

Test particle motion in modified gravity theories

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We derive the equations of motion of an electrically neutral test particle for modified gravity theories for which the covariant divergence of the ordinary matter energy-momentum tensor does not vanish (i.e., $\nabla_{\mu}T^{\mu\nu} \neq 0$). In fact, we generalize the Mathisson-Papapetrou equations by deriving a general form for the equations of motion of a test particle. Furthermore, using the generalized Mathisson-Papapetrou equations, we investigate the equations of motion of a pole-dipole (spinning) particle in the context of modified gravity.

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I. INTRODUCTION

Deriving the equations of motion of a test particle from the field equations of general relativity has a long history; see, for example, the account in Ref. [1]. It is now well established in the literature that the equations of motion need not be postulated separately, but can be derived from the field equations of general relativity. More precisely, the conservation of the ordinary matter energy-momentum tensor, i.e., $\nabla_{\mu}T^{\mu\nu} = 0$, is enough to find the geodesic equation and also the equations of motion of a spinning test particle (pole-dipole particle). In general relativity, a pole-dipole particle is described by the four-momentum and the tensor of spin, and the dynamics is governed by the Mathisson-Papapetrou equations [2,3]. Combining these equations with a proper supplementary condition, one gets a self-consistent set of equations for describing the motion of a pole-dipole particle in a given space-time; for example, see Refs. [4–8]. The case in which both gravitational and electromagnetic fields are present and the particle is charged was studied by Dixon and Souriau [9,10]. The equations of motion of a charged spinning test particle are known as Dixon-Souriau equations. For some solutions of the Dixon-Souriau equations in specific backgrounds, see Refs. [11–13].

The multipole approximation method of Refs. [2,3] has been used to investigate the motion of a test particle in the context of some modified gravity theories; for example, see Refs. [14–16]. In this paper, using the multipole method, we derive the equations of motion of a pole-dipole test particle in modified gravity theories in which the covariant divergence of the ordinary matter energy-momentum tensor is given by $\nabla_{\mu}T^{\mu\nu} = A^{\nu}$, where the nonminimal coupling term A^{ν} is an arbitrary covariant vector, and in each modified gravity theory this vector can be determined exactly when the particular way by which the gravitational fields of the theory couple with the ordinary matter is specified. It is obvious that A^{ν} leads to a modification of the equation of motion of the test particle. In fact, we generalize the Mathisson-Papapetrou equations in such a way that they can be used in any torsionless nonmetric

theory of gravity. By metric theories of gravity, we mean theories which postulate that (i) space-time is endowed with a metric $g_{\mu\nu}$, (ii) the world lines of single-pole particles are geodesics of that metric, and (iii) in the freely falling frames, the nongravitational laws of physics are those of special relativity [17]. It should be noted that, in metric theories of gravity, the ordinary matter energy-momentum tensor is conserved; the nonmetricity $Q_{\mu\nu\lambda} = -\nabla_{\mu}g_{\nu\lambda}$ is zero; and consequently the single-pole particles' motion is described by the geodesic equation, and the pole-dipole particles' motion is governed by the Mathisson-Papapetrou equations. Furthermore, we make use of the generalized Mathisson-Papapetrou equations in order to investigate the test particle equations of motion in modified gravity (MOG) [18]. In addition, we derive some conserved quantities for the test particle motion.

II. ENERGY-MOMENTUM CONSERVATION

Let us study the test particle motion in a modified gravity theory without torsion in which the usual conservation law of the ordinary matter energy-momentum tensor is replaced by

$$\nabla_{\mu}T^{\mu\lambda} = A^{\lambda}. \quad (1)$$

We know that modified gravity theories which yield to such an identity for the energy-momentum tensor would violate the Einstein equivalence principle. In fact, one can show that the covariant divergence of the energy-momentum tensor is zero if the gravitational fields (such as scalar fields or vector fields) of the theory do not interact directly with the matter. In the presence of the test particle, the components of the metric tensor are $\hat{g}_{\mu\lambda} = g_{\mu\lambda} + \delta g_{\mu\lambda}$, where $g_{\mu\lambda}$ is the background metric and $\delta g_{\mu\lambda}$ is the metric perturbation caused by the test particle, and it is supposed to be small. Also, let the energy-momentum tensor be $\hat{T}^{\mu\lambda} = T^{\mu\lambda} + \delta T^{\mu\lambda}$ and the nonminimal coupling term be $\hat{A}^{\lambda} = A^{\lambda} + \delta A^{\lambda}$, where the tensor $\delta T^{\mu\lambda}$ describes the material distribution inside the test particle and δA^{λ} stands for the test particle's contribution to the coupling term. If we assume that the test particle's trajectory is outside the

massive bodies, then the background energy-momentum tensor $T^{\mu\lambda}$ vanishes inside as well as near the test particle. Thus, using the identity in Eq. (1), we get

$$\nabla_{\mu}\tau^{\mu\lambda} = \delta f^{\lambda}, \quad (2)$$

where $\tau^{\mu\lambda} = \sqrt{-g}\delta T^{\mu\lambda}$ and $\delta f^{\lambda} = \sqrt{-g}\delta A^{\lambda}$. As we mentioned before, the use of Eq. (2) alone is sufficient to derive the equations of motion of the test particle, and we do not need the field equations of the theory. In the following sections, by using this equation, we derive the generalized version of the geodesic equation. Furthermore, we find the generalized Mathisson-Papapetrou equations for describing a spinning test particle.

III. THE SINGLE-POLE PARTICLE

In this section, we derive the equations of motion for a single-pole test body. Our analysis is based on the integration of the conservation law [Eq. (2)] over the world tube of the test body. This procedure is independent of a specific choice of energy-momentum tensor for the test particle. In fact, in the four-dimensional diagram, the interior of the particle can be considered as a tubelike region (world tube) contained in the three-dimensional timelike hypersurface $\Sigma(t)$. A representative continuous curve through the tube is parametrized by $X^{\lambda}(t)$. Coordinates within the world tube with respect to a coordinate system centered on $X^{\lambda}(t)$ are labeled by x^{λ} ; see Ref. [3] for details.

