3 - \alpha_3 S_4\right)\Psi_2 \\ &\quad = T^{4n}\left(\frac{3}{4}\Theta_2\Psi_3 - \Theta_3\Psi_2\right). \end{aligned}$$

This gives us an expression for T^{4n} [which we check reduces to Eq. (45)]. After some algebra this gives the following implicit form for the dimension:

$$(n+2)(2n+2)S_1 - 3(2n+2)(3n+2)S_2 + 3(3n+2) \times (4n+2)S_3 - (4n+2)(5n+2)S_4 = 0. \quad (57)$$

Importantly, in the absence of curvature the S_k are all equal and the left-hand side is identically zero. We therefore need to be more careful in order to obtain the Myrheim-Meyer dimension in the flat spacetime limit. Using $U_k = (k+2) \times ((k+1)n+2)S_k$ we may write the expression more succinctly as

$$U_1 - 3U_2 + 3U_3 - U_4 = 0. \qquad (58)$$

It is interesting to note the appearance of the binomial coefficients $(-1)^k\binom{r-1}{k}$ for $r = 4$ in the above expression for the dimension estimator, as well as in the expressions for $R(0)$ and T for $r = 3$.

Since the S_k themselves explicitly contain dimension information via the ζ_k, it is more useful to expand the expression to

$$(n+2)(2n+2)\left(\frac{\langle C_1\rangle}{\chi_1}\right)^4 - 3(2n+2)(3n+2)\left(\frac{\langle C_2\rangle}{\chi_2}\right)^2 + 3(3n+2)(4n+2)\left(\frac{\langle C_3\rangle}{\chi_3}\right)^{4/3} - (4n+2)(5n+2)\left(\frac{\langle C_4\rangle}{\chi_4}\right) = 0, \qquad (59)$$

or defining

$$\omega_k \equiv (-)^{k-1}\binom{3}{k-1}\frac{(kn+2)((k+1)n+2)}{\chi_k^{4/k}}, \qquad (60)$$

we get our final expression for the dimension estimator in curved spacetime

$$\sum_{k=1}^{4}\omega_k(n)\langle C_k\rangle^{4/k} = 0. \qquad (61)$$

Again, in the flat spacetime limit, the left-hand side of the equation reduces to zero. However, in order to recover the Myrheim-Meyer dimension estimator $\mathsf{f}_0(n)$, we must remind ourselves of its definition Eq. (11), which suggests that we divide Eq. (61) throughout by $\omega_2(n)\langle C_1\rangle^4$ to get

$$\mathsf{f}_0^2(n)\left(-\frac{1}{3}\frac{(n+2)}{(3n+2)} - \frac{(4n+2)}{(2n+2)}\left(\frac{\langle C_3\rangle}{\chi_3}\right)^{\frac{4}{3}}\frac{1}{\langle C_1\rangle^4} + \frac{1}{3}\frac{(4n+2)(5n+2)}{(2n+2)(3n+2)}\frac{\langle C_4\rangle}{\chi_4}\frac{1}{\langle C_1\rangle^4}\right) = -\frac{\langle C_2\rangle^2}{\langle C_1\rangle^4}. \qquad (62)$$

In the flat spacetime case using Eq. (8), the above equation reduces to

$$\mathsf{f}_0^2(n)\left(-\frac{1}{3}\frac{(n+2)}{(3n+2)} - \frac{(4n+2)}{(2n+2)} + \frac{1}{3}\frac{(4n+2)(5n+2)}{(2n+2)(3n+2)}\right) = -\frac{\langle C_2\rangle^2}{\langle C_1\rangle^4} \Rightarrow \mathsf{f}_0^2(n) = \frac{\langle C_2\rangle^2}{\langle C_1\rangle^4}, \qquad (63)$$

which is the Myrheim-Meyer dimension estimator.

VI. CALCULATING THE ERRORS

We expect that as the sprinkling density ρ increases, our curvature and dimension estimators should do a better job of reproducing the continuum results. While the geometric parameters themselves do not depend on ρ, it is clear that the error will. The (rms) error $\delta C_k = \sqrt{\Delta C_k} = \sqrt{\langle C_k^2\rangle - \langle C_k\rangle^2}$, where ΔC_k is the variance. We follow the analysis in Ref. [9] to find the dependence of ΔC_k on ρ in the RNC. Unlike the flat spacetime case, however, it is not the continuum volume that must be increased to improve accuracy since this region must still be "small" for the RNC to be valid. Instead, it is the sprinkling density ρ that should be increased for reducing the error.

