

Supersymmetric Q -balls and boson stars in $(d + 1)$ dimensions

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We construct supersymmetric Q -balls and boson stars in $(d + 1)$ dimensions. These nontopological solitons are solutions of a scalar field model with global $U(1)$ symmetry and a scalar field potential that appears in gauge-mediated supersymmetry breaking in the minimal supersymmetric extension of the standard model. We are interested in both the asymptotically flat as well as in the asymptotically anti-de Sitter (AdS) solutions. In particular, we show that for our choice of the potential gravitating, asymptotically flat boson stars exist in $(2 + 1)$ dimensions. We observe that the behavior of the mass and charge of the asymptotically flat solutions at the approach of the maximal frequency depends strongly on the number of spatial dimensions. In particular, we find that in the “thick-wall limit” Q -balls are always unstable in flat space-time, but that they can become stable in AdS. Moreover, for the asymptotically AdS solutions the model on the conformal boundary can be interpreted as describing d -dimensional condensates of scalar glueballs.

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I. INTRODUCTION

A number of nonlinear field theories possess solitonic-like solutions. These have broad applications in many branches of physics and constitute localized, globally regular structures with finite energy. Topological solitons [1] possess a conserved topological charge that is connected to the existence of noncontractible loops in the theory. Nontopological solitons [2,3] on the other hand appear in models with symmetries and possess a locally conserved Noether current and a globally conserved Noether charge. An example of such a nontopological soliton is the Q -ball [4] and its generalization in curved space-time, the boson star [5–10]. These are solutions of models with self-interacting complex scalar fields and the conserved Noether charge Q is then related to the global phase invariance of the theory and is directly proportional to the frequency of the harmonic time dependence. Q can e.g., be interpreted as particle number [2]. As such, these solutions have been constructed in $(3 + 1)$ -dimensional models with nonrenormalizable Φ^6 potential [11–13], but also in supersymmetric extensions to the standard model [14]. In the latter case, several scalar fields interact via complicated potentials. It was shown that cubic interaction terms that result from Yukawa couplings in the superpotential and supersymmetry (SUSY) breaking terms lead to the existence of Q -balls with nonvanishing baryon or lepton number or electric charge. These supersymmetric Q -balls have been considered as possible candidates for baryonic dark matter [15] and their astrophysical implications have been discussed [16]. In Ref. [17], these objects have been constructed numerically using the exact form of a scalar potential that results from gauge-mediated SUSY

breaking. However, this potential is nondifferentiable at the SUSY breaking scale. In Ref. [18] a differentiable approximation of this potential was suggested and the properties of the corresponding Q -balls have been investigated. Most models of a quantum theory of gravity need more than $(3 + 1)$ dimensions and as such, it is surely of interest to investigate the properties of soliton solutions in higher dimensions. The first study of Q -balls in higher dimensional space-time has been done in Refs. [19,20]. In Ref. [19] a mixture of analytical and numerical tools was used, while in Ref. [20] only a linearized version of the Lagrangian and equations of motion depending only in zeroth order on the ratio between the typical energy scale and the Planck mass has been used. In this case, an analytical solution can be given, however, the model does not capture the nonlinear phenomena such as e.g., the behavior of the mass and charge at the maximal frequency. A similar study has been done in Ref. [21] for $(2 + 1)$ dimensions. Q -balls and boson star solutions of the full system of coupled nonlinear equations in $(4 + 1)$ -dimensional asymptotically flat space-time have been investigated in Ref. [22]. Interestingly, it was found that the behavior of the mass and charge at the approach of the maximal possible frequency is different for $d = 4$ as compared to $d = 3$. This was related to a scaling behavior of the solutions at this critical approach and different dimensions of the spatial integrals. Spinning generalizations of these solutions can also be constructed [11–13,22–24]. These solutions possess a quantized angular momentum that is an integer multiple of the Noether charge.

Topological and nontopological solitons in anti-de Sitter (AdS) space-time have been investigated intensively recently. The interest in these objects is related to the AdS/CFT correspondence [25,26] which states that a gravity theory in a d -dimensional anti-de Sitter (AdS) space-time is equivalent to a conformal field theory (CFT) on the

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$(d - 1)$ -dimensional boundary of AdS. Interestingly, this is a weak-strong coupling duality that can be used to describe strongly coupled quantum field theories with the help of weakly coupled gravity theories. This has been applied to a modeling of high temperature superconductivity with the help of classical black hole and soliton solutions in AdS [27–29]. The basic models use a scalar field coupled to a $U(1)$ gauge field and the observation that close to the horizon of the black hole the effective mass of the scalar field can become negative with masses below the Breitenlohner-Freedman bound [30] such that the scalar field becomes unstable and possesses a nonvanishing value on and close to the horizon of the black hole. When computing the conductivities it turns out that the formation of a scalar field on a charged black hole corresponds to a phase transition from a conductor to a superconductor. However, insulator/superconductor phase transitions also play an important role in high temperature superconductivity and as such models including solitons have been suggested that describe this phenomenon [31–33]. The AdS soliton is related to the black hole by a double Wick rotation with one of the coordinates compactified to a circle and has originally been suggested to describe a confining vacuum in the dual gauge theory [34,35] since it possesses a mass gap. For solutions with Ricci-flat horizons there is a phase transition between the AdS black hole and the AdS soliton [36] which was interpreted as a confining/deconfining phase transition in the dual gauge theory. Note that this is different for black holes in global AdS where the black hole decays to global AdS space-time when lowering the temperature [37].

In the limit of vanishing gauge coupling, the soliton solutions correspond to planar boson stars in AdS space-time. Since the scalar field is uncharged the interpretation in terms of insulators/superconductors is difficult in this case. However, since the AdS/CFT correspondence connects strongly coupled CFTs to weakly coupled gravity theories the prototype example of a strongly coupled field theory comes to mind—quantum chromodynamics (QCD). As such the planar boson stars in AdS have been interpreted as Bose-Einstein condensates of glueballs. Glueballs are color-neutral bound states of gluons predicted by QCD and the scalar glueball (which is also the lightest possible glueball) is predicted to have a mass of 1–2 GeV (see e.g., Ref. [38] for an overview on experimental results). Since these glueballs appear due to nonlinear interactions and as such cannot be described by a perturbative approach, it is very difficult to make predictions within the framework of quantum field theory. However, holographic methods have been applied to make predictions about glueball spectra (see e.g., Ref. [39] and references therein).

Nonspinning boson stars in $(d + 1)$ -dimensional AdS space-time have been studied before using a massive scalar field without self-interaction [40] and in $(3 + 1)$

dimensions with an exponential self-interaction potential [41]. Spinning solutions in $(2 + 1)$ and $(3 + 1)$ dimensions have been constructed in Ref. [24,42], respectively.

In this paper, we are interested in Q -balls and boson stars in both asymptotically flat as well as asymptotically AdS space-time with $(d + 1)$ dimensions. We use an exponential scalar field potential already employed in Ref. [41]. We will consider first the asymptotically flat case generalizing some of the results obtained in Ref. [22] to higher dimensions and then also consider asymptotically AdS solutions.

Our paper is organised as follows: In Sec. II we give the model, equations of motion and boundary conditions. In Sec. III, we present our numerical results and conclude in Sec. IV. Appendices A and B contain results on the generalization of an exact solution first found in Ref. [42] to $(d + 1)$ dimensions and on the existence of asymptotically flat boson stars in $(2 + 1)$ dimensions, respectively.