Equation (2) can be rewritten as

$$\partial_{\mu}\tau^{\mu\lambda}(x) + \Gamma_{\mu\alpha}^{\lambda}(x)\tau^{\mu\alpha}(x) = \delta f^{\lambda}(x), \quad (3)$$

where $\Gamma_{\mu\alpha}^{\lambda}(x)$ are the Christoffel symbols corresponding to the background metric $g_{\mu\nu}$. Since $g_{\mu\nu}$ changes very little inside the test particle, the Christoffel symbols can be developed in Taylor series around the point $X^{\lambda}(t)$:

$$\Gamma_{\mu\alpha}^{\lambda}(x) = \Gamma_{\mu\alpha}^{\lambda}(X) + \delta x^{\sigma}\partial_{\sigma}\Gamma_{\mu\alpha}^{\lambda}(X) + \dots, \quad (4)$$

where $\delta x^{\sigma} = x^{\sigma} - X^{\sigma}$. The single-pole particle has the simplest internal structure, and its dipole, as well as higher multipole moments, are zero:

$$\int_{\Sigma} \delta x^{\sigma}\tau^{\mu\lambda}d^3x = 0, \quad \int_{\Sigma} \delta x^{\sigma}\delta x^{\alpha}\tau^{\mu\lambda}d^3x = 0, \dots \quad (5)$$

Note that the integrals are taken over the three-dimensional hyperspace $\Sigma(t)$, which denotes the interior of the test particle at time t . Substituting Eq. (4) into Eq. (3) and taking into account that we are considering a single-pole particle, we get

$$\partial_{\mu}\tau^{\mu\lambda}(x) + \Gamma_{\mu\alpha}^{\lambda}(X)\tau^{\mu\alpha}(x) = \delta f^{\lambda}(x). \quad (6)$$

Integrating this equation over the hypersurface Σ and taking into account that $\tau^{\mu\nu}$ are zero outside Σ , we find

$$\frac{d}{dt} \int \tau^{\lambda 0} d^3x + \Gamma_{\mu\alpha}^{\lambda}(X) \int \tau^{\mu\alpha} d^3x = \int \delta f^{\lambda} d^3x. \quad (7)$$

On the other hand, multiplication of Eq. (6) by x^{ν} yields

$$\tau^{\nu\lambda} = \partial_{\mu}(x^{\nu}\tau^{\mu\lambda}) + x^{\nu}\Gamma_{\mu\alpha}^{\lambda}(X)\tau^{\mu\alpha} - x^{\nu}\delta f^{\lambda}. \quad (8)$$

Integrating this equation over Σ , we derive

$$\int \tau^{\nu\lambda} d^3x = \frac{d}{dt} \int x^{\nu}\tau^{\lambda 0} d^3x + \Gamma_{\mu\alpha}^{\lambda}(X) \int x^{\nu}\tau^{\mu\alpha} d^3x - \int x^{\nu}\delta f^{\lambda} d^3x. \quad (9)$$

Furthermore, using the requirements of Eq. (5), we can write

$$\begin{aligned} \int x^{\nu}\tau^{\lambda 0} d^3x &= X^{\nu}(t) \int \tau^{\lambda 0} d^3x, \\ \int x^{\nu}\tau^{\mu\alpha} d^3x &= X^{\nu}(t) \int \tau^{\mu\alpha} d^3x, \\ \int x^{\nu}\delta f^{\lambda} d^3x &= X^{\nu}(t) \int \delta f^{\lambda} d^3x. \end{aligned} \quad (10)$$

Substituting these equations into Eq. (10) and using Eq. (7), one gets

$$\int \tau^{\nu\lambda} d^3x = \frac{dX^{\nu}}{dt} \int \tau^{\lambda 0} d^3x. \quad (11)$$

With the help of the $\lambda = 0$ component, we rewrite Eq. (11) as follows:

$$\int \tau^{\nu\lambda} d^3x = \frac{dX^{\nu}}{dt} \frac{dX^{\lambda}}{dt} \int \tau^{00} d^3x. \quad (12)$$

Combining Eqs. (7) and (12), we find

$$\begin{aligned} \frac{d}{dt} \left(\frac{dX^{\lambda}}{dt} \int \tau^{00} d^3x \right) + \Gamma_{\mu\alpha}^{\lambda}(X) \frac{dX^{\mu}}{dt} \frac{dX^{\alpha}}{dt} \int \tau^{00} d^3x \\ = \int \delta f^{\lambda} d^3x. \end{aligned} \quad (13)$$

On the other hand, the line element of the curve $X^{\lambda}(t)$ is given by $ds^2 = g_{\mu\nu}dX^{\mu}dX^{\nu}$. Also, the corresponding four-velocity is $u^{\lambda} = \frac{dX^{\lambda}}{ds}$. Thus, Eq. (13) becomes

$$\frac{d}{ds} (mu^{\lambda}) + m\Gamma_{\mu\alpha}^{\lambda}u^{\mu}u^{\alpha} = u^0 \int \delta f^{\lambda} d^3x, \quad (14)$$

where m is defined as follows:

$$m = \frac{1}{u^0} \int \tau^{00} d^3x. \quad (15)$$

Multiplying Eq. (14) by u_{λ} and taking into account that $u^{\lambda}\nabla_{\mu}u_{\lambda} = 0$, we find

$$\frac{dm}{ds} = u^0 u_{\lambda} \int \delta f^{\lambda} d^3x. \quad (16)$$

In general relativity, the right-hand side of this equation is zero, and so m is a constant of the motion. In this case, m is

interpreted as the rest mass of the particle [3]. Substituting Eq. (16) into Eq. (14), we find

$$m\left(\frac{du^\lambda}{ds} + \Gamma_{\mu\alpha}^\lambda u^\mu u^\alpha\right) = (\delta_\sigma^\lambda - u^\lambda u_\sigma)\Delta^\sigma; \quad (17)$$

here we introduced Δ^σ as

$$\Delta^\sigma = u^0 \int \delta f^\sigma d^3x. \quad (18)$$

Finally, Eq. (17) can be brought into a more compact form, namely

$$m \frac{Du^\lambda}{ds} = (\delta_\sigma^\lambda - u^\lambda u_\sigma)\Delta^\sigma, \quad (19)$$

where $\frac{D}{ds} = u^\mu \nabla_\mu$ is the directional covariant derivative. Equation (19) is the generalization of the geodesic equation. As we expect, orbits of single-pole particles are not the geodesics of the background metric. From Eq. (19), one can immediately read off the additional contribution to the equations of motion due to the nonzero covariant divergence of the matter energy-momentum tensor. One can interpret the left-hand side of Eq. (19) as an extra force—“extra” in comparison to the case $\Delta^\sigma = 0$ —influencing the test particle motion. Such an extra force, due to the nonminimal coupling term on the right-hand side of Eq. (19), may provide an interpretation for the observed mass discrepancy in spiral galaxies and clusters of galaxies [19,20], or in the context of the so-called Pioneer anomaly [21]. For some attempts to construct consistent modified gravity theories that might yield to an appropriate extra force for explaining the aforementioned problems, see Refs. [18,22] and references therein.