Let us begin with $k = 2$, so that $\Delta C_2 = \langle C_2^2\rangle - \langle C_2\rangle^2$. Now $\langle C_2^2\rangle$ is the probability of finding two sets of 2-chains in $\mathbf{I}[p,q]$, with the possibility that some of the elements can coincide. Let us call the points of these two 2-chains x, x' and y, y', with $x \prec x'$ and $y \prec y'$. Thus, $\langle C_2^2\rangle$ gets contributions from each type of coincidence. The first is simply that there are no coincidences, i.e., that all four points x, x', y, y' are distinct, which gives a contribution $\langle C_2\rangle^2$. Although this term $\sim\rho^4$, it cancels out in the expression for the variance and therefore plays no role. The next type is the one-coincidence case. For this there are two types: (i) $x = y'$ or $y = x'$, so that the two 2-chains collapse to a single 3-chain, and (ii) $x = y$ or $x' = y'$, corresponding to the probability for a three element "V" or "Λ" shaped causal set. The contribution from (i) is clearly twice that of $\langle C_3\rangle$ whose ρ dependence is $\sim\rho^3$, while the contribution from (ii) is

$$2\rho^3\int_{\mathbf{I}[p,q]}dx\sqrt{-g(x)}\int_{J^+(x)\cap\mathbf{I}[p,q]}dx'\sqrt{-g(x')}\int_{J^+(x)\cap\mathbf{I}[p,q]}dy'\sqrt{-g(y')} = 2\rho\int_{\mathbf{I}[p,q]}dx\langle C_1(x)\rangle^2 \sim \rho^3. \qquad (64)$$

Finally, there is a contribution from the two coincidences $x = y$, $x' = y'$, which is just $\langle C_2\rangle$ which goes as $\sim\rho^2$. Thus, $\Delta C_2 \sim \rho^3$ or $\delta C_2 \sim \rho^{\frac{3}{2}}$, with the dominant contribution coming from the one-coincidence case.

In order to calculate the error for all our geometric parameters, we need to perform a similar analysis for $k = 3, 4$. In each such case, the dominant contribution to the error comes from the one-coincidence case, since the no-coincidence contributions simply cancel out.

For $k = 3$, if $x \prec x' \prec x''$ and $y \prec y' \prec y''$, the no-coincidence case is again simply $\langle C_3^2\rangle$ and again cancels out. The one-coincidence terms include (i) $x = y''$ or $y = x''$, which is the 5-chain, $\langle C_5\rangle \sim \rho^5$, (ii) $x = y'$ or $y = x'$ or $x'' = y'$ or $y'' = x'$ which contribute

$$4\rho^2\int_{\mathbf{I}[p,q]}dy\sqrt{-g(y)}\int_{J^+(y)\cap\mathbf{I}[p,q]}dx\sqrt{-g(x)}\langle C_2(x)\rangle\langle C_1(x)\rangle \sim \rho^5, \qquad (65)$$

(iii) $x = y$ or $x'' = y''$ which contribute

$$2\rho\int_{\mathbf{I}[p,q]}dx\sqrt{-g(x)}\langle C_2(x)\rangle^2 \sim \rho^5, \qquad (66)$$

and finally (iv) $x' = y'$ which contribute

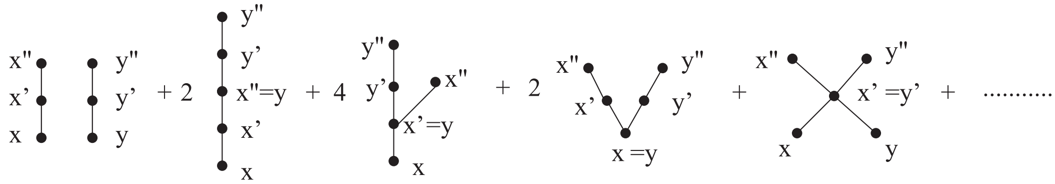

FIG. 2. The contributions to $\langle C_3^2\rangle$ from no coincidences and one coincidence are shown.

$$\rho^3 \int_{\mathbf{I}[p,q]} dx\sqrt{-g(x)} \int_{\mathbf{I}[p,q]} dy\sqrt{-g(y)} \int_{J^+(x)\cap J^+(y)\cap \mathbf{I}[p,q]} \times dx'\sqrt{-g(x')}\langle C_1(x)\rangle^2 \sim \rho^5. \tag{67}$$

Figure 2 shows the contributions to $\langle C_3^2\rangle$ from the case of no coincidence and the case of one coincidence. Since the no-coincidence term cancels out, $\delta C_3 \sim \rho^{\frac{5}{2}}$.

