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Topological implications of inhomogeneity

Boudewijn F. Roukema

Toruń Centre for Astronomy, Faculty of Physics, Astronomy and Informatics, Nicolaus Copernicus University, ulica Gagarina 11, 87-100 Toruń, Poland and Centre de Recherche Astrophysique de Lyon, Observatoire de Lyon, CNRS UMR 5574: Université Lyon 1, École Normale Supérieure de Lyon, Université de Lyon, 9 avenue Charles André, Saint-Genis-Laval F-69230, France[*]

Vincent Blanlœil

Département de Mathématiques, Université de Strasbourg, 7 rue René Descartes, 67084 Strasbourg cedex, France

Jan J. Ostrowski

Toruń Centre for Astronomy, Faculty of Physics, Astronomy and Informatics, Nicolaus Copernicus University, ulica Gagarina 11, 87-100 Toruń, Poland

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The *approximate* homogeneity of spatial sections of the Universe is well supported observationally, but the inhomogeneity of the spatial sections is even better supported. Here, we consider the implications of inhomogeneity in dust models for the connectedness of spatial sections at early times. We consider a nonglobal Lemaître-Tolman-Bondi (LTB) model designed to match observations, a more general, heuristic model motivated by the former, and two specific, global LTB models. We propose that the generic class of solutions of the Einstein equations projected back in time from the spatial section at the present epoch includes subclasses in which the spatial section evolves (with increasing time) smoothly (i) from being disconnected to being connected, or (ii) from being simply connected to being multiply connected, where the coordinate system is comoving and synchronous. We show that (i) and (ii) each contain at least one exact solution. These subclasses exist because the Einstein equations allow nonsimultaneous big bang times. The two types of topology evolution occur over time slices that include significantly postquantum epochs if the bang time varies by much more than a Planck time. In this sense, it is possible for cosmic topology evolution to be ''mostly'' classical.

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I. INTRODUCTION

The approximate homogeneity of spatial sections (hypersurfaces) of the Universe is well supported observationally. Both the assumption of homogeneity and fact of *inhomogeneity* play an important role in relativistic cosmological models. The Friedmann-Lemaître-Robertson-Walker (FLRW) models [1–5] are solutions of the Einstein equations in which the density is constant in any comoving spatial section. With the concordance model parameters of the metric [6], the FLRW models provide reasonably good fits to observational data [faint galaxy number counts (e.g., Refs. [7,8]), gravitational lensing (e.g., Ref. [9]), supernovae type Ia magnitude-redshift relations (e.g., Refs. [10,11])]. However, there is no serious question of whether the Universe is inhomogeneous: the Earth, galaxies, and galaxy clusters exist. The real question is whether the homogeneous, heuristic approach gives a sufficiently accurate approximation. The forcing of an FLRW model onto late-epoch observations requires a nonzero ''dark energy'' parameter Ω_Λ, suggesting that the latter is most simply interpreted as an artefact of forcing an oversimplified model onto the data (e.g., Refs. [12–14]).

The near homogeneity is also a key element of the ''horizon problem'' for noninflationary FLRW models: how was it possible for causally disconnected (but spatially connected) regions of the spatial section of the Universe to homogenize? In the context of dust models with comoving spatial sections, this question implicitly assumes that universe models with initially inhomogeneous spatial sections are relativistically valid and only have a problem with causal disconnectedness, not with comoving spatial disconnectedness. Is this assumption correct?

Although many families of inhomogeneous, exact, cosmological solutions of the Einstein equations are known (see the extensive compilation in Ref. [15] and a recent review in Ref. [16]), no generic model of exact solutions is known. The Lemaître-Tolman-Bondi (LTB) family of exact solutions [17–19] is one well-known family of exact solutions. These solutions consist of exact solutions to the Einstein equations that are radially inhomogeneous and spherically symmetric with respect to an origin. In analogy with the way that the FLRW model is interpreted to apply to a three-dimensionally averaged spatial section, an LTB solution can be interpreted to apply to a spatial section that has been averaged over every infinitesimally thin spherical shell, i.e., two-dimensionally, prior to solving the Einstein

[*]During visiting lectureship.

equations. However, the Einstein equations do not imply their averaged equivalent: $\mathbf{G}(\mathbf{g}) = 8\pi\mathbf{T}(\rho) \not\Rightarrow \mathbf{G}(\langle\mathbf{g}\rangle) = 8\pi\mathbf{T}(\langle\rho\rangle)$ [20] (e.g., Ref. [21]). That is, although homogeneity is often described in terms of the cosmological "principle," the application of either an FLRW or an LTB solution to the real Universe sis better seen as a *heuristic calculational strategy* rather than a physical principle, with the risk of averaging-related artefacts occurring in both cases. LTB solutions provide an intermediate step between the FLRW solutions that force full homogeneity and a more realistic, unknown family of generic solutions.