IV. THE POLE-DIPOLE PARTICLE

In this section, we derive the equations of motion of a pole-dipole particle. The pole-dipole particle is a particle for which all integrals $\int \delta x^{\rho_1} \delta x^{\rho_2} \dots \delta x^{\rho_n} \tau^{\mu\nu} d^3x$ with more than one factor δx^ρ vanish. Remembering that the test particle is a pole-dipole particle, we substitute Eq. (4) into Eq. (3) and then integrate over the hypersurface Σ . The result is

$$\begin{aligned} & \frac{d}{dt} \int \tau^{\lambda 0} d^3x + \Gamma_{\mu\alpha}^\lambda(X) \int \tau^{\mu\alpha} d^3x + \partial_\sigma \Gamma_{\mu\alpha}^\lambda(X) \\ & \times \int \delta x^\sigma \tau^{\mu\alpha} d^3x = \int \delta f^\lambda d^3x. \end{aligned} \quad (20)$$

Also, multiplying Eq. (3) by x^ν and integrating over Σ , it follows that

$$\begin{aligned} \int \tau^{\nu\lambda} d^3x &= \frac{d}{dt} \int \delta x^\nu \tau^{\lambda 0} d^3x + \frac{dX^\nu}{dt} \int \tau^{\lambda 0} d^3x \\ &+ \Gamma_{\mu\alpha}^\lambda(X) \int \delta x^\nu \tau^{\mu\alpha} d^3x - \int \delta x^\nu \delta f^\lambda d^3x. \end{aligned} \quad (21)$$

Here we have used Eq. (20) to simplify Eq. (21). Another useful equation can be obtained by multiplying Eq. (6) by $x^\alpha x^\beta$ and integrating over Σ . The result is

$$\begin{aligned} & \frac{dX^\alpha}{dt} \int \delta x^\beta \tau^{\lambda 0} d^3x + \frac{dX^\beta}{dt} \int \delta x^\alpha \tau^{\lambda 0} d^3x \\ &= \int \delta x^\alpha \tau^{\beta\lambda} d^3x + \int \delta x^\beta \tau^{\alpha\lambda} d^3x. \end{aligned} \quad (22)$$

Equations (20)–(22) contain enough information to describe the motion of pole-dipole particles. In the following, we will bring these equations into a more familiar form. To do so, analogously to Ref. [3], we introduce the quantities $M^{\lambda\alpha\beta}$, $S^{\alpha\beta}$, and $M^{\alpha\beta}$ as follows:

$$M^{\lambda\alpha\beta} = -u^0 \int \delta x^\lambda \tau^{\alpha\beta} d^3x, \quad (23)$$

$$M^{\alpha\beta} = u^0 \int \tau^{\alpha\beta} d^3x, \quad (24)$$

$$S^{\alpha\beta} = -\frac{1}{u^0} (M^{\alpha\beta 0} - M^{\beta\alpha 0}), \quad (25)$$

where $S^{\alpha\beta}$ is the total angular momentum or macroscopic spin of the test particle. Furthermore, one should note that $M^{0\alpha\beta} = 0$ (because δx^0 is zero) and $M^{\alpha 00} = -u^0 S^{\alpha 0}$. Also, we define the new quantity $\Delta^{\alpha\beta}$ as follows:

$$\Delta^{\alpha\beta} = u^0 \int \delta x^\alpha \delta f^\beta d^3x. \quad (26)$$

With these definitions, Eqs. (20)–(22) become

$$\frac{d}{ds} \left(\frac{M^{\lambda 0}}{u^0} \right) + \Gamma_{\mu\alpha}^\lambda M^{\mu\alpha} - \partial_\sigma \Gamma_{\mu\alpha}^\lambda M^{\sigma\mu\alpha} = \Delta^\lambda, \quad (27)$$

$$M^{\nu\lambda} = \frac{u^\nu}{u^0} M^{\lambda 0} - \frac{d}{ds} \left(\frac{M^{\nu\lambda 0}}{u^0} \right) - \Gamma_{\mu\alpha}^\lambda M^{\nu\mu\alpha} - \Delta^{\nu\lambda}, \quad (28)$$

$$u^0 (M^{\alpha\beta\lambda} + M^{\beta\alpha\lambda}) = u^\alpha M^{\beta\lambda 0} + u^\beta M^{\alpha\lambda 0}. \quad (29)$$

Cyclic permutation of the indices in Eq. (29) and subtraction of the second from the combination of the first and the third of the permutations yields

$$M^{\alpha\beta\gamma} = -\frac{1}{2} (S^{\alpha\beta} u^\gamma + S^{\alpha\gamma} u^\beta) + \frac{1}{2} \frac{u^\alpha}{u^0} (S^{0\beta} u^\gamma + S^{0\gamma} u^\beta). \quad (30)$$

By setting $\lambda = 0$ in Eq. (28), we find

$$M^{\nu 0} = \frac{u^\nu}{u^0} M^{00} - \frac{DS^{\nu 0}}{ds} - \Gamma_{\mu\alpha}^0 M^{\nu\mu\alpha} - \Delta^{\nu 0}. \quad (31)$$

If we contract Eq. (31) with u_ν and use the same choice for the mass as in Ref. [3], namely

$$m = \frac{1}{u^0} (M^{\nu 0} + \Gamma_{\gamma\sigma}^\nu u^\gamma S^{\sigma 0}) u_\nu, \quad (32)$$

we obtain

$$m = \frac{1}{(u^0)^2} (M^{00} + \Gamma_{\mu\nu}^0 S^{\mu 0} u^\nu) + \frac{u_\rho}{u^0} \frac{DS^{\rho 0}}{ds} - \frac{u_\rho}{u^0} \Delta^{\rho 0}. \quad (33)$$

This mass parameter is sometimes called the ‘‘kinematical’’ or ‘‘monopole’’ rest mass of the particle [6]. Using Eqs. (30), (31), and (33), the equation (28) takes the following form:

$$\begin{aligned} M^{\nu\lambda} = & mu^\nu u^\lambda + \frac{u^\nu}{u^0} \left(\frac{DS^{\lambda 0}}{ds} - u^\lambda u_\rho \frac{DS^{\rho 0}}{ds} \right) + \Gamma_{\mu\alpha}^\lambda S^{\nu\mu} u^\alpha \\ & + \frac{1}{2} \frac{DS^{\nu\lambda}}{ds} + \frac{d}{ds} \left(\frac{S^{(\nu 0} u^{\lambda)}}{u^0} \right) \\ & - \left(\Delta^{\nu\lambda} + \frac{u^\nu}{u^0} \Delta^{\lambda 0} + \frac{u^\nu u^\lambda}{u^0} u_\rho \Delta^{\rho 0} \right), \end{aligned} \quad (34)$$

where the round brackets in $S^{(\nu 0} u^{\lambda)}$ denote symmetrization. Since $M^{\nu\lambda}$ is symmetric, the antisymmetric part of the right-hand side of Eq. (28) vanishes. Hence, one can immediately verify

$$\begin{aligned} & \frac{u^\nu}{u^0} \frac{DS^{\lambda 0}}{ds} - \frac{u^\lambda}{u^0} \frac{DS^{\nu 0}}{ds} + \frac{DS^{\nu\lambda}}{ds} \\ & = \Delta^{\nu\lambda} - \Delta^{\lambda\nu} + \frac{1}{u^0} (u^\nu \Delta^{\lambda 0} - u^\lambda \Delta^{\nu 0}). \end{aligned} \quad (35)$$