A similar analysis shows that for $\langle C_4^2\rangle$ the no-coincidence term cancels out and the one-coincidence cases lead to a dependence $\sim\rho^7$, so that $\delta C_4 \sim \rho^{\frac{7}{2}}$ This generalizes in a straightforward manner to

$$\delta C_k \sim \rho^{\frac{2k-1}{2}}. \tag{68}$$

We are now in a position to calculate the dependence on ρ of the errors in the Q_k and S_k:

$$\delta Q_k \sim \frac{3}{k\rho^3}\langle C_k\rangle^{3/k-1}\delta C_k \sim \rho^{-1/2}, \tag{69}$$

and

$$\delta S_k \sim \frac{4}{k\rho^4}\langle C_k\rangle^{4/k-1}\delta C_k \sim \rho^{-1/2}. \tag{70}$$

This immediately means that the errors δT^{3n}, $\delta R(0)$, $\delta R_{00}(0)$ go as $\rho^{-1/2}$ and hence become smaller as ρ increases. The error in the dimension estimator is similarly given by

$$\begin{aligned}\delta n((4n+6)S_1 - 3(12n+10)S_2 \\ + 3(24n+14)S_3 - (40n+18)S_4) \\ = -((n+2)(2n+2)\delta S_1 - 3(2n+2)(3n+2)\delta S_2 \\ + 3(3n+2)(4n+2)\delta S_3 - (4n+2)(5n+2)\delta S_4) \\ \Rightarrow \delta n \sim \rho^{-1/2}.\end{aligned} \tag{71}$$

VII. CONCLUSIONS AND REMARKS

In this work we have found expressions for the proper time T [Eq. (45)], the scalar curvature $R(0)$ [Eq. (47)], the time-time component of the Ricci tensor $R_{00}(0)$ [Eq. (50)], and a new dimension estimator [Eq. (61)] from a causal set underlying a small causal diamond $\mathbf{I}[p,q]$ in a generic spacetime in arbitrary dimensions. We find that the errors in these estimators goes as $\rho^{-1/2}$, thus becoming smaller as the sprinkling density is increased, while keeping the volume of $\mathbf{I}[p,q]$ fixed. Our results not only verify the deep relationship between order and Lorentzian geometry but also provide new observables that can be used to assess whether a causal set is manifold-like or not.

Our calculation moreover brings to light an intriguing connection between two seemingly disparate order theoretic structures in a causal set. While our expression for the scalar curvature is purely in terms of the abundance of k-chains, the Benincasa-Dowker (BD) scalar curvature $R_{\rm BD}$ [14] is constructed from the abundance of k-inclusive intervals. In four dimensions for example, their expression for the scalar curvature is

$$R_{\rm BD}(0) = \frac{2}{6\sqrt{\rho}}(1 - (N_2(0) - 9N_3(0) + 16N_4(0) - 8N_5(0))), \tag{72}$$

where $N_k(0)$ are the number of k-inclusive intervals $I_k[x,0]$, where $x \prec 0$.[2] As noted in Sec. II, k-chains and k-inclusive intervals are very distinct order theoretic structures. The continuum geometry, however, seems to link them via the scalar curvature. If such a relationship exists, does it, for example, indicate manifold-likeness in a causal set? This and several related questions remain to be investigated.

Further comparisons between the two expressions for $R(0)$ are warranted. Both contain alternating sums, although the coefficients differ markedly. For one, the BD expression appears to have a strong dimension dependence in the number of terms required—for $n = 2$ the sum is truncated at $k = 4$, while for $n = 4$ it is truncated at $k = 5$. Our expression for $R(0)$ is in this sense independent of n—it requires $\langle C_1\rangle$, $\langle C_2\rangle$, $\langle C_3\rangle$ in *all* dimensions.[3] Moreover, the systematic determination of the coefficients for $R_{\rm BD}$ for arbitrary n is fairly involved, whereas the coefficients in Eqs. (47) and (58) are simply the binomial coefficients $(-1)^k\binom{2}{k}$ and $(-1)^k\binom{3}{k}$, respectively, in all dimensions.

