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No asymptotically highly damped quasinormal modes without horizons?

Cecilia Chirenti,[1,*] Alberto Saa,[2,†] and Jozef Skákala[1,‡]

[1]*Centro de Matemática, Computação e Cognição, UFABC, 09210-170 Santo André, São Paulo, Brazil*

[2]*Departamento de Matemática Aplicada, Universidade Estadual de Campinas, 13083-859 Campinas, São Paulo, Brazil*

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We explore the question of what happens with the asymptotically highly damped quasinormal modes (ℓ fixed, $|\omega_I| \to \infty$) when the underlying spacetime has no event horizons. We consider the characteristic oscillations of a scalar field in a large class of asymptotically flat, spherically symmetric, static spacetimes without (absolute) horizons, such that the class accommodates the cases that are known to be of some sort of physical interest. The question of the asymptotic quasinormal modes in such spacetimes is relevant to elucidating the connection between the behavior of the asymptotic quasinormal modes and the quantum properties of event horizons, as put forward in some recent important conjectures. We prove for a large class of asymptotically flat spacetimes without horizons that the scalar field asymptotically highly damped modes do not exist. This provides in our view additional evidence that there is indeed a close link between the asymptotically highly damped modes and the existence of spacetime horizons (and their properties).

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I. INTRODUCTION

It is a known result [1,2] that in generic *static* spacetimes (globally hyperbolic, or not) one can always define (for reasonable enough initial data) a sensible time evolution of a scalar field represented by a self-adjoint operator on a suitable Hilbert space. Many important features of a typical scattering can be described by a set of characteristic complex frequencies ($\omega = \omega_R + i\omega_I$), the quasinormal modes (QNMs). In spherically symmetric spacetimes, the quasinormal modes are labeled by the wave mode number ℓ and a discrete number n, such that n grows with the decreasing damping time of the modes. More than a decade ago, it was conjectured by Hod [3] that, due to Bohr's correspondence principle, the asymptotically highly damped modes (with the wave mode numbers fixed and $|\omega_I| \to \infty$) might carry important information about the quantum properties of the black hole horizon. The original conjecture of Hod [3] was modified by Maggiore [4], but the essence of the conjectures remains the same. The conjecture of Maggiore [4] was successfully used in the case of many black hole spacetimes. (For the spherically symmetric black hole spacetimes, see, for example, Ref. [5].)

Furthermore, one can generically observe that in the case of static, spherically symmetric *black hole* spacetimes the asymptotically highly damped modes always exist and fulfill certain general patterns [6–9]. This fact can be seen as a strong support for the conjecture of Maggiore. Thus, one might be interested in the question of what is going to happen with the asymptotic highly damped modes in static, spherically symmetric spacetimes in the case in which there are *no* (absolute) horizons. It is very intuitive to expect that from the point of view of the conjectures in question [3,4], such asymptotically highly modes might *not* exist. The nonexistence of such modes was shown in one particular case of a spacetime without horizons, in the case of a Reissner-Nordström naked singularity [10]. Thus, the result of Chirenti *et al*. [10] can be seen as supporting the conjectures that the QNMs might generally relate to the spacetime horizons in an essential way.[1] The aim of this short paper is to generalize this result to large classes of asymptotically flat, spherically symmetric, static spacetimes *without* horizons, accommodating all the known interesting cases. As a result of this fact, this paper should add much more evidence to support the intuition based on the mentioned conjectures.

II. STATIC, SPHERICALLY SYMMETRIC SPACETIMES WITHOUT (ABSOLUTE) HORIZONS

The metric of a general spherically symmetric, static spacetime can be cast in the form

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + h^2(r)d\Omega^2. \qquad (1)$$

We assume $f(r)$, $h'(r)$ being continuous and almost everywhere 1-differentiable with bounded first derivatives everywhere outside the infinitesimal neighborhood of zero. Furthermore, since the spacetime has no event horizon, $f(r)$ is positive on $(0, \infty)$. In order to guarantee an asymptotically flat spacetime, one needs the further restrictions $f(r) \to 1$ and $h(r) \sim r$ for $r \to \infty$. Under the geometry of Eq. (1), the area of the spherical surface with radius r is

*cecilia.chirenti@ufabc.edu.br
†asaa@ime.unicamp.br
‡jozef.skakala@ufabc.edu.br

[1]Another example of a spacetime without horizons is the pure anti–de Sitter spacetime, where no quasinormal modes exist; hence, no asymptotic quasinormal modes exist.

given by $4\pi h^2(r)$, and hence it is natural also to assume $h(0) = 0$ and $h'(r) > 0$.