After some elementary calculations, this equation can be rewritten as

$$\begin{aligned} & \frac{DS^{\nu\lambda}}{ds} + u^\nu u_\rho \frac{DS^{\lambda\rho}}{ds} - u^\lambda u_\rho \frac{DS^{\nu\rho}}{ds} \\ & = (\Delta^{\nu\lambda} - \Delta^{\lambda\nu}) + u^\nu u_\rho (\Delta^{\lambda\rho} - \Delta^{\rho\lambda}) \\ & \quad - u^\lambda u_\rho (\Delta^{\nu\rho} - \Delta^{\rho\nu}). \end{aligned} \quad (36)$$

This equation should be compared to Eq. (5.3) in Ref. [3]. With the help of Eqs. (34)–(36), we derive the following equations, which are useful in simplifying Eq. (27):

$$M^{\nu 0} = u^0 \left[mu^\nu + u_\rho \frac{DS^{\nu\rho}}{ds} - \Gamma_{\gamma\sigma}^\nu \frac{u^\gamma}{u^0} S^{\sigma 0} - u_\rho (\Delta^{\nu\rho} - \Delta^{\rho\nu}) \right], \quad (37)$$

$$\begin{aligned} \Gamma_{\nu\lambda}^\alpha M^{\nu\lambda} = & \Gamma_{\nu\lambda}^\alpha \left(mu^\nu u^\lambda + u^\nu u_\rho \frac{DS^{\lambda\rho}}{ds} \right) \\ & + \Gamma_{\nu\lambda}^\alpha \Gamma_{\mu\rho}^\lambda S^{\nu\mu} u^\rho + \Gamma_{\nu\lambda}^\alpha \frac{d}{ds} \left(\frac{S^{\nu 0} u^\lambda}{u^0} \right) \\ & - \Gamma_{\nu\lambda}^\alpha (\Delta^{\nu\lambda} + u^\nu u_\rho (\Delta^{\lambda\rho} - \Delta^{\rho\lambda})). \end{aligned} \quad (38)$$

By using Eqs. (37) and (38), we can bring Eq. (27) into its final covariant form:

$$\begin{aligned} & \frac{D}{ds} \left(mu^\nu + u_\rho \frac{DS^{\nu\rho}}{ds} \right) + \frac{1}{2} S^{\mu\alpha} u^\sigma R^\nu{}_{\alpha\sigma\mu} \\ & = \frac{D}{ds} (u_\rho (\Delta^{\nu\rho} - \Delta^{\rho\nu})) + \Delta^\nu + u^0 \int \nabla_\alpha (\delta x^\alpha \delta f^\nu) d^3 x, \end{aligned} \quad (39)$$

where $R^\nu{}_{\alpha\sigma\mu}$ is the curvature tensor. Also, note that it is straightforward to show

$$u^0 \int \nabla_\alpha (\delta x^\alpha \delta f^\nu) d^3 x \simeq \Gamma_{\alpha\beta}^\nu (X) \Delta^{\alpha\beta}. \quad (40)$$

Equation (39) should be compared to Eq. (5.7) in Ref. [3]. In general relativity, the right-hand side of Eq. (39) is zero [3].

It is important to note that the mass parameter m is not necessarily a constant of motion. To show this in more detail, let us define the ‘‘generalized momentum’’ p^ν as follows:

$$p^\nu = mu^\nu + \frac{DS^{\nu\alpha}}{ds} u_\alpha + 2u_\rho \Delta^{[\rho\nu]}, \quad (41)$$

where the square brackets denote antisymmetrization. Using this equation and taking into account that $m = u_\nu p^\nu$, it is straightforward to rewrite Eqs. (36) and (39) as follows:

$$\dot{S}^{\nu\lambda} = 2(p^{[\nu} u^{\lambda]}) + \Delta^{[\nu\lambda]}, \quad (42)$$

$$\dot{p}^\nu = -\frac{1}{2} S^{\mu\alpha} u^\sigma R^\nu{}_{\alpha\sigma\mu} + (\Delta^\nu + \Gamma_{\alpha\beta}^\nu \Delta^{\alpha\beta}), \quad (43)$$

where the dot denotes the covariant derivative with respect to the proper time, D/ds . The main result of this section is embodied in the Eqs. (42) and (43). These equations should be compared to the well-known Mathisson-Papapetrou equations for pole-dipole test particles in general relativity. It is obvious that if we assume that there is no coupling between matter and the gravitational fields, then Δ^α and $\Delta^{\alpha\beta}$ are zero, and Eqs. (42) and (43) recover the Mathisson-Papapetrou equations.

Using Eq. (43), one can easily show

$$\dot{m} = \left(\frac{D}{ds} (S^{\nu\rho} u_\rho) + 2u_\rho \Delta^{[\rho\nu]} \right) \dot{u}_\nu + u_\nu (\Delta^\nu + \Gamma_{\alpha\beta}^\nu \Delta^{\alpha\beta}). \quad (44)$$

We can also define another mass parameter by $M^2 = p_\alpha p^\alpha$, which in general is different from m . The mass parameter M is sometimes called the ‘‘dynamical,’’ ‘‘total,’’ or ‘‘effective’’ rest mass of the test particle [6]. If we contract Eq. (42) with $p_\nu \dot{p}_\lambda$, we get

$$\dot{M} = -\frac{\dot{p}_\lambda}{Mm} \frac{D}{ds} (S^{\nu\lambda} p_\nu) + \frac{M}{m} (\Delta^\nu + \Gamma_{\alpha\beta}^\nu \Delta^{\alpha\beta}) u_\nu. \quad (45)$$

Thus, M is not also necessarily a constant of motion.

It is also necessary to mention that, just as in general relativity, Eqs. (42) and (43) are insufficient to describe the motion of the pole-dipole particle. In fact, we have 14 unknown variables (x^α , u^α , and $S^{\alpha\beta}$) and only 11 equations; i.e., Eqs. (43) and (42), and $u^\alpha u_\alpha = 1$. Note that $u^\alpha u_\alpha = 1$ is the mere constraint due to the choice of parametrization of the trajectories. Thus, we need three more equations to make the model self-consistent.