II. THE MODEL

In the following we will study nonspinning Q -balls and boson stars in a $(d + 1)$ -dimensional anti-de Sitter (AdS) space time. The action S reads

$$S = \int \sqrt{-g} d^{d+1}x \left(\frac{R - 2\Lambda}{16\pi G_{d+1}} + \mathcal{L}_m \right) + \frac{1}{8\pi G_{d+1}} \int d^d x \sqrt{-h} K, \quad (1)$$

where R is the Ricci scalar, G_{d+1} denotes the $(d + 1)$ -dimensional Newton's constant, and Λ is the negative cosmological constant related to the AdS radius ℓ by $\Lambda = -d(d - 1)/(2\ell^2)$. The second term on the right-hand side of (1) is the Gibbons-Hawking surface term [43] with h the induced metric and K the trace of the extrinsic curvature on the AdS boundary. \mathcal{L}_m is the matter Lagrangian given by

$$\mathcal{L}_m = -\partial_M \Phi \partial^M \Phi^* - U(|\Phi|), \quad M = 0, 1, \dots, d, \quad (2)$$

where Φ denotes a complex scalar field and we choose the metric to have mainly positive signature. $U(|\Phi|)$ is the potential

$$U(|\Phi|) = m^2 \eta_{\text{susy}}^2 \left(1 - \exp\left(-\frac{|\Phi|^2}{\eta_{\text{susy}}^2}\right) \right). \quad (3)$$

This potential is motivated by supersymmetric extensions of the standard model [17,18]. Here η_{susy} is a parameter such that $\eta_{\text{susy}}^{2/(d-1)}$ corresponds to the energy scale below which supersymmetry is broken, while m denotes the scalar boson mass.

The coupled system of ordinary differential equations is then given by the Einstein equations

$$G_{MN} + \Lambda g_{MN} = 8\pi G_{d+1} T_{MN}, \quad M, N = 0, 1, \dots, d \quad (4)$$

with the energy-momentum tensor

$$\begin{aligned}
 T_{MN} &= g_{MN} \mathcal{L} - 2 \frac{\partial \mathcal{L}}{\partial g^{MN}} \\
 &= -g_{MN} \left[\frac{1}{2} g^{KL} (\partial_K \Phi^* \partial_L \Phi + \partial_L \Phi^* \partial_K \Phi) + U(\Phi) \right] \\
 &\quad + \partial_M \Phi^* \partial_N \Phi + \partial_N \Phi^* \partial_M \Phi
 \end{aligned} \tag{5}$$

and the Klein-Gordon equation

$$\left(\square - \frac{\partial U}{\partial |\Phi|^2} \right) \Phi = 0. \tag{6}$$

The matter Lagrangian \mathcal{L}_m (2) is invariant under the global $U(1)$ transformation

$$\Phi \rightarrow \Phi e^{i\chi}. \tag{7}$$

As such the locally conserved Noether current j^M , $M = 0, 1, \dots, d$ associated to this symmetry is given by

$$j^M = -\frac{i}{2} (\Phi^* \partial^M \Phi - \Phi \partial^M \Phi^*) \quad \text{with} \quad j^M_M = 0. \tag{8}$$

The globally conserved Noether charge Q of the system then reads

$$Q = - \int d^d x \sqrt{-g} j^0. \tag{9}$$

A. Ansatz and equations

For the metric we use the following ansatz in spherical Schwarzschild-like coordinates:

$$ds^2 = -A^2(r)N(r)dt^2 + \frac{1}{N(r)}dr^2 + r^2 d\Omega_{d-1}^2, \tag{10}$$

where

$$N(r) = 1 - \frac{2n(r)}{r^{d-2}} - \frac{2\Lambda}{(d-1)d} r^2 \tag{11}$$

and $d\Omega_{d-1}^2$ is the line element of a $(d-1)$ -dimensional unit sphere. Note that the gravitational constant is chosen such that the $d=2$ case has no Newtonian limit [44]. As such the metric function $n(r)$ is well behaved also in the $d=2$ case. Note that for weak and static gravitational fields $g_{tt} \sim -(1+2\psi(r))$, where $\psi(r)$ is the Newtonian potential. Then the behavior of $\psi(r)$ in $d=2$ which is $\psi(r) \sim -\ln(r)$ would imply the divergence of $n(r)$ if that limit would exist within our ansatz.

For the complex scalar field, we use a stationary ansatz that contains a periodic dependence of the time-coordinate t :

$$\Phi(t, r) = e^{i\omega t} \phi(r), \tag{12}$$

where ω is a constant and denotes the frequency.

In order to be able to use dimensionless quantities we introduce the following rescalings:

$$\begin{aligned}
 r &\rightarrow \frac{r}{m}, & \omega &\rightarrow m\omega, & \ell &\rightarrow \ell/m, \\
 \phi &\rightarrow \eta_{\text{susy}} \phi, & n &\rightarrow n/m^{d-2}
 \end{aligned} \tag{13}$$

and find that the equations depend only on the dimensionless coupling constants

$$\kappa = 8\pi G_{d+1} \eta_{\text{susy}}^2 = 8\pi \frac{\eta_{\text{susy}}^2}{M_{\text{pl},d+1}^{d-1}}, \tag{14}$$

where $M_{\text{pl},d+1}$ is the $(d+1)$ -dimensional Planck mass. Note that with these rescalings the scalar boson mass $m_B \equiv m$ becomes equal to unity. In these rescaled variables and coupling constants the coupled system of nonlinear ordinary differential equations reads

$$n' = \kappa \frac{r^{d-1}}{2} \left(N \phi'^2 + U(\phi) + \frac{\omega^2 \phi^2}{A^2 N} \right), \tag{15}$$

$$A' = \kappa r \left(A \phi'^2 + \frac{\omega^2 \phi^2}{AN^2} \right), \tag{16}$$

$$(r^{d-1} AN \phi')' = r^{d-1} A \left(\frac{1}{2} \frac{\partial U}{\partial \phi} - \frac{\omega^2 \phi}{NA^2} \right). \tag{17}$$

These equations have to be solved numerically subject to appropriate boundary conditions. We want to construct globally regular solutions with finite energy. At the origin we hence require

$$\phi'(0) = 0, \quad n(0) = 0, \tag{18}$$

while we choose $A(\infty) = 1$ (any other choice would just result in a rescaling of the time coordinate). Moreover, while the scalar field function falls off exponentially for $\Lambda = 0$ with

$$\phi(r \gg 1) \sim \frac{1}{r^{\frac{d-1}{2}}} \exp(-\sqrt{1-\omega^2}r) + \dots, \tag{19}$$

it falls off power law for $\Lambda < 0$ with

$$\phi(r \gg 1) = \frac{\phi_\Delta}{r^\Delta}, \quad \Delta = \frac{d}{2} + \sqrt{\frac{d^2}{4} + \ell^2}. \tag{20}$$

When solving the equations numerically, we will choose as fourth boundary condition $\phi(\infty) = 0$ for $\Lambda = 0$, while for $\Lambda < 0$ we will choose the falls off given in (20). ϕ_Δ is then a constant that has to be determined numerically and which can be interpreted via the AdS/CFT correspondence as the value of the condensate of glueballs in the dual theory living on the d -dimensional boundary of global AdS.