[2]Our notation differs from Ref. [14] where k is replaced by $k-1$, in keeping with our definition of a k-chain.

[3]As may be already evident to the astute reader, one could replace these three values $k = 1, 2, 3$ with any other k_1, k_2, k_3, to get an expression for $R(0)$ in terms of $\langle C_{k_1}\rangle$, $\langle C_{k_2}\rangle$, $\langle C_{k_3}\rangle$. What is important is that this choice is not dimension dependent—every choice works for every n.

Apart from these, there are deeper differences in the two expressions for $R(0)$. $R_{\rm BD}(0)$ is constructed from *all* inclusive intervals of the form $I_k[x, 0]$ in the causal set for $k = 1, \ldots 5$. It is therefore an essentially nonlocal expression since it depends on the structure of the causal set throughout a past (or future) neighborhood of the element 0 in the causal set. The expression Eq. (47) on the other hand is *only* valid in a small causal diamond and hence is strictly local and dependent on a proper choice of neighborhood of the element in the causal set. This means that the BD form for $R(0)$, being "neighborhood independent," can be readily used to obtain an action for the entire causal set, which has strong implications for causal set theory. Unless the definition of a small neighborhood can be made entirely order theoretically in the causal set, our expression for $R(0)$ on the other hand cannot be used to obtain an action in a simple manner. Of course, in the *specific* case when the entire causal set is approximated by a small causal diamond $\mathbf{I}[p, q]$, the action is simply

$$S/\hbar = \sum_{s=1}^{N} R(e_s) = NR, \tag{73}$$

where e_s denotes an element in $\mathbf{C}$, since to this approximation $R(x)$ is the same throughout $\mathbf{I}[p, q]$.

Nevertheless, the geometric estimators we have found in this work take a big step towards the order-Lorentzian geometry correspondence. We have a new dimension estimator which can determine the manifold dimension for a curved spacetime with greater accuracy than the currently available flat spacetime Myrheim-Meyer estimator. The local nature of the expression for the scalar curvature can also be seen as an advantage since only a small neighborhood of an element in the causal set is required to determine the curvature, rather than its entire past. An obvious next step is to follow up with a numerical analysis of causal sets which are approximated by different curved spacetimes and to see how well our estimators work in these cases [19].

These observables can also be used as additional tests of manifold-likeness of a causal set. In Ref. [18] the expectation values of a range of such observables, including the abundance N_k of k-element inclusive intervals was found from Monte Carlo simulations of two-dimensional causal set quantum gravity. Comparing with the flat spacetime distribution of the N_k, these observables were used to demonstrate that the dominant contribution to the causal set path integral in two dimensions comes from flat spacetime suggesting that manifold-like behavior is emergent. The estimators we have obtained in this current work could well be employed to calculate more accurately how close to flatness one is—is the two-dimensional universe just a little positively curved, for example?

The extended hope of the current analysis is also that as more of the order theoretic basis of geometry is uncovered it may be possible to find a meaningful order theoretic definition of locality which translates to our commonly held (albeit Riemannian geometry based) notions of locality in the continuum.

APPENDIX

To evaluate the $\langle C_k \rangle$ we find it convenient to transform from the Cartesian coordinates x^i to spherical polar coordinates $x^i = rf^i(\Omega)$, where

$$f^i(\theta_1, \theta_2 \ldots \theta_{n-2}) = \begin{cases} \prod_{k=1}^{n-i-1} \sin\theta_k \cos\theta_{n-i} & (i > 1), \\ \prod_{k=1}^{n-2} \sin\theta_k & (i = 1). \end{cases}$$

It is also useful to express the radial coordinate for any $x_1 \in \mathbf{I}[p, q]$ in light-cone coordinates:

$$r_1 = \frac{v_1 - u_1}{2} = \frac{1}{2}\left(\left(\frac{T}{2} - u_1\right) - \left(\frac{T}{2} - v_1\right)\right) \Rightarrow r_1^l = \frac{1}{2^l}\sum_{k=0}(-1)^k\binom{l}{k}\left(\frac{T}{2} - v_1\right)^k\left(\frac{T}{2} - u_1\right)^{l-k}.$$