Let us recall the origin of this apparent inconsistency. The above procedure for determining the motion of test particles consists of choosing a representative point [or a representative world line $X^\lambda(t)$] in the particle and taking the moments of Eq. (1) about that point. However, we have not specified how to choose this representative point in the test particle. In other words, there is an arbitrary step in the above method, and it is the choosing of a representative point in the body. Thus, one can expect that if such a point is not uniquely specified, the motion of the representative point will not be fully determined. In general relativity, the proper specification of such a representative world line yields to a supplementary condition to Mathisson-Papapetrou equations which determines the particle's motion [23]. The two most widely used supplementary conditions are the Frenkel condition [24] (for massless particles)

$$S^{\alpha\beta} u_\beta = 0 \quad (46)$$

and the Tulczyjew condition [25] (for massive particles)

$$S^{\alpha\beta} p_\beta = 0. \quad (47)$$

In general relativity, with the help of Eqs. (44) and (45), one can immediately verify that m is constant when the Frenkel condition [Eq. (46)] is assumed and the mass parameter M is constant for the Tulczyjew condition [Eq. (47)]. We do not want to derive the corresponding supplementary equations which may yield to constant parameters m and M in the context of modified gravity theories here, because this may require the exact form of the nonminimal coupling term δf^ν . However, we shall go into this issue for MOG in the next sections.

V. TEST PARTICLE MOTION IN MOG

In this section, we consider the test particle motion in MOG, and we make some comments about this theory. MOG is an alternative theory of gravity, and it is claimed in the literature that this theory can address the dark matter problem in spiral galaxies and clusters of galaxies [18,26,27]. More specifically, MOG is a scalar-tensor-vector gravity theory which postulates, in addition to the metric tensor, three dynamical scalar gravitational fields G , μ , and ω , and also a dynamical massive four-vector

gravitational field ϕ^μ . (See the Appendix for more detail.) The vector field ϕ^μ is coupled universally to matter. As we mentioned before, such a coupling between matter and the vector field yields to a nonzero covariant divergence of the ordinary matter energy-momentum tensor, and consequently this coupling leads to a modification of the law of gravitation. Here, by the law of gravitation, we mean the gravitational force law between two point particles in the weak-field approximation of the given modified gravity theory. We know that general relativity in the Newtonian limit coincides with Newtonian gravity, but this is not the case for other gravity theories. It should be emphasized that, for a modification to the law of gravitation, it is not necessary to construct a theory with nonminimal coupling terms between matter and gravitational fields. In fact, it is well known that modified gravity theories which yield to a conserved energy-momentum tensor can also provide a modification to the law of gravitation. For example, see metric $f(R)$ theory [28,29].

In order to investigate the test particle equations of motion, we need the exact form of the conservation law [Eq. (1)]. We have found the conservation law in the Appendix. The result, from Eq. (A31), is

$$\nabla_\mu T^{\mu\nu} = B_{\alpha}{}^\nu J^\alpha + \nabla^\mu \left(g_{\mu\alpha} \omega \phi^\nu \frac{\partial W(\phi)}{\partial \phi_\alpha} - 2\omega g^{\nu\rho} \frac{\partial W(\phi)}{\partial g^{\mu\rho}} \right), \quad (48)$$

where $B_{\mu\nu} = \nabla_\mu \phi_\nu - \nabla_\nu \phi_\mu$, the term J^α is a ‘‘fifth force’’ matter current defined by Eq. (A20), and the self-interaction potential $W(\phi)$ is given by Eq. (A30). As we have discussed, the right-hand side of this equation provides the modification of the law of gravitation. It seems that this theory can provide a wide range of modifications to the law of gravitation. In other words, by choosing different self-interaction potentials $W(\phi)$, one can introduce different modifications. In Ref. [18], a special modification has been postulated [see Eq. (32) in Ref. [18]]. This postulation can be considered as a special choice for $W(\phi)$. However, we show that there is no potential $W(\phi)$ corresponding to the extra force postulated in Ref. [18]. In other words, we show that the test particle equation of motion postulated in Ref. [18] is not compatible with the field equations of MOG. Albeit, as we shall discuss, this issue may not change the main results of MOG.

In this theory, the covariant derivatives of the vector field ϕ^μ do not appear in $W(\phi)$. Thus, the most general form for $W(\phi)$ can be given as

$$W(\phi) = \sum_n c_n (\phi^\alpha \phi_\alpha)^n, \quad (49)$$

where c_n are constant coefficients. For such a potential, it is straightforward to show that the second term on the right-hand side of Eq. (48) vanishes. Finally, the conservation

law of the ordinary matter energy-momentum tensor in MOG reads

$$\nabla_{\mu} T^{\mu\nu} = B_{\alpha}{}^{\nu} J^{\alpha}. \quad (50)$$

Comparing this equation to Eq. (1), we get $\delta f^{\nu} = \sqrt{-g} B_{\alpha}{}^{\nu} \delta J^{\alpha}$. In the presence of the test particle, the matter current density is given by $\hat{J}^{\nu} = J^{\nu} + \delta J^{\nu}$. On the other hand, since the particle is moving outside of the massive bodies, J^{ν} is zero near and inside the test particle. Furthermore, since it is assumed that the matter current density is conserved (i.e., $\nabla_{\alpha} J^{\alpha} = 0$), one can immediately infer that $\nabla_{\alpha} \delta J^{\alpha} = 0$. Consequently, it is straightforward to show that $Q = \int \sqrt{-g} \delta J^0 d^3x$ is a time-independent quantity. Hence, we may introduce the matter current density δJ^{α} as follows (see Ref. [30]):

$$\delta J^{\alpha} = \frac{Q}{\sqrt{-g}} \int \delta^4(x - X) dx^{\alpha}. \quad (51)$$

Therefore, it is easy to show that Δ^{σ} is given by

$$\Delta^{\sigma} = q_5 B^{\sigma}{}_{\alpha} u^{\alpha}, \quad (52)$$

where $q_5 = -Q$ is the test particle's fifth force charge. In this theory, the fifth force charge is assumed to be proportional to the particle's mass m , such that $q_5 = \kappa m$. Finally, using Eq. (19), the equation of motion of the single-pole particle is given by

$$m \dot{u}^{\lambda} = q_5 B_{\alpha}^{\lambda} u^{\alpha}. \quad (53)$$

Taking into account Eq. (16), it is clear that, as in general relativity, m is a constant of motion, and can be considered as the rest mass of the test particle. This equation of motion [Eq. (53)] is different from that postulated in Eq. (31) of Ref. [18]. In fact, as is obvious from Eq. (53), the scalar field $\omega(x)$ and its derivatives do not appear in the equation of motion. As we mentioned before, the vector field ϕ^{μ} is coupled to ordinary matter; thus, we expect that the vector field or its derivatives will exist in the equation of motion. On the other hand, there is a coupling between ϕ^{μ} and $\omega(x)$ [see Eq. (A3)], and consequently one may expect that the scalar field $\omega(x)$ should appear in the equation of motion. However, as is clear from Eq. (53), that is not the case in this theory. Furthermore, one can easily show that the test particle action governing the equation of motion [Eq. (53)] is given by

$$S_{TP} = - \int (m + q_5 \phi_{\mu} u^{\mu}) ds, \quad (54)$$

where s is the proper time along the world line of the test particle. This action is also different from what is postulated in Eq. (30) of Ref. [18].