The explicit expression for the Noether charge reads

$$Q = \frac{2\pi^{d/2}}{\Gamma(d/2)} \int_0^\infty dr r^{d-1} \frac{\omega \phi^2}{AN}. \tag{21}$$

For $\kappa \neq 0$, we can determine the mass M from the behavior of the metric function $n(r)$ at infinity. This reads [40]

$$n(r \gg 1) = M + n_1 r^{2\Delta+d} + \dots, \quad (22)$$

where n_1 is a constant that depends on ℓ .

For $\kappa = 0$ we have $A \equiv 1$ and $n \equiv 0$. Then the mass M corresponds to the integral of the energy density T_0^0 and reads

$$M = \frac{2\pi^{d/2}}{\Gamma(d/2)} \int_0^\infty dr r^{d-1} \left(N \phi'^2 + \frac{\omega^2 \phi^2}{N} + U(\phi) \right). \quad (23)$$

Note that this expression is perfectly finite in AdS. While N contains a term $\propto r^2$ the falloff of the scalar function ϕ guarantees that M has a finite value. Using the expression for the charge Q (21), we can give a relation between the charge and the mass:

$$M = \omega Q + \frac{2\pi^{d/2}}{\Gamma(d/2)} \int_0^\infty dr r^{d-1} (N \phi'^2 + U(\phi)). \quad (24)$$

For $\Lambda = 0$ it was found [14,18] that in the ‘‘thin-wall’’ approximation which corresponds to $\omega \sim 0$ the mass and charge are related as follows:

$$\frac{M}{\omega Q} \sim \frac{d+1}{d}. \quad (25)$$

This has been used to draw conclusions on the stability of the Q -balls. On the other hand, when ω becomes comparable to the scalar boson mass $m_B \equiv m$ which corresponds to ω being close to its maximal possible value the Q -balls become very spread out, they possess hence a small charge Q and we can use the so-called ‘‘thick-wall’’ approximation [14,18,19]. This has been done for Q -balls in Minkowski space-time with a self-interaction potential of the form $U(\phi) \sim m^2 \phi^2 - a \phi^p$ for an arbitrary power p , arbitrary constant a and in arbitrary spatial dimensions d . Since in the thick wall approximation ϕ is very small we can approximate our exponential potential by this potential for $p = 4$. The main result of Ref. [19] is that the following condition has to be fulfilled in order to have stable Q -balls in the thick-wall limit: $4 - 2d > 0$. We see immediately that for all $d \geq 2$ this is not fulfilled and we would expect that the thick-wall Q -balls are unstable. This was also found in Ref. [18] in $d = 3$ and is confirmed by our numerical findings for general $d \geq 2$ (see Sec. III below). Since a negative cosmological constant acts as an attractive force, we would expect that Q -balls can become stable for sufficiently negative Λ in the thick-wall limit. Analytical results are very difficult to find here and are beyond the scope of this paper. Our numerical results show that Q -balls in AdS can become stable for sufficiently large Q , i.e., in the ‘‘thick-wall limit’’ (see below).

III. NUMERICAL RESULTS

The solutions to the coupled system of nonlinear differential equations are only known numerically. We

have solved these equations using the ordinary differential equation solver COLSYS [45]. The solutions have relative errors on the order of 10^{-6} – 10^{-10} .

A. Q -balls

We have first studied the case $\kappa = 0$. This corresponds to Q -balls in a Minkowski ($\Lambda = 0$) or AdS background ($\Lambda < 0$), respectively. In this case, the Einstein equations decouple from the system and $n(r) \equiv 0$, while $A(r) \equiv 1$.

It is known that Q -balls in $d = 3$ exist on a limited interval of the frequency $\omega \in [\omega_{\min}; \omega_{\max}]$. For $\Lambda = 0$ the mass and charge diverge at both boundaries [12,13], while for $\Lambda \neq 0$ this is still true at ω_{\min} , but now the two quantities tend to zero at ω_{\max} [41]. This is related to the fact that in the limit $\omega \rightarrow \omega_{\max}$ the scalar field function tends to zero everywhere. In Minkowski space-time this still leads to an infinite value of the integral since the space-time is infinite, however AdS space-time acts as a confining box and as such the integral becomes zero.

I. $\Lambda = 0$

Q -balls with an exponential interaction potential of the form (3) have been studied in Refs. [18,41] in $(3+1)$ dimensions. Here, we extend these results to $d \neq 3$ and show that the behavior of the mass M and the charge depends crucially on the space dimension d .

Our results for the mass M and charge Q in dependence on ω are given in Fig. 1 for $d = 2, 3, 4, 5, 6$. We observe that the mass and charge diverge at $\omega = \omega_{\max}$ for $d \geq 3$, while for $d = 2$ they tend to finite values. Moreover, since the potential and effective potential $U_{\text{eff}} := \omega^2 \phi^2 - U(\phi)$ do not depend on d the arguments employed in Ref. [11] can also be used here, such that $\omega_{\max} \equiv 1$ and $\omega_{\min} \equiv 0$ do not depend on d . This is clearly seen in Fig. 1.

In order to get an idea about the stability of these objects we can compare the mass M with that of Q free scalar bosons of mass $m_B \equiv m$. Because of our rescalings $m \equiv 1$ and the mass of Q free scalar bosons is just equal to Q . Any solution with $M < Q$ would hence be stable to decay into Q free bosons. Our results for $d = 2, 3, 4, 5, 6$ are shown in Fig. 2. In $d = 3$ it was found [12,13,41] that there exist two solutions with different M for a given charge Q , one of which is stable to the decay into Q free bosons, while the other is unstable. Here we observe that this is also true for $d > 3$: a stable branch exists up to a maximal value of the charge and then extends backwards to form a second branch that for some critical value of the charge becomes unstable. We observe that the bigger d the bigger is the value of the charge at which the solutions become unstable. On the other hand, for $d = 2$ we find that the solutions are always stable with respect to the decay into Q free bosons. The fact that Q -balls are stable for small values of ω was pointed out already in Ref. [18]. In this so-called ‘‘thin-wall limit’’ the Q -balls fulfill the relation $M \simeq ((d+1)/d)\omega Q$. Since ω is small $M < Q$ in this limit

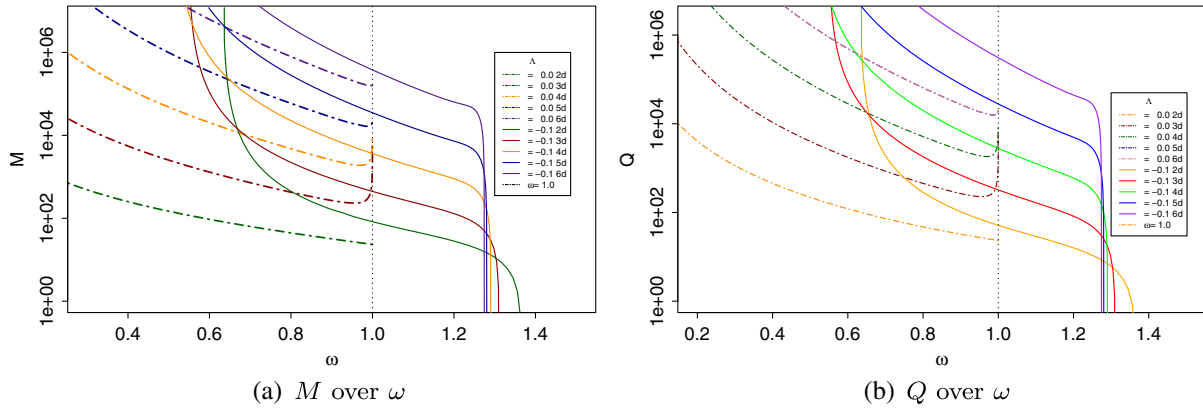


FIG. 1 (color online). The value of the mass M (left) and the charge Q (right) of the Q -balls in dependence on the frequency ω in Minkowski space-time ($\Lambda = 0$) and AdS space-time ($\Lambda = -0.1$) for different values of d .