Since the metric $\eta(u, v)$ in $\mathbf{I}_0[p, q]$ in light-cone coordinates is

$$ds^2 = -dudv + \left(\frac{v - u}{2}\right)^2 d\Omega^2, \tag{A1}$$

the associated measure of any integral over $\mathbf{I}_0[p, q]$ is

$$\sqrt{-\eta(u, v)} = \frac{1}{2}\left(\frac{v - u}{2}\right)^{n-2}\left(\prod_{i=1}^{n-3}\sin^{(n-i-2)}\theta_i\right) = \frac{1}{2^{n-1}}\left(\prod_{i=1}^{n-3}\sin^{(n-i-2)}\theta_i\right)\sum_{k=0}^{n-2}(-1)^k\binom{n-2}{k} \times\left(\frac{T}{2} - v\right)^k\left(\frac{T}{2} - u\right)^{n-2-k}. \tag{A2}$$

Finally, we will find it useful to define the following pair of integrals:

$$\begin{aligned} \mathrm{L}(\mathsf{a}, v, T) &= \int_{-\frac{T}{2}}^{v} du\left(\frac{T}{2} - u\right)^{\mathsf{a}} = \frac{1}{\mathsf{a}+1}\left(T^{\mathsf{a}+1} - \left(\frac{T}{2} - v\right)^{\mathsf{a}+1}\right), \\ \mathrm{L}(\mathsf{a}, \mathsf{b}, T) &= \int_{-\frac{T}{2}}^{\frac{T}{2}} dv\left(\frac{T}{2} - v\right)^{\mathsf{b}}\mathrm{L}(\mathsf{a}, v, T) = \frac{T^{\mathsf{a}+\mathsf{b}+2}}{(\mathsf{b}+1)(\mathsf{a}+\mathsf{b}+2)}. \end{aligned} \tag{A3}$$

The above identities will now be used to evaluate the set of integrals $\mathrm{I}_{1,2,3,4}(m)$ required for the calculation of $\langle C_k \rangle$ [Eqs. (17), (28), and (33)].

(1)
$$\mathrm{I}_1(m)=\int dx_1 T_1^m=\int_{-\frac{T}{2}}^{\frac{T}{2}} dv_1\int_{-\frac{T}{2}}^{v_1} du_1\int d\Omega\sqrt{-\eta(u_1,v_1)}T_1^m$$
$$=\frac{A_{n-2}}{2^{n-1}}\sum_{k=0}^{n-2}\binom{n-2}{k}\mathrm{L}\left(\frac{m}{2}+n-k-2,\frac{m}{2}+k,T\right)=\frac{A_{n-2}T^{n+m}}{2^{n-1}}\frac{(n-2)!(\frac{m}{2})!}{(n+m)(n+\frac{m}{2}-1)!}. \tag{A4}$$

(2) For $m+2$ the above equation gives us the relation

$$\mathrm{I}_1(m+2)=\mathrm{I}_1(m)T^2\frac{(n+m)(\frac{m}{2}+1)}{(n+m+2)(n+\frac{m}{2})}=\mathrm{I}_1(m)T^2 g_1(m). \tag{A5}$$

(3) As noted in Sec. III any integral of the form

$$I_2(m)=\int_{\mathbf{I}_0[p,q]} dx_1(x_1^i)^2T_1^m \tag{A6}$$

is independent of the spatial direction i because of the spatial symmetry of $\mathbf{I}_0[p,q]$. Thus, we can choose $x^{n-1}=r\cos\theta_1$ to simplify our calculation so that

$$\mathrm{I}_2(m)=\int dx_1T_1^m r_1^2\cos^2\theta_1=\int_{-\frac{T}{2}}^{\frac{T}{2}}dv_1\int_{-\frac{T}{2}}^{v_1}du_1\int d\Omega\frac{\cos^2\theta_1}{2^{n+1}}\left(\frac{T}{2}-u_1\right)^{\frac{m}{2}}\left(\frac{T}{2}-v_1\right)^{\frac{m}{2}}(v_1-u_1)^n$$
$$=\frac{A_{n-2}}{2^{n+1}}\frac{T^{n+m+2}}{(n-1)(n+m+2)}\frac{n!(\frac{m}{2})!}{(n+\frac{m}{2}+1)!}=\mathrm{I}_1(m)T^2\frac{n(n+m)}{4(n+m+2)(n+\frac{m}{2}+1)(n+\frac{m}{2})}=\mathrm{I}_1(m)T^2f_2(m). \tag{A7}$$