In Ref. [18], it has been assumed that, for the vacuum solution around a spherically symmetric pointlike mass,

$\omega(x)$ is nearly constant. This assumption is also consistent with the numerical solution of the vacuum field equations [31]. With this unnecessary assumption, the postulated equation of motion can be written as

$$m \dot{u}^{\lambda} = \lambda \omega B_{\alpha}^{\lambda} u^{\alpha}, \quad (55)$$

where λ denotes a coupling constant, and its magnitude can be fixed using the experimental observations. Equation (55) may still seem different from Eq. (53) because of the presence of constant ω . However, since λ has been assumed to be an arbitrary constant, one can choose it to be $\lambda = -\frac{Q}{\omega}$. Consequently, in this case Eq. (55) coincides with Eq. (53), and the main result of this theory related to the dark matter problem will not change.

Now let us consider the spinning particle equations of motion in MOG. It is straightforward to verify that $\Delta^{\alpha\beta}$ is zero in MOG. Therefore, the equations of motion [Eqs. (42) and (43)] can be simplified as follows:

$$\dot{S}^{\nu\lambda} = 2p^{[\nu} u^{\lambda]}, \quad (56)$$

$$\dot{p}^{\nu} = -\frac{1}{2} S^{\mu\alpha} u^{\sigma} R^{\nu}{}_{\alpha\sigma\mu} + q_5 B^{\nu}{}_{\alpha} u^{\alpha}, \quad (57)$$

where $p^{\nu} = m u^{\nu} + \dot{S}^{\nu\alpha} u_{\alpha}$. Equation (56) is the same as in general relativity, but the extra term $q_5 B^{\nu}{}_{\alpha} u^{\alpha}$ exists on the right-hand side of Eq. (57) relative to the Mathisson-Papapetrou equations. It is obvious from Eqs. (56) and (57), which describe the motion of electrically neutral pole-dipole particles in MOG, that there is a similarity between these equations and the Dixon-Souriau equations which determine the motion of charged pole-dipole particles in the presence of electromagnetic fields in the context of general relativity. Mathematically, the origin of this similarity lies in Eq. (50). In fact, using Einstein-Maxwell equations, one can easily show that the covariant divergence of the ordinary matter energy-momentum tensor is given by

$$\nabla_{\mu} T^{\mu\nu} = F_{\alpha}{}^{\nu} J^{\alpha}, \quad (58)$$

where $F_{\mu\nu}$ is the field strength tensor, $J^{\mu} = (\rho, J^x, J^y, J^z)$ is the current four-vector, and ρ is the electric charge density. The close similarity between Eqs. (50) and (58) yields to a similarity between the equations of motion of test particles in MOG and Dixon-Souriau equations. However, one should note that Eqs. (56) and (57) are not exactly similar to the Dixon-Souriau equations. In fact, mathematically, they are similar to the equations of motion of a charged spinning particle without magnetic moment; see Eq. (9) in Ref. [13].

VI. CONSERVED QUANTITIES

It is well known that the symmetries of the background space-time may yield to the existence of conserved quantities. For example, in general relativity, for a charged test

particle moving under the influence of an electromagnetic field, if the electromagnetic field satisfies some consistency conditions, then the space-time symmetries guarantee the existence of some conserved quantities. In this section, we consider the conserved quantities related to the existence of the Killing vectors and the Killing tensors. We recall that a Killing vector ξ^μ and a totally symmetric rank- n Killing tensor $K_{\beta_1 \dots \beta_n}$ satisfy

$$\nabla_\alpha \nabla_\beta \xi_\mu = R_{\beta\mu\alpha}{}^\lambda \xi_\lambda, \quad \nabla_{(\alpha} \xi_{\beta)} = 0, \quad (59)$$

and

$$\nabla_{(\alpha} K_{\beta_1 \dots \beta_n)} = 0. \quad (60)$$

Let us first consider the conserved quantities for the motion of a single-pole particle. The equation of motion is given by Eq. (53). If we assume that the background geometry possesses a Killing vector ξ_μ , then one can straightforwardly show

$$\frac{D}{ds} (\xi_\alpha p^\alpha + q_5 \xi_\alpha \phi^\alpha) = q_5 u^\mu L_\xi \phi_\mu, \quad (61)$$

where $L_\xi \phi_\mu$ is the Lie derivative of the vector field ϕ_μ with respect to ξ . Thus, the quantity

$$Q_\xi = \xi_\alpha p^\alpha + q_5 \xi_\alpha \phi^\alpha \quad (62)$$

is conserved if the vector field ϕ_μ is Lie conserved along ξ ; i.e., $L_\xi \phi_\mu = 0$. Similarly, one can verify that the component of the velocity along ξ ,

$$u_\xi = \xi_\alpha u^\alpha, \quad (63)$$

is conserved if

$$\xi_\alpha B^\alpha{}_\beta = 0. \quad (64)$$

This condition can be generalized to quantities nonlinear in velocities if the space-time has appropriate Killing tensors. Let $K_{\beta_1 \dots \beta_n}$ be a rank- n Killing tensor satisfying Eq. (60). Then, it is straightforward to show

$$\dot{Q}_K = -\frac{q_5}{m} K^\sigma{}_{(\beta_1 \dots \beta_{n-1}} B_{\beta_n)\sigma} u^{\beta_1} \dots u^{\beta_n}. \quad (65)$$

Thus, Q_K is conserved if

$$K^\sigma{}_{(\beta_1 \dots \beta_{n-1}} B_{\beta_n)\sigma} = 0. \quad (66)$$

Since the metric $g_{\mu\nu}$ is a rank-two Killing tensor satisfying trivially the condition in Eq. (66), we find conservation of the norm of the test particle's velocity; i.e., $g_{\mu\nu} u^\mu u^\nu$.