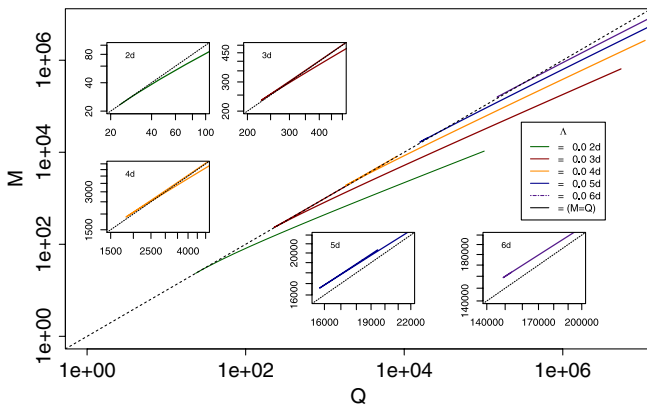


FIG. 2 (color online). We show the value of the mass M of the Q -balls in dependence on their charge Q for different values of d in Minkowski space-time. The small subplots show the behavior close to the minimal value of Q .

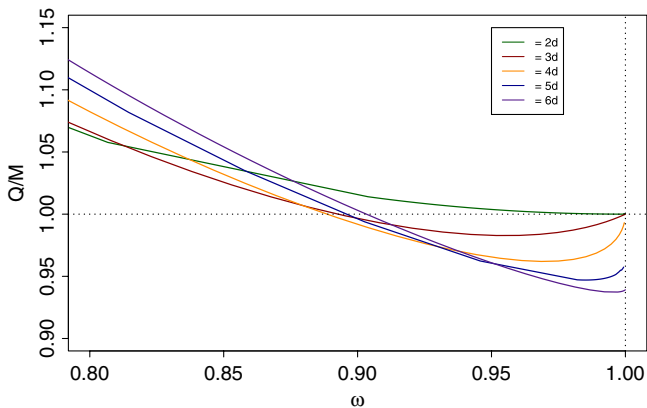


FIG. 3 (color online). We show the value of Q/M in dependence on ω for Q -balls close to the thick-wall limit. Q -balls are stable to decay into Q scalar bosons of mass m for $Q/M > 1$. Clearly, the Q -balls are unstable in the thick-wall limit for $d \geq 3$, while $M \rightarrow Q$ for $d = 2$ in this limit.

and the Q -balls are stable with respect to a decay to Q free bosons. On the other hand, for the thick-wall limit it is more difficult to make analytical statements, but the results in Ref. [18] again agree with our numerical findings. This is shown in Fig. 3, where we give M/Q as a function of ω . For ω close to unity the Q -balls are in the thick-wall limit. Clearly, the Q -balls are unstable in this case since $Q < M$.

2. $\Lambda \neq 0$

This case corresponds to Q -balls in a fixed AdS background and has been studied for the exponential potential and $d = 3$ in Ref. [41]. Our results are shown in Fig. 2 for $\Lambda = -0.1$ and $d = 2, 3, 4, 5, 6$. Similar to $d = 3$ the value of the mass M and charge Q tend to infinity at $\omega_{\min} = 0$ independent of d . On the other hand, the mass and charge tend to zero at ω_{\max} . Moreover, we find that ω_{\max} decreases with increasing d having the largest value for $d = 2$. For $\Lambda \neq 0$ and $\kappa = 0$ it is known that in the $d = 3$ case and a particular choice of potential exact solutions to the scalar field equation exist [42]. We show in Appendix A that this generalizes to d dimensions. While the potential necessary to obtain this result is not of the form chosen in this paper, however, our numerical results for ω_{\max} agree quite well with the analytic expression given by $\omega_{\max} = \Delta/\ell$. This is related to the fact that for $\omega \rightarrow \omega_{\max}$ the function $\phi(r)$ tends to zero everywhere. Hence, all higher order terms in the potential become negligible and the ansatz made in Ref. [42] gives a good result.

Our numerical results for ω_{\max} in dependence on Λ and d are shown in Fig. 4 together with the value of Δ/ℓ . As is apparent from this figure the analytical result agrees quite well with our numerical values. Moreover, we observe as expected from the analytical result that ω_{\max} increases with decreasing Λ and decreases with increasing d .

In Fig. 5 we show the value of the mass M in dependence on Q for $d = 2, 3, 4, 5, 6$ and $\Lambda = -0.1$, while in Fig. 6 we present Q/M in dependence on Q for the thick-wall limit.

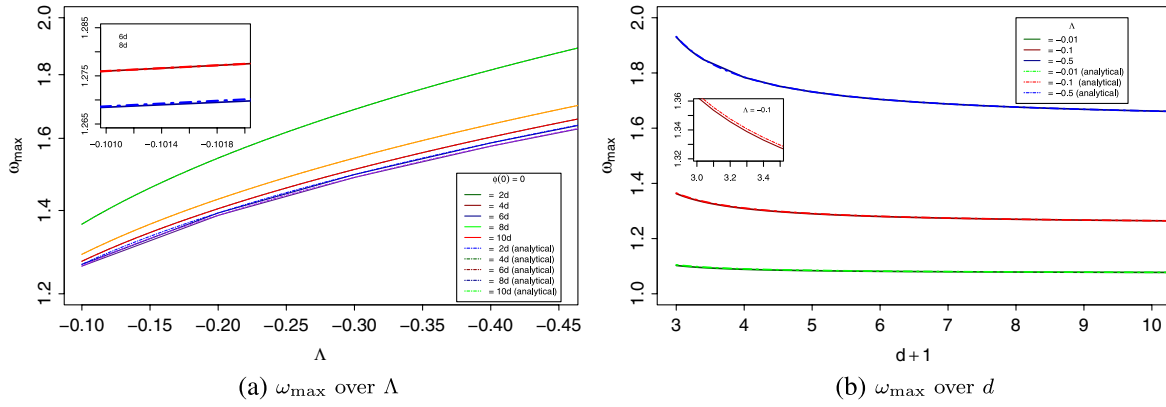


FIG. 4 (color online). The value of ω_{\max} in dependence on Λ (left) and in dependence on d (right). Though we plot d here as a continuous parameter, we should only read of the value for $d \in \mathbb{N}$. We also give the value of Δ/ℓ and find that it gives a good approximation to our numerical data.

Very similar to $d = 3$ the Q -balls have mass M larger than Q and are hence unstable. This is clearly seen in Fig. 6, while for sufficiently large Q they become stable with respect to this decay. The value of $Q = Q_{\text{crit}}$ at which this transition happens depends on Λ and it was found that Q_{crit} increases with decreasing Λ [41]. We find that the number of spatial dimensions d also has an influence on the value of Q_{crit} . We find that Q_{crit} increases with increasing d .