The next integral can be split into three parts

$$\mathrm{I}_3(m)=\int dx_1t_1^2T_1^m=\mathrm{I}_3^a+\mathrm{I}_3^b+\mathrm{I}_3^c, \tag{A8}$$

where, using $\mathsf{a}_k=n-2-k+\frac{m}{2}$, $\mathsf{b}_k=k+\frac{m}{2}$

$$\mathrm{I}_3^a(m)=\frac{1}{4}\int dx_1u_1^2T_1^m$$
$$=\frac{A_{n-2}}{2^{n+1}}\sum_{k=0}^{n-2}(-1)^k\binom{n-2}{k}\left(\frac{T^2}{4}\mathrm{L}(\mathsf{a}_k,\mathsf{b}_k,T)-T\mathrm{L}(\mathsf{a}_k+1,\mathsf{b}_k,T)+\mathrm{L}(\mathsf{a}_k+2,\mathsf{b},T)\right), \tag{A9}$$

$$\mathrm{I}_3^b(m)=\frac{1}{4}\int dx_1v_1^2T_1^m$$
$$=\frac{A_{n-2}}{2^{n+1}}\sum_{k=0}^{n-2}(-1)^k\binom{n-2}{k}\left(\frac{T^2}{4}\mathrm{L}(\mathsf{a}_k,\mathsf{b}_k,T)-T\mathrm{L}(\mathsf{a}_k,\mathsf{b}_k+1,T)+\mathrm{L}(\mathsf{a}_k,\mathsf{b}+2,T)\right), \tag{A10}$$

and

$$\mathrm{I}_3^c(m)=\frac{2}{4}\int dx_1u_1v_1T_1^m$$
$$=\frac{A_{n-2}}{2^n}\sum_{k=0}^{n-2}(-1)^k\binom{n-2}{k}\left(\frac{T^2}{4}\mathrm{L}(\mathsf{a}_k,\mathsf{b}_k,T)-\frac{T}{2}\mathrm{L}(\mathsf{a}_k+1,\mathsf{b}_k,T)-\frac{T}{2}\mathrm{L}(\mathsf{a}_k,\mathsf{b}_k+1,T)+\mathrm{L}(\mathsf{a}_k+1,\mathsf{b}+1,T)\right). \tag{A11}$$

After some algebra, we find that

$$\mathrm{I}_3(m)=\mathrm{I}_1(m)T^2\frac{8n+m(m+2)(m+n+2)}{4(2+m+n)(2n+m)(2n+m+2)}=\mathrm{I}_1(m)T^2f_3(m). \tag{A12}$$

(4) Finally,

$$\mathrm{I}_4(m)=\int dx_1\left(\frac{T}{2}-t_1\right)^2T_1^m=\frac{T^2}{4}\mathrm{I}_1(m)+\mathrm{I}_3(m)-T\tilde{\mathrm{I}}_4(m). \tag{A13}$$

Evaluating

$$\tilde{\mathrm{I}}_4(m) = \int dx_1 t_1 T^m = \mathrm{I}_1(m)\frac{T}{2}\bigg(1 - \frac{m+n}{m+n+1} - \frac{(\frac{m}{2}+1)(m+n)}{(m+n+1)(\frac{m}{2}+n)}\bigg) = \mathrm{I}_1(m)T\tilde{f}_4(m), \tag{A14}$$

we can then express

$$\mathrm{I}_4(m) = \mathrm{I}_1(m)T^2 f_4(m), \tag{A15}$$

where $f_4(m) = \frac{1}{4} + \tilde{f}_4(m) + f_3(m)$.

Gathering these expressions

$$\begin{aligned} g_1(m) &= \frac{(n+m)(\frac{m}{2}+1)}{(n+m+2)(n+\frac{m}{2})}, \\ f_2(m) &= \frac{n(n+m)}{4(n+m+2)(n+\frac{m}{2}+1)(n+\frac{m}{2})}, \\ f_3(m) &= \frac{8n + m(m+2)(m+n+2)}{4(2+m+n)(2n+m)(2n+m+2)}, \\ f_4(m) &= \frac{1}{4} - \tilde{f}_4(m) + f_3(m). \end{aligned} \tag{A16}$$

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