Thus, here we primarily consider LTB solutions. We first examine an LTB fit to recent observational data to see what it implies for the connectedness of spatial sections at early times (Sec. II). In Sec. III, a less restricted situation is discussed by supposing that a solution with a Gaussian bang time function (11) exists and considering its topology evolution. In Sec. IV, we formally generalize to a wider class of inhomogeneous solutions that contains two subclasses of distinct types of topology evolution (Definition 1) and present a conjecture and a corollary regarding one of the subclasses. We give case examples in Secs. IV A and IV B, using an LTB solution found previously [22], to show that the two subclasses given in Definition 1 are nonempty (Theorem 1). Interpretations are discussed in Sec. V and the second subclass is considered further in Sec. V B. Conclusions are given in Sec. VI. Unless otherwise stated, we only consider relativistic (non-quantum), comoving, dust solutions with a zero cosmological constant (the first fit of inhomogeneous exact cosmological solutions to supernovae type Ia data used radially inhomogeneous pressure solutions [23]).

II. OBSERVATIONAL ESTIMATE

There are many different fits of LTB models to observations—see Ref. [16] for a list of direct and inverse fits. Here, we consider a recent paper [12] that phenomenologically used the inverse method to find an LTB model. That is, the authors started with functions implied by the FLRW model with concordance model values of the metric parameters [6] and inferred LTB functions. A similar method and result are given in Ref. [24]. By construction, the two fits found in Ref. [12] provide good fits to the observed supernovae type Ia angular-diameter-distance—redshift relation and to an observational estimate of Hubble parameter evolution with redshift, based on differential stellar ages of the oldest passively evolving galaxies at different redshifts [25].

The LTB models are comoving, synchronous, dust models with a metric that is normally written [26]

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \lim_{\hat{r}\to r} \frac{R_{,r}^2(t,\hat{r})}{1+2E(\hat{r})}\,\mathrm{d}r^2 + R^2(t,r)(\mathrm{d}\theta^2 + \cos^2\theta\,\mathrm{d}\phi^2), \tag{1}$$

where $c = 1$, the gravitational constant G is written explicitly, and $E(r)$ is a curvature-related function, e.g., (1), (21), (23) of Ref. [19]; (2.1) of Ref. [27]; (1) of Ref. [12]; (1) of Ref. [28]. A solution to the Einstein equations exists if

$$R_{,t}^2 = 2E + \frac{2GM}{R}, \tag{2}$$

$$\rho = \frac{M_{,r}}{4\pi R^2 R_{,r}}, \tag{3}$$

for which

$$R(t,r) = -\frac{GM(r)}{\chi(r)} f(t,r), \tag{4}$$

$$t - t_{\mathrm{B}}(r) = \frac{GM(r)}{[\chi(r)]^{3/2}}\,\xi(t,r), \tag{5}$$

and

$$\begin{aligned} &\chi(r) = -2E, \qquad f(t,r) = 1-\cos\eta,\\ &\xi(t,r) = \eta - \sin\eta \quad \text{if } E(r) < 0, \qquad \chi(r) = 1,\\ &f(t,r) = \eta^2/2, \qquad \xi(t,r) = \eta^3/6 \quad \text{if } E(r) = 0,\\ &\chi(r) = 2E, \qquad f(t,r) = \cosh\eta - 1,\\ &\xi(t,r) = \sinh\eta - \eta \quad \text{if } E(r) > 0, \end{aligned} \tag{6}$$

where $M(r')$ is a weighted integral [via (3)] of the density over $0 \le r \le r'$, $t_{\mathrm{B}}(r)$ is called the "bang time," and $\xi = \xi(t,r)$ and $\eta = \eta(t,r)$ are auxiliary functions. Equations (5), (3), (6), and (4), imply that [29]

$$\text{as } t - t_{\mathrm{B}}(r) \to 0^+ \quad \text{at fixed } r, \qquad R(t,r) \to 0^+. \tag{7}$$

Thus, as $t - t_{\mathrm{B}}(r) \to 0^+$ at some given r, the surface area of a spherical (S^2) shell at r approaches zero.

In other words, the LTB family allows the age of the universe in a given universe model in a comoving spatial section to be a function $t - t_{\mathrm{B}}(r)$ that varies with the radial coordinate r. Thus, since the authors deliberately aimed to avoid making arbitrary assumptions, Figs. 3 and 12 of Ref. [12] show, unsurprisingly, that the $t_{\mathrm{B}}(r)$ solutions are not constant. In a comoving section at the present epoch t_0, the age of the universe increases from t_0 at the observer to $\sim t_0 + 2$ Gyr on shells at an areal distance [30] of about 3.7 Gpc.

What