It is also known that radially excited Q -ball solutions exist which possess a number $k \in \mathbb{N}$ of zeros in the scalar field function. In Fig. 7(a) we show the mass M as a function of ω and as a function of Q , respectively, for $\Lambda = -0.1$ and $d = 3, 4$ and $k = 0, 1, 2$. We observe that for fixed d the value of ω_{\max} increases with the increase of k . Moreover, the bigger k the bigger is the difference between ω_{\max} for $d = 3$ and $d = 4$. The dependence of the mass M on Q shown in Fig. 7(b) indicates that all solutions with

nodes are unstable to decay into Q free bosons. This is not surprising since these can be seen as excited Q -balls in AdS space-time.

As pointed out in Ref. [32], the field theory on the boundary of AdS describes condensates of scalar glueballs. This was further investigated in Ref. [41], where Q -balls in $(3 + 1)$ -dimensional asymptotically global AdS have been studied. In Fig. 8 we show our results for different values of Λ and d . Apparently, the expectation value of the dual operator $\langle O \rangle^{1/\Delta}$, which corresponds to the value of the condensate of scalar glueballs decreases for increasing d when fixing $\phi(0)$. This is related to the fact that the scalar field can spread into more dimensions when d is increased and hence less condensate is collected. Furthermore, the value of the condensate increases with decreasing Λ . This is connected to the fact that the “AdS box” decreases in size for decreasing Λ and as such the value of the condensate becomes bigger.

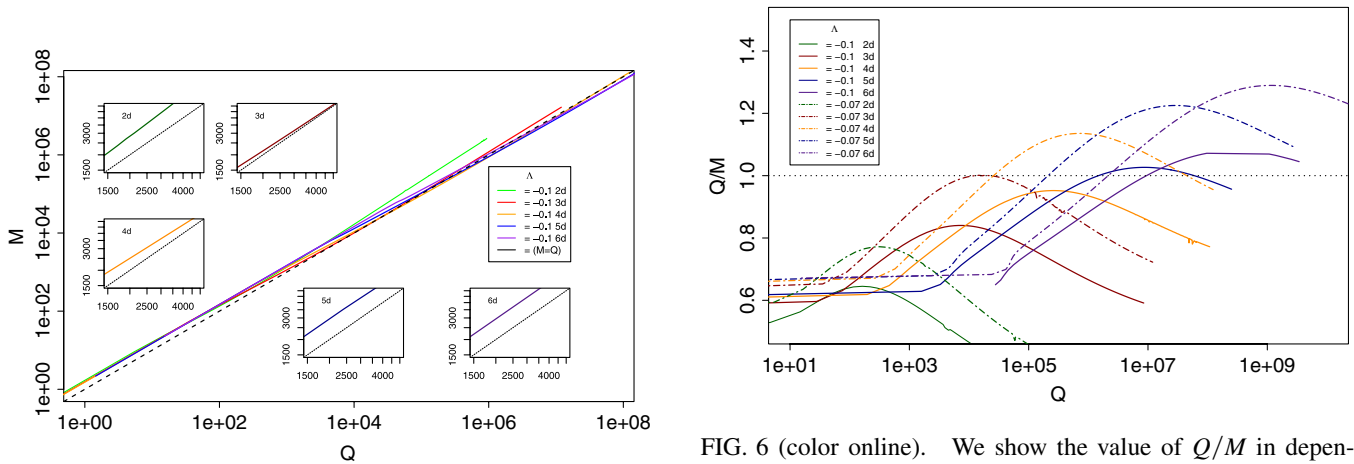


FIG. 5 (color online). We show the value of the mass M of the Q -balls in dependence on their charge Q for different values of d in AdS space-time with $\Lambda = -0.1$.

FIG. 6 (color online). We show the value of Q/M in dependence on Q for Q -balls close to the thick-wall limit. Q -balls are stable to decay into Q scalar bosons of mass m for $Q/M > 1$. Clearly, the Q -balls can become stable in the thick-wall limit for sufficiently large d and/or $|\Lambda|$.

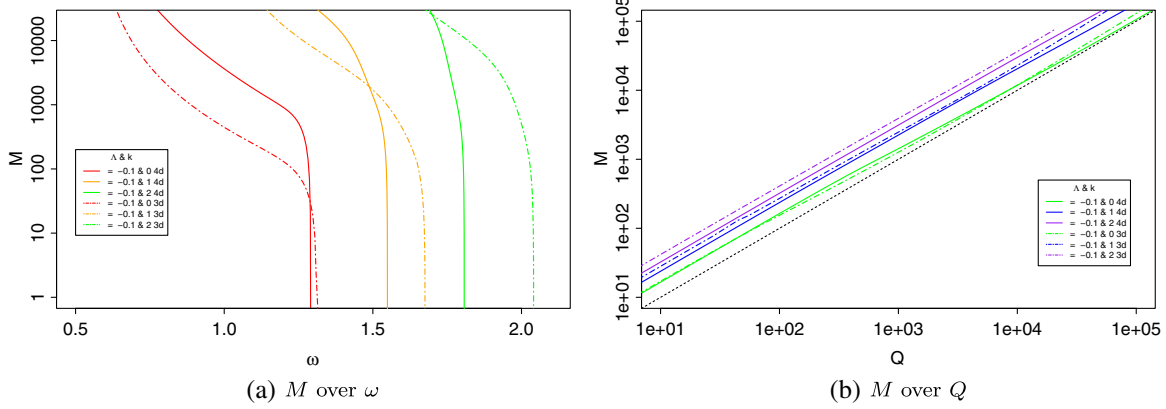


FIG. 7 (color online). The value of the mass M of the Q -balls in dependence on ω (left) and in dependence on the charge Q (right) in AdS space-time for different values of d and number of nodes k of the scalar field function.

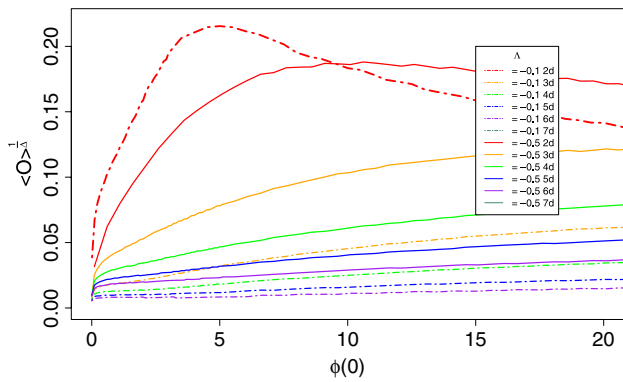


FIG. 8 (color online). We show the expectation value of the dual operator on the AdS boundary $\langle O \rangle^{1/\Delta}$ corresponding to the value of the condensate of scalar glueballs in dependence on $\phi(0)$ for different values of Λ and d .

B. Boson stars

We now discuss the case $\kappa \neq 0$. This corresponds to boson stars in an asymptotically flat ($\Lambda = 0$) or asymptotically AdS space-time ($\Lambda < 0$), respectively.

I. $\Lambda = 0$

As pointed out in Ref. [40], boson stars in $(2 + 1)$ -dimensional, asymptotically flat space-time do not exist for massive scalar fields without self-interaction. In Appendix B we show that this is different in our case and that gravitating, asymptotically flat boson star solutions in $(2 + 1)$ do exist in our model with an exponential self-interaction potential.