Now let us consider the conserved quantities for the motion of a spinning test particle in MOG. Using Eqs. (44), (45), and (56), we find

$$\begin{aligned} \dot{m} &= \frac{D}{ds} (S^{\alpha\beta} u_\beta) \dot{u}_\alpha, & m M \dot{M} &= \frac{D}{ds} (S^{\alpha\beta} p_\beta) \dot{p}_\alpha, \\ S \dot{S} &= p^{[\alpha} u^{\beta]} S_{\alpha\beta}, \end{aligned} \quad (67)$$

where S is the magnitude of the spin, defined by $S^2 = \frac{1}{2} S_{\alpha\beta} S^{\alpha\beta}$. Thus, m is a constant of the motion when the Frenkel condition [Eq. (46)] is assumed, and accordingly M is constant when the Tulczyjew condition [Eq. (47)] is assumed. Also, the magnitude S of the spin is a constant of the motion for both supplementary conditions. Furthermore, if we assume that the background geometry possesses a Killing vector ξ , then using Eqs. (56) and (57), we get

$$\frac{D}{ds} \left(\xi_\alpha p^\alpha + q_5 \xi_\alpha \phi^\alpha + \frac{1}{2} S^{\alpha\beta} \nabla_\alpha \xi_\beta \right) = q_5 u^\mu L_\xi \phi_\mu. \quad (68)$$

Thus, for a spinning test particle, the quantity

$$Q_\xi = \xi_\alpha p^\alpha + q_5 \xi_\alpha \phi^\alpha + \frac{1}{2} S^{\alpha\beta} \nabla_\alpha \xi_\beta \quad (69)$$

is conserved if the vector field ϕ_μ is Lie conserved along ξ .

VII. CONCLUSIONS

In this paper, we derived the generalized Mathisson-Papapetrou equations in the realm of modified gravity theories allowing a nonminimal coupling between matter and geometry. In other words, we derived the equations of motion of the single-pole and the pole-dipole (spinning) test particles using the multipole approximation method. Our results are consistent with the results already found in the literature. In the second part of the paper, using the generalized Mathisson-Papapetrou equations, we analyzed the equations of motion of a pole-dipole particle in the context of MOG. Furthermore, we found some conserved quantities for the motion of a test particle in MOG.

In the context of MOG, our results are different from the results already found in the literature. In fact, the equations of motion for the test particle postulated in MOG [18] are not true, and they should be replaced by Eq. (53). However, as we discussed in Sec. V, this point does not change the main features of this theory considering the dark matter problem.

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APPENDIX: COVARIANT DIVERGENCE OF $T^{\mu\nu}$ IN MOG

The action of the theory is given by

$$S = S_{\text{Grav}} + S_\phi + S_\chi + S_\psi + S_\omega + S_M, \quad (A1)$$

where S_M is the action for ordinary matter, and

$$S_{\text{Grav}} = \frac{1}{16\pi} \int \sqrt{-g} d^4x f(\chi) R, \quad (A2)$$

$$S_\phi = - \int \sqrt{-g} d^4x \omega \left(\frac{1}{4} B^{\mu\nu} B_{\mu\nu} + V_\phi(\phi, g_{\mu\nu}, \psi) \right), \quad (\text{A3})$$

$$S_\chi = \int \sqrt{-g} d^4x \left(\frac{1}{2} g^{\mu\nu} \nabla_\mu \chi \nabla_\nu \chi - V_\chi(\chi) \right), \quad (\text{A4})$$

$$S_\psi = \int \sqrt{-g} d^4x f(\chi) \left(\frac{1}{2} g^{\mu\nu} \nabla_\mu \psi \nabla_\nu \psi - V_\psi(\psi) \right), \quad (\text{A5})$$

$$S_\omega = \int \sqrt{-g} d^4x f(\chi) \left(\frac{1}{2} g^{\mu\nu} \nabla_\mu \omega \nabla_\nu \omega - V_\omega(\omega) \right), \quad (\text{A6})$$

where R is the Ricci scalar, $B_{\mu\nu} = \nabla_\mu \phi_\nu - \nabla_\nu \phi_\mu$, and the scalar fields χ and ψ are related to the scalar fields G and μ of the original paper [18] by

$$f(\chi) = \frac{\chi^2}{2} = \frac{1}{G}, \quad \psi = \ln \mu; \quad (\text{A7})$$

also, V_χ and V_ψ are related to V_G and V_μ by

$$V_\chi = \frac{V_G}{G^3}, \quad V_\psi = \frac{V_\mu}{\mu^2}. \quad (\text{A8})$$

In this theory, it has been assumed that $\frac{\partial V_\phi}{\partial \nabla_\gamma \phi_\alpha} = 0$. In other words, the covariant derivatives of ϕ^α do not appear in V_ϕ .

The total energy-momentum tensor is given by

$$T_{\mu\nu}^{\text{total}} = T_{\mu\nu} + T_{\mu\nu}^\phi + T_{\mu\nu}^\chi + T_{\mu\nu}^\psi + T_{\mu\nu}^\omega, \quad (\text{A9})$$

where

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}}, \quad T_{\mu\nu}^Q = \frac{-2}{\sqrt{-g}} \frac{\delta S_Q}{\delta g^{\mu\nu}}, \quad (\text{A10})$$

and Q can be ϕ , χ , ψ , or ω . Using Eqs. (A2)–(A6) and (A10), one can easily verify

$$T_{\mu\nu}^\phi = \omega \left(B_{\mu}{}^\alpha B_{\nu\alpha} - g_{\mu\nu} \left(\frac{1}{4} B^{\rho\sigma} B_{\rho\sigma} + V_\phi \right) \right) + 2 \frac{\partial V_\phi}{\partial g^{\mu\nu}}, \quad (\text{A11})$$

$$T_{\mu\nu}^\chi = - \left(\nabla_\mu \chi \nabla_\nu \chi - g_{\mu\nu} \left(\frac{1}{2} \nabla_\alpha \chi \nabla^\alpha \chi - V_\chi \right) \right), \quad (\text{A12})$$

$$T_{\mu\nu}^\psi = -f(\chi) \left(\nabla_\mu \psi \nabla_\nu \psi - g_{\mu\nu} \left(\frac{1}{2} \nabla_\alpha \psi \nabla^\alpha \psi - V_\psi \right) \right), \quad (\text{A13})$$

$$T_{\mu\nu}^\omega = -f(\chi) \left(\nabla_\mu \omega \nabla_\nu \omega - g_{\mu\nu} \left(\frac{1}{2} \nabla_\alpha \omega \nabla^\alpha \omega - V_\omega \right) \right). \quad (\text{A14})$$