The massless Klein-Gordon time-independent equation for the metric in Eq. (1) can be cast by introducing the usual tortoise coordinate $x(r)$:

$$\frac{dr}{dx} = f(r), \tag{2}$$

as

$$(\partial_x^2 + \omega^2 - V(r))\phi_\ell = 0, \tag{3}$$

where $\psi_\ell(t, r) = \exp(-i\omega t)\phi_\ell(r)/h(r)$, with ψ_ℓ standing for the spherical harmonic component of the scalar field, and

$$V(r) = f(r)\left[\frac{\ell(\ell+1)}{h^2(r)} + \frac{1}{h(r)}\frac{d}{dr}\left(f(r)\frac{d}{dr}h(r)\right)\right]. \tag{4}$$

On general grounds, one can expect $V(r) \to 0$ for $r \to \infty$ due to the asymptotic flatness, and a diverging $V(r)$ for $r \to 0$ due to the centrifugal barrier (at least).

Let us explore the poles of the scattering amplitude that do not belong to the bound states, the (QNMs). Some superposition of QNMs dominates the time evolution of an arbitrary perturbation at a given point within some specific time scale. In the scattering problem in a spacetime without horizons, the QNMs are determined by the normal-mode boundary condition at zero, and at the same time by the purely outgoing radiation condition at infinity $\sim e^{i\omega x}$. The reason for this is the following: The Green function is composed of two solutions, such that each of them fulfills the given boundary condition at one of the "ends." (This is because one requires boundedness of the Green function in the spatial variables.) Then, the poles of the Green function (QNMs) occur where the two solutions coincide. (For a more detailed argumentation, see Ref. [10]. Note also that by considering only scattering that is reflective, one can immediately see that two such boundary conditions *cannot* be fulfilled at the same time for *real* frequencies ω.)

Unfortunately, since the potential is typically noncompact, and the frequencies ω have negative imaginary parts (unless instabilities occur), one of the solutions is exponentially suppressed; whereas the other one, corresponding to the "outgoing" wave, is exponentially growing, and one has to give the "outgoing radiation condition" a clearer meaning. (If the potential had a compact support, one could claim the solutions to be directly proportional to the outgoing waves, but since the plane waves are only approximations to the solutions as one approaches infinity, one needs exponential precision in the error to rule out the "incoming" wave solution.) On the other hand, the outgoing/incoming wave solutions behave as $A_\pm e^{\pm i\omega x}(1 + O(1/x))$ (see Ref. [11]), so let us follow the suggestion of Motl and Neitzke [12] by analytically continuing the solutions in the complex plane in x and picking the purely outgoing radiation condition on the line $\mathrm{Im}(\omega x) = 0$ as $\omega x \to \infty$. (This means we pick the solution in the region where the two solutions are purely oscillatory and of comparable sizes.)

The asymptotically highly damped quasinormal frequencies are, with our normal mode convention, characterized by $-\omega_I \to \infty$ and ℓ fixed, implying that the highly damped quasinormal modes should obey the approximate equation

$$(\partial_x^2 + \omega^2)\phi_\ell = 0, \tag{5}$$

everywhere apart from a small neighborhood of $r = 0$. Of course, the two linearly independent solutions of Eq. (5) are plane waves:

$$\phi_\ell = C_1 e^{i\omega x} + C_2 e^{-i\omega x}. \tag{6}$$

The pure outgoing mode at infinity then corresponds to $\psi_\ell(t, r) \sim e^{-i\omega(t - x(r))}/r$; hence it is picked by $C_2 = 0$. For $|\omega| \gg 1$, the potential [Eq. (4)] will effectively affect the dynamics of the scalar field only in a region near $r = 0$. The crucial point in our analysis is that the dynamics near $r = 0$ can be solved and will determine the constants C_1 and C_2 for $|\omega| \gg 1$ in a way that will prevent the appearance of the quasinormal oscillations. (This is because one can approximate the scalar field equation everywhere by the form of the equation near zero, since in the region in which the approximation of the potential near zero ceases to hold, one can neglect the potential as a whole with respect to the ω^2 term.) Relaxing the condition $|\omega| \gg 1$, the solution in Eq. (6) will be a good approximation for Eq. (3) only for large r, and the natural boundary condition selected by the dynamics near $r = 0$ can be translated in a ω-dependent relation between the coefficients C_1 and C_2, giving rise to a characteristic equation $C_2(\omega) = 0$ for the quasinormal modes.

We have basically two different kinds of interesting spacetimes in our analysis: spherically symmetric stellar spacetimes [with $f(0)$ finite and positive] and naked singularities [$f(r) \sim r^{-\epsilon}$, $\epsilon \neq 0 \wedge \epsilon > -1$, for $r \to 0$]. We will consider them separately, although the analyses are similar. Some of the cases corresponding to $f(0) = 0$ will be also left for the concluding section.

A. Stellar spacetimes

The choices $f(r) \sim A + Br^\epsilon$ and $h(r) \sim Dr^\delta$ for $r \to 0$, with positive A, D, ϵ, and δ, seem generic enough to accommodate all known stellar spacetimes. In fact, there are further restrictions among such constants. The divergent terms of the Ricci scalar for such choices of $f(r)$ and $h(r)$ read

$$R \sim \frac{2}{(Dr^\delta)^2} - 2\delta(3\delta - 2)\frac{A}{r^2}, \tag{7}$$

for $r \to 0$. The only way to assure that R is finite for $r \to 0$ is by demanding that $\delta = 1$ and $A = D^{-2}$. (The stellar spacetimes must also ensure that the other curvature invariants will be finite, but we will perform a more general analysis by applying only the given Ricci curvature finiteness condition, which automatically contains regular stellar spacetimes as its subcase.) With these values, the potential [Eq. (4)] is given by

$$V(r) \sim \frac{\ell(\ell+1)}{D^4 r^2} + \frac{\epsilon B}{D^2 r^{2-\epsilon}}, \tag{8}$$

for $r \to 0$. Notice that the condition that $f(0) = D^{-2}$ must be finite assures from Eq. (2) that $x \sim D^2 r + C$ for $r \to 0$. Let us further choose $C = 0$, hence $x(0) = 0$. Since $\epsilon > 0$, the dominant term of the potential near the origin is the centrifugal barrier,

$$V(x) = \frac{\ell(\ell+1)}{x^2}. \tag{9}$$

The general solution of Eq. (3) with the potential of Eq. (9) is given in terms of Bessel functions:

$$\phi(x) = \sqrt{\omega x}(C_3 J_\beta(\omega x) + C_4 J_{-\beta}(\omega x)), \tag{10}$$

where in this case $\beta = \ell + 1/2$ and is a noninteger. To impose the boundary condition at $\mathrm{Im}(\omega x) = 0$, we understand the solution [Eq. (10)] to be analytically continued into the complex plane in $\omega \cdot x$. Since the solutions are multivalued functions of the complex variable $\omega \cdot x$ around zero, their analytical extensions require taking a convenient branch cut. The branch cut can and will be taken along the half line where $\omega \cdot x$ is real and negative.