We have also studied the dependence of the mass and charge on the frequency ω . Our results are shown in Fig. 9 for the mass. The curves look qualitatively similar for the charge Q ; this is why we do not show them here. We observe that the behavior at ω_{\max} depends crucially on the number of spatial dimensions d . For $d = 3$ the mass and charge tend to zero, while for $d = 4$ they tend to a

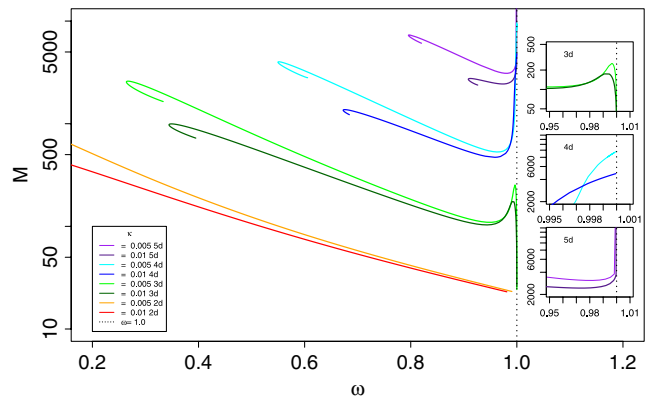


FIG. 9 (color online). The value of the mass M of the boson stars in dependence on the frequency ω for $\Lambda = 0$ and different values of d and κ . The small subfigures show the behavior of M at the approach of ω_{\max} for $d = 3, 4, 5$ (from top to bottom). Note that the curves for the charge Q look qualitatively very similar; this is why we do not give them here.

finite value. This has already been observed before and is confirmed with our type of potential. For $d = 5$ we find that now the mass and charge tend to infinity at the approach of ω_{\max} . We have integrated up to values of the mass and charge of 10^7 and believe that approaching ω_{\max} even closer, these values would further increase. This can be understood using the argument employed for $d = 4$ in Ref. [22]. As noticed in this latter paper, the scalar field function and radial coordinate show a scaling behavior that is equal in $d = 3$ and $d = 4$. We find that this is also the case here and generalizes to $d > 4$ such that the behavior is $\phi(r) \rightarrow \hat{\phi}(\hat{r})$, $n \rightarrow 0$, $A \rightarrow 1$ for $\omega \rightarrow \omega_{\max}$ with

$$\hat{\phi} = \phi/\phi_0, \quad \hat{r} = (\phi_0 \kappa^{1/2})^{1/2} r \quad (26)$$

where ϕ_0 a constant that tends to zero in the limit $\omega \rightarrow \omega_{\max}$. Now this implies e.g., for the charge Q (the argument works similarly for the mass M):

$$\begin{aligned}
 Q &= \frac{2\pi^{d/2}}{\Gamma(d/2)} \omega \int_0^\infty dr r^{d-1} \phi^2 \rightarrow Q \\
 &= \frac{2\pi^{d/2}}{\Gamma(d/2)} \omega_{\max} \phi_0^{2-d/2} \kappa^{-d/4} \int_0^\infty d\hat{r} \hat{r}^{d-1} \hat{\phi}^2. \quad (27)
 \end{aligned}$$

For $d \leq 3$ this tends obviously to zero, for $d = 4$ this becomes constant, and for $d \geq 5$ this tends to infinity, respectively, for $\phi_0 \rightarrow 0$. In addition to this we observe that the approach to ω_{\max} is not smooth in $d = 5$. This is shown in Fig. 10. For $d = 3$ and $d = 4$ the mass tends smoothly to zero and a finite value, respectively. For $d = 5$ we observe that the mass tends to a finite value on a lower branch of solutions, but that close to ω_{\max} new branches of solutions exist. These are quite small in extent and in fact are barely noticeable for $d > 5$. As such, a second branch of solutions extends backwards from ω_{\max} down to a critical value of ω and then bends backwards to tend to infinity. Our conclusion hence is that while these solutions can exist for arbitrarily large values of the mass (and charge) there exists a mass gap in which solutions are not allowed. Furthermore, there is a small interval of ω in which up to three solutions with different masses exist. To understand this pattern, we plot the three solutions for ω close to ω_{\max} in Fig. 11. We observe that for the same value of ω the three solutions are distinguished by the value of $\phi(0)$ with $\phi(0)$ decreasing from the first to the third branch. Moreover, the solution spreads out more and more over r . On the first branch, the solution is still quite localized around the origin, while it becomes very delocalized on the third branch.

We further observe that for all d the solutions exist down to a minimal value of the frequency ω_{\min} . At this value of ω , a second branch of solutions exists that has lower mass

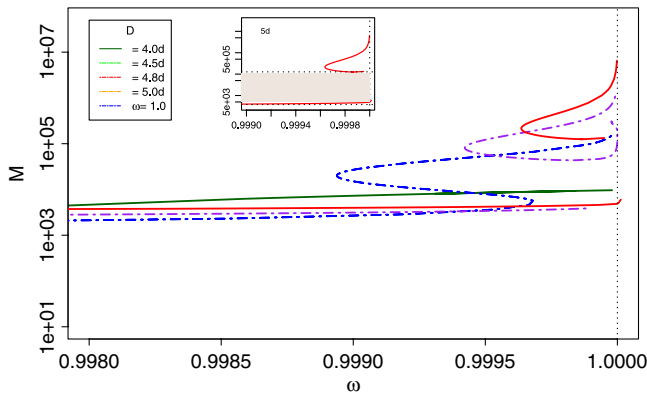


FIG. 10 (color online). We show the value of the mass M of the boson stars in asymptotically flat space-time ($\Lambda = 0$) in dependence on the frequency ω close to ω_{\max} . Note that only $d = 4$ and $d = 5$ are physical values, but that the dimension is a parameter in our numerical program that can also have non-integer values. Here we demonstrate how the mass evolves when going from $d = 4$ to $d = 5$. The subplot further demonstrates that there exists a mass gap in $d = 5$.

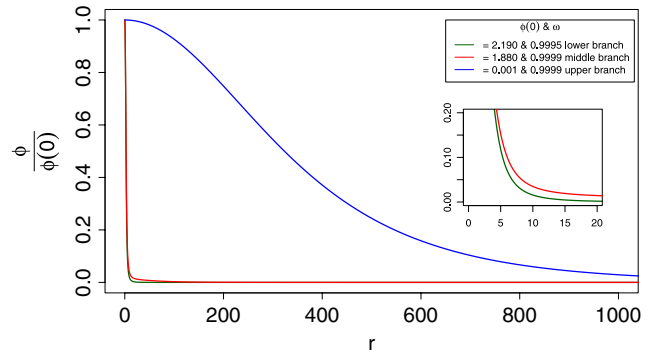


FIG. 11 (color online). We show the profiles of the scalar field function $\phi(r)/\phi(0)$ for the case where three branches of solutions exist close to ω_{\max} in $d = 5$. Here $\kappa = 0.001$.

and charge for a fixed value of ω . This second branch extends back to a critical value of ω and then forms another branch. This spiraling has been observed before in $d = 3$ and is apparently also present in $d \neq 3$.

We can also read off the dependence of ω_{\min} and ω_{\max} on d and κ from Fig. 9. As is apparent, $\omega_{\max} = 1$ does neither depend on κ nor on d . On the other hand, the value of the minimal frequency ω_{\min} depends strongly on κ and d . For fixed d it increases with increasing κ , i.e., the stronger the interaction between the gravitational field and the scalar field the larger we have to choose the value of the frequency to find solutions. This is true for all d that we have studied. We also notice that the decrease in ω is bigger in higher d when increasing κ by the same amount. For fixed κ our results indicate that ω_{\min} increases with increasing d . Normally, we would expect gravity in higher d to become weaker since it can leak into extra dimensions; however, here we are keeping G_{d+1} , i.e., the strength of the gravitational interaction constant.