Furthermore, variation of Eq. (A1) with respect to $g^{\mu\nu}$, ϕ_α , χ , ψ , and ω yields the following field equations, respectively:

$$f(\chi) R_{\mu\nu} - \frac{1}{2} f(\chi) R g_{\mu\nu} = (\nabla_\mu \nabla_\nu - g_{\mu\nu} \square) f(\chi) + 8\pi T_{\mu\nu}^{\text{total}}, \quad (\text{A15})$$

$$\omega \nabla_\mu B^{\alpha\mu} + \nabla_\mu \omega B^{\alpha\mu} + \omega \frac{\partial V_\phi}{\partial \phi_\alpha} = -J^\alpha, \quad (\text{A16})$$

$$\square \chi + V'_\chi = \frac{f'(\chi)}{16\pi} + f'(\chi) \left(\frac{1}{2} g^{\mu\nu} (\nabla_\mu \psi \nabla_\nu \psi + \nabla_\mu \omega \nabla_\nu \omega) - (V_\psi + V_\omega) \right), \quad (\text{A17})$$

$$f(\chi) (\square \psi + V'_\psi) = -f'(\chi) \nabla_\gamma \chi \nabla^\gamma \psi - \omega \frac{\partial V_\phi}{\partial \psi}, \quad (\text{A18})$$

$$f(\chi) (\square \omega + V'_\omega) = -f'(\chi) \nabla_\gamma \chi \nabla^\gamma \omega - \frac{1}{4} B^{\rho\sigma} B_{\rho\sigma} - V_\phi, \quad (\text{A19})$$

where the prime stands for $H'(x) = \frac{dH}{dx}$, and J^α is a ‘‘fifth force’’ matter current defined as

$$J^\alpha = - \frac{1}{\sqrt{-g}} \frac{\delta S_M}{\delta \phi_\alpha}. \quad (\text{A20})$$

Taking the covariant divergence on both sides of Eq. (A15) yields

$$\nabla^\mu T_{\mu\nu} = - \frac{f'(\chi)}{16\pi} R \nabla_\nu \chi - (\nabla^\mu T_{\mu\nu}^\phi + \nabla^\mu T_{\mu\nu}^\chi + \nabla^\mu T_{\mu\nu}^\psi + \nabla^\mu T_{\mu\nu}^\omega). \quad (\text{A21})$$

Note that on purely geometrical grounds, $(\square \nabla_\mu - \nabla_\mu \square) f = R_{\mu\nu} \nabla^\nu f$ and $\nabla^\mu G_{\mu\nu} = 0$, where f is an arbitrary scalar function and $G_{\mu\nu}$ is the Einstein tensor. On the other hand, taking the covariant divergence of Eqs. (A11)–(A14) and using the field equations (A16)–(A19), we find

$$\begin{aligned} \nabla^\mu T_{\mu\nu}^\phi &= B_{\nu\alpha} J^\alpha - \nabla_\nu \omega \left(\frac{1}{4} B^{\rho\sigma} B_{\rho\sigma} + V_\phi \right) \\ &\quad + 2 \nabla^\mu \left(\omega \frac{\partial V_\phi}{\partial g^{\mu\nu}} \right) - \omega \nabla_\nu V_\phi + \omega B_{\nu\alpha} \frac{\partial V_\phi}{\partial \phi_\alpha}, \end{aligned} \quad (\text{A22})$$

$$\begin{aligned} \nabla^\mu T_{\mu\nu}^\chi &= - \frac{f'(\chi)}{16\pi} R \nabla_\nu \chi - f'(\chi) \nabla_\nu \chi \left(\frac{1}{2} g^{\alpha\beta} (\nabla_\alpha \psi \nabla_\beta \psi + \nabla_\alpha \omega \nabla_\beta \omega) - (V_\psi + V_\omega) \right), \end{aligned} \quad (\text{A23})$$

$$\nabla^\mu T_{\mu\nu}^\psi = f'(\chi) \left(\frac{1}{2} \nabla_\alpha \psi \nabla_\alpha \psi - V_\psi \right) \nabla_\nu \chi + \omega \frac{\partial V_\phi}{\partial \psi} \nabla_\nu \psi, \quad (\text{A24})$$

$$\begin{aligned} \nabla^\mu T_{\mu\nu}^\omega &= f'(\chi) \left(\frac{1}{2} \nabla_\alpha \omega \nabla_\alpha \omega - V_\omega \right) \nabla_\nu \chi \\ &+ \left(\frac{1}{4} B^{\rho\sigma} B_{\rho\sigma} + V_\phi \right) \nabla_\nu \omega. \end{aligned} \quad (\text{A25})$$

In Eq. (A22), we used the following equation:

$$B_{\mu}{}^\alpha \nabla^\mu B_{\nu\alpha} - \frac{1}{4} \nabla_\nu (B^{\rho\sigma} B_{\rho\sigma}) = 0. \quad (\text{A26})$$

Equation (A26) can be easily verified by using the definition of $B_{\mu\nu}$. Now, substituting Eqs. (A22)–(A25) into Eq. (A21), we obtain

$$\nabla^\mu T_{\mu\nu} = B_{\alpha\nu} J^\alpha + \left(\omega \nabla_\alpha \phi_\nu \frac{\partial V_\phi}{\partial \phi_\alpha} - 2 \nabla^\mu \left(\omega \frac{\partial V_\phi}{\partial g^{\mu\nu}} \right) \right). \quad (\text{A27})$$

On the other hand, taking the covariant divergence of Eq. (A16) and assuming $\nabla_\alpha J^\alpha = 0$, one gets

$$\nabla_\alpha \left(\omega \frac{\partial V_\phi}{\partial \phi_\alpha} \right) = 0. \quad (\text{A28})$$

With the help of this equation, we can rewrite Eq. (A27) as

$$\nabla^\mu T_{\mu\nu} = B_{\alpha\nu} J^\alpha + \nabla^\mu \left(g_{\mu\alpha} \omega \phi_\nu \frac{\partial V_\phi}{\partial \phi_\alpha} - 2 \omega \frac{\partial V_\phi}{\partial g^{\mu\nu}} \right). \quad (\text{A29})$$

Finally, we choose the following form for the potential V_ϕ (see Eq. (20) in Ref. [18]):

$$V_\phi = -\frac{1}{2} e^{2\psi} \phi^\alpha \phi_\alpha + W(\phi), \quad (\text{A30})$$

where $W(\phi)$ is the vector field ϕ^α self-interaction contribution. Substituting this potential into Eq. (A30), we get

$$\nabla^\mu T_{\mu\nu} = B_{\alpha\nu} J^\alpha + \nabla^\mu \left(g_{\mu\alpha} \omega \phi_\nu \frac{\partial W(\phi)}{\partial \phi_\alpha} - 2 \omega \frac{\partial W(\phi)}{\partial g^{\mu\nu}} \right). \quad (\text{A31})$$

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