Notice that the regularity of the scalar field ψ_ℓ at the origin and the properties of the Bessel functions close to zero automatically select the reflection condition $\phi_\ell(0) = 0$, which corresponds to $C_4 = 0$. The solution picked at the origin then behaves for $\omega x \to \infty$ as

$$\sqrt{\omega x} \cdot J_{\ell+1/2}(\omega x) \approx \sqrt{\left(\frac{2}{\pi}\right)} \cos\left(\omega x - \frac{(\ell+1)\cdot\pi}{2}\right) + O\left(\frac{1}{x}\right). \tag{11}$$

This means that the reflection condition imposed on Eq. (6) implies $C_1 = \pm C_2$, leaving no room for the fulfillment of the highly damped quasinormal mode condition $C_2 = 0$.

B. Naked singularities

In this case, one has $f(r) = Br^{-\epsilon}$ and $h(r) = Dr^\delta$ for $r \to 0$ with positive B, D, δ, and $\epsilon > -1$, $\epsilon \neq 0$. (Again, such conditions should be fulfilled for a fairly generic naked singularity, and such a metric is of a Szekeres-Iyer form [13].) The tortoise coordinate in this case is given near the radial center as $x = [B(\epsilon+1)]^{-1} r^{\epsilon+1}$ [again by setting $x(0) = 0$], and the potential $V(x)$ has two terms near zero:

$$V(x) = \frac{P_1}{x^\mu} + \frac{P_2}{x^2}, \tag{12}$$

where $\mu = \frac{2\delta+\epsilon}{\epsilon+1}$, $P_1 = \frac{\ell(\ell-1)}{D^2[B(\epsilon+1)]^\mu}$, and $P_2 = -\frac{\delta}{\epsilon+1} \times (-\frac{\delta}{\epsilon+1} + 1)$. In the case $\mu \leq 2$, the potential again behaves close to zero as $\sim x^{-2}$. This case corresponds to $\epsilon/2 + 1 \geq \delta$. In the case $\mu > 2$, the scalar field equation has an irregular singular point at the radial center, and much less can be said about the solutions. We will consider only the case $\mu \leq 2$, but this case already includes all the well-known naked singularity solutions. For $\mu \leq 2$, unless β is an integer, the solutions are again given by Eq. (10). If β is an integer, the solutions are given as

$$\sqrt{\omega x}[C_3 J_\beta(\omega x) + C_4 Y_\beta(\omega x)],$$

where Y_β is a Bessel function of the second kind. The parameter β is given in the naked singularity case for $\mu < 2$ as

$$\beta = \left|\frac{1}{2} - \frac{\delta}{\epsilon+1}\right|,$$

and for $\mu = 2$ and $P_1 + P_2 > -1/4$ as

$$\beta = \sqrt{1/4 + P_1 + P_2}.$$

To determine the quasinormal modes, one has to first select the normal modes at the origin such that $\psi_\ell(0)$ is finite. This can be uniquely done for $\delta \geq (\epsilon+1)/2$, which means that in such case the time evolution of the scalar field is *unique*. In case the singularity is too strong and $\delta < (\epsilon+1)/2$, both of the solutions ψ_ℓ are convergent at zero, and the time evolution is nonunique. (This means the singularity has "hair," and the quasinormal modes depend on the "hair.") In such case, one can still uniquely choose normal modes such that fulfill $\psi_\ell(0) = 0$, corresponding to a reflective boundary condition at the singularity. This represents a preferred choice for the time evolution, in the language of operators corresponding to what is called a Friedrich's self-adjoint extension of a symmetric operator. In case the time evolution is nonunique, we will automatically impose this most natural choice of time evolution. So in each of the cases, let us proceed exactly as before and impose the $\psi_\ell(0) = 0$ condition at the radial center, which leads (whether β is an integer or noninteger) to the condition $C_4 = 0$. But by using the asymptotic behavior at $\omega x \to \infty$ given by Eq. (11), the choice of normal modes leads, exactly as before, to a nontrivial linear combination of outgoing and incoming waves, such that it is independent of ω. This means that one cannot fulfill at the same time the boundary conditions at both "ends." (For the sake of completeness, let us say that one can repeat the same analysis with the same results also for subcases of the case $\epsilon = 0$ and $\delta \neq 1$.)