We show the dependence of M on Q for different values of κ and d in Fig. 12. As expected boson stars in asymptotically flat space-time are stable to decay into Q free bosons

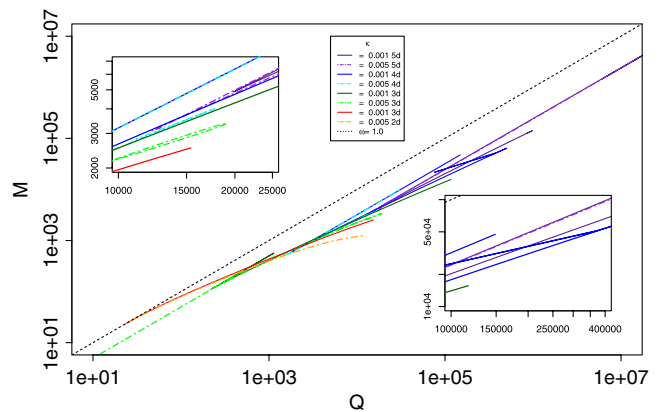


FIG. 12 (color online). We show the value of the mass M of the boson stars in asymptotically flat space-time ($\Lambda = 0$) in dependence on their charge Q for different values of κ and d .

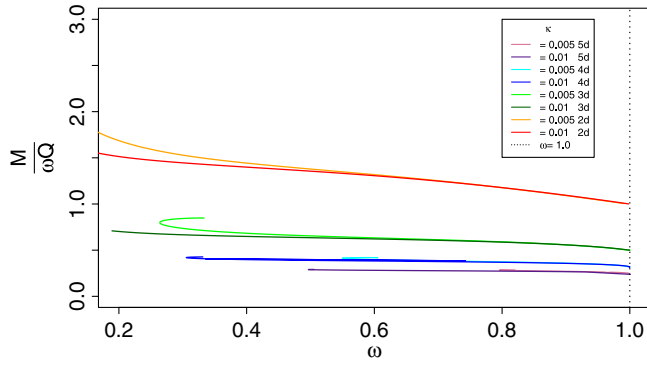


FIG. 13 (color online). We show the value of the mass $M/(\omega Q)$ of the boson stars in asymptotically flat space-time ($\Lambda = 0$) in dependence on the frequency ω for different values of κ and d .

since these objects are gravitationally bound. This is also true for the case $d \geq 5$, where no bound on the mass M and charge Q exists. As is clearly seen from Fig. 12 we find that also for $d = 5$ the curve is always below $M = Q$.

For $\kappa = 0$ it was found that for small values of ω there is a relation between M and Q that depends on ω and d [18]. In Fig. 13 we plot $M/(\omega Q)$ as a function of ω and find that only for $d = 2$ the approximation of flat space-time $M/(\omega Q) \sim (d + 1)/d$ is a good approximation for small ω . For larger values of d the solutions do not exist for small ω and $M/(\omega Q)$ is always smaller than one.

2. $\Lambda \neq 0$

We have also studied boson stars in AdS space-time. Our results for the mass M and charge Q in dependence on the frequency ω are shown in Fig. 14 for $\Lambda = -0.1$.

As for $\kappa = 0$, we find that the mass tends to zero at $\omega \rightarrow \omega_{\max}$, while at $\omega \rightarrow \omega_{\min}$ we find the spiraling behavior typical for boson star solutions. This is true for all $d \geq 3$. Hence, while boson stars in asymptotically flat space-time with $d \geq 5$ dimensions can have arbitrarily large masses and charges, their mass and charge are bounded from above in asymptotically AdS space-time.

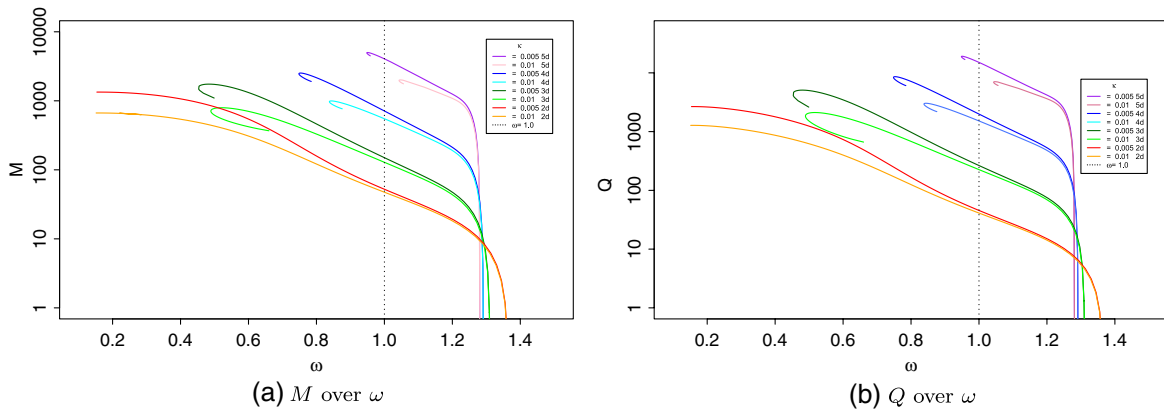


FIG. 14 (color online). The value of the mass M (left) and the charge Q (right) of the boson stars in dependence on the frequency ω in asymptotically AdS space-time ($\Lambda = -0.1$) for different values of d and κ .

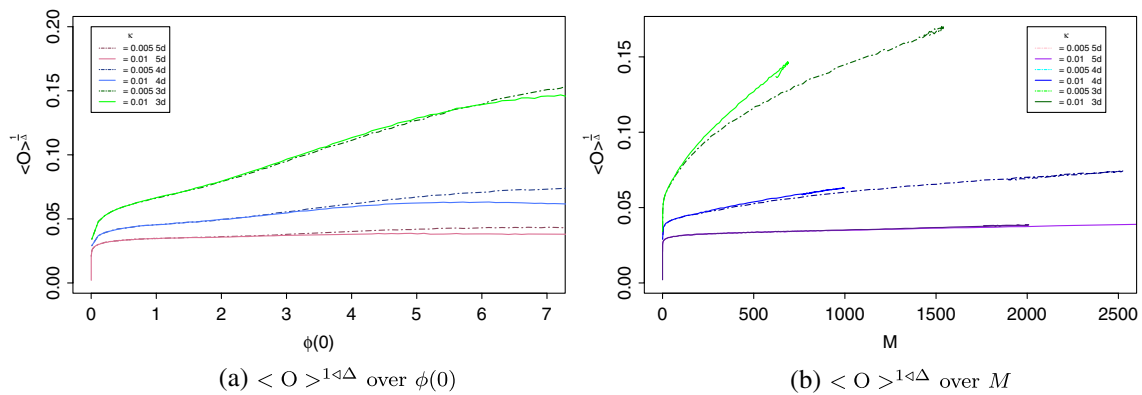


FIG. 15 (color online). We show the expectation value of the dual operator on the AdS boundary $\langle O \rangle^{1/\Delta}$ corresponding to the value of the condensate of scalar glueballs in dependence on $\phi(0)$ (left) and in dependence on M (right) for different values of κ and d with $\Lambda = -0.1$.

The value of ω_{\max} depends on Λ and d , but not on κ . It increases with decreasing Λ and decreasing d . Again, this can be explained with the fact that the maximal value of ω can be very well approximated by $\omega_{\max} = \Delta/\ell$.

In terms of the holographic interpretation our solutions describe scalar glueball condensates including backreaction. Our results for the expectation value of the dual operator on the AdS boundary are given in Fig. 15.

We observe that the value of the condensate at a fixed value of the mass M increases with increasing κ , i.e., backreaction of the space-time. This is true for all d . Moreover, there exists a maximal possible value of the condensate at the maximal value of the mass corresponding to intermediate values of $\phi(0)$.

IV. CONCLUSIONS AND OUTLOOK

In this paper, we have studied $(d+1)$ -dimensional Q -balls and boson stars in (asymptotically) flat as well as AdS space-time. We have used an exponential self-interaction potential for the scalar field motivated from minimal supersymmetric extensions of the standard model. We find that the behavior of the solutions in asymptotically flat space-time at the approach of the maximal frequency depends strongly on the number of spatial dimensions d such that for $d = 2, 3, 4$ boson stars exist only up to a maximal value of the mass and charge, while for $d \geq 5$ the mass and charge of these objects can become arbitrarily large. However, our numerical results also indicate that for $d = 5$ a mass gap for the solutions exists. On the other hand, the value of the maximal and minimal frequency does not depend on d in this case. We show that the existence of boson stars in $(2+1)$ -dimensional asymptotically flat space-time depends crucially on the choice of the scalar potential. For our choice of potential, $(2+1)$ -dimensional boson stars always exist.

We also generalize the exact Q -ball solution in AdS space-time found in Ref. [42] to d dimensions and find that the conclusion about the maximal value of ω drawn from this solution gives a good approximation to our numerical results. As such, the maximal value of the frequency ω decreases with decreasing AdS radius and with d .

We can also draw conclusions about the stability of these objects with respect to the decay into Q free bosons. While there are stable as well as unstable Q -balls in Minkowski space-time for $d \geq 3$, they are always stable in $d = 2$. In AdS space-time Q -balls are stable for small values of the charge and unstable for large values of the charge. Boson stars in asymptotically flat space-time are always stable, while for asymptotically AdS we find that only for $d = 2$ and above a critical ω the solutions become stable. For all other d the solutions are stable.

With view to the holographic interpretation spinning solutions in asymptotically AdS space-time play an important role. These solutions have been constructed recently in

$d = 3$ [42]. However, the ansatz was chosen such that the boundary theory possesses no rotation and hence describes static glueball condensates. It is surely of interest to generalize this to find a holographic description of scalar glueballs possessing angular momentum.

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APPENDIX A: AN EXACT SOLUTION AND THE VALUE OF ω_{\max}

In analogy to the $d = 3$ case [42] we find that there exists an exact solution when choosing the scalar field potential of the form $\tilde{U}(\phi) = \mu^2 \phi^2 - \lambda \phi^{2k}$ and $\kappa = 0$ such that

$$\phi(r) = \left(\frac{\mu^2}{\lambda} \frac{\Delta^2 - \ell^2 \omega^2}{(\Delta - d)(\Delta + 1)} \right)^{\Delta/2} \left(1 + \frac{r^2}{\ell^2} \right)^{-\Delta/2}, \quad (\text{A1})$$

where $k = 1 + 1/\Delta$. In the limit $\omega \rightarrow \omega_{\max}$ it is known that the function $\phi(r)$ spreads out and becomes zero everywhere. Hence, we can read off the value of ω_{\max} from (A1) which gives

$$\omega_{\max} = \Delta/\ell. \quad (\text{A2})$$

APPENDIX B: EXISTENCE OF GRAVITATING, ASYMPTOTICALLY FLAT BOSON STARS IN 2 + 1 DIMENSIONS

We follow the calculations in Ref. [40] and employ a scaling argument. For this, we use the reduced action which reads

$$S = \int_0^\infty dr \left(\frac{(d-1)}{8\pi G_{d+1}} n' A - r^{d-1} A \left(N \phi'^2 - \frac{\omega^2 \phi^2}{NA^2} + U(\phi) \right) \right), \quad (\text{B1})$$

where the prime denotes the derivative with respect to r . We now rescale the variable $r \rightarrow \mu r$ and require that $(dS/d\mu)|_{\mu=1} = 0$. This gives

$$\int_0^\infty dr r^{d-1} A \left[(d-2) \left(1 - \frac{2\Lambda r^2}{(d-1)(d-2)} \right) \phi'^2 + d \cdot U(\phi) - \left(d - \frac{4(d-1)n}{r^{d-2}} - \frac{2(d-2)\Lambda r^2}{d(d-1)} \right) \frac{\omega^2 \phi^2}{N^2 A^2} \right] = 0. \quad (\text{B2})$$

The equation for ϕ (17) can be rewritten as follows:

$$\frac{1}{2} (r^{d-1} AN(\phi^2)')' = r^{d-1} A \left(N(\phi')^2 + \frac{1}{2} \frac{\partial U}{\partial \phi} \phi - \frac{\omega^2 \phi^2}{NA^2} \right). \quad (\text{B3})$$

Integrating this with the help of the boundary conditions (18) and (19), we find that

$$\int_0^\infty dr r^{d-1} A \left(N(\phi')^2 + \frac{1}{2} \frac{\partial U}{\partial \phi} \phi - \frac{\omega^2 \phi^2}{NA^2} \right) = 0. \quad (\text{B4})$$

Combining with (B2) we get

$$\int_0^\infty dr r^{d-1} A \left[\left(1 - \frac{d \cdot n}{r^{d-2}} \right) \phi'^2 + \frac{d}{2} \left(\frac{1}{2} \frac{\partial U}{\partial \phi} \phi - U(\phi) \right) + \frac{\omega^2 \phi^2 e^{2\delta}}{N^2} \left(\frac{2\Lambda r^2}{d(d-1)} - \frac{(d-2)n}{r^{d-2}} \right) \right] = 0. \quad (\text{B5})$$

For $\Lambda = 0$ and $d = 2$ this reads

$$\int_0^\infty dr r A \left[N \phi'^2 + \left(\frac{1}{2} \frac{\partial U}{\partial \phi} \phi - U(\phi) \right) \right] = 0. \quad (\text{B6})$$

Now $N\phi'^2$ is always positive. In Ref. [40] it was argued that for $U(\phi) = m^2 \phi^2$ no gravitating asymptotically flat

boson stars in $d = 2$ exist. This can be immediately seen when realizing that in this case the potential term vanishes and the integral can never become zero. However, for our case with $U(\phi) = 1 - \exp(-\phi^2)$ the potential term gives

$$\left(\frac{1}{2} \frac{\partial U}{\partial \phi} \phi - U(\phi) \right) = \exp(-\phi^2)(\phi^2 + 1) - 1. \quad (\text{B7})$$

This expression is negative definite (and vanishes for $\phi \equiv 0$) such that we can have gravitating asymptotically flat boson stars in $d = 2$. Note that for the ϕ^6 self-interaction potential the negative definiteness of this expression depends crucially on the choice of potential parameters. As such, our choice of potential also seems ‘‘more natural’’ from this point of view, since we can have solutions in all possible limits and dimensions of our model without ‘‘fine-tuning’’ the potential.

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