Let us mention three types of naked singularities of some interest that fall under our analysis: The negative-mass Schwarzschild singularity, the vacuum solution of the Einstein equations, has the parameters given as $\epsilon = \delta = 1$. This means $\mu = 3/2 < 2$ and $\delta/(\epsilon+1) = 1/2$, which means that the time evolution is in this case unique. The Reissner-Nordström naked singularity, arising from a coupling between gravity and an electromagnetic field, has the parameters given as $\epsilon = 2$ and $\delta = 1$, which means $\mu = 4/3 < 2$ and $\delta/(\epsilon + 1) = 1/3 < 1/2$. This means that in the case of a Reissner-Nordström naked singularity, the time evolution is nonunique. The Wyman [14] solution, that provides a naked singularity arising from a minimal coupling of gravity to a charged massive scalar field, has $(0 < \alpha < 1)$, $\delta = (1 - \alpha)/2$ and $\epsilon = -\alpha$. Here α is simply related to the scalar field mass and the scalar field charge. This means $\mu = 1 - \alpha/(1 - \alpha) < 2$ and $\delta/(\epsilon + 1) = 1/2$. Thus, in the case of the Wyman solution, the time evolution is unique. Let us mention here that the same results concerning the uniqueness of the scalar field time evolution were obtained for these three naked singularities via the language and methods of functional analysis in Ref. [15]. (Reference [15] also provides more details about the naked singularities in question.)

III. CONCLUSIONS

Note that the case $\epsilon \leq -1$, for $f(r) \sim r^{-\epsilon}$ as $r \rightarrow 0$, is qualitatively different from the cases analyzed before, since one has $x(r) \rightarrow -\infty$ for $r \rightarrow 0$. In this case, the light rays reach the radial center in infinite coordinate time. One cannot explore the asymptotic properties of the solutions of Eq. (3) to select the natural boundary conditions at $r = 0$ in the same way, and we indeed may have room for the asymptotically highly damped quasinormal modes. An infinitesimal-mass Schwarzschild black hole belongs to this class, for instance.

In this paper, we generalized our previous result from Ref. [10] to (at least) large classes of spherically symmetric, asymptotically flat, static spacetimes *without* (absolute) horizons. Our results provide evidence that for the class of spacetimes in question, the asymptotically highly damped modes (ℓ fixed and $|\omega_I| \rightarrow \infty$) do *not* exist. This means that the nonexistence of (absolute) spacetime horizons could be conjectured to mean the nonexistence of the asymptotic modes. As mentioned in the Introduction, it is widely observed [6–9] that the existence of (absolute) spacetime horizons leads to the existence of the asymptotic quasinormal modes. One can then *conjecture* that the equivalence existence of (absolute) spacetime horizons $\leftrightarrow$ existence of asymptotically highly damped modes might be completely general, confirming the intuition one might have from the currently popular conjectures linking the modes to the horizon properties. To our knowledge, our result already covers all the static, spherically symmetric (asymptotically flat) spacetimes *without* horizons that are of some sort of interest: the (regular) stellar spacetimes of the given type and the most "famous" static naked singularity solutions. Note that in the case of spacetimes of a static neutron star, the computed behavior of the imaginary parts of the (axial) w-modes (for the fixed ℓ) already suggests (see some of the plots of Ref. [16]) that the absolute values of the imaginary parts might be upper bounded.

Let us also add that models of a theoretical interest from the point of view of the topic analyzed in this paper are gravastars. They behave close to the radial center as the de Sitter spacetime [17], and hence fall in the stellar case of this paper, and the quasinormal mode spectrum for fixed ℓ has to be bounded in the imaginary part. One might be interested in what happens if a sequence of static gravastar solutions approaches the black hole horizon (that is, a sequence of mass M gravastars with surface radius $R \rightarrow 2M^+$). One of the possibilities is that the upper bound of the ($\pm$) quasinormal frequency imaginary part grows to infinity as the horizon is approached; the other possibility is that there will be a sudden discontinuous qualitative change once the black hole horizon is reached. (Such a qualitative change might be seen as a form of "phase transition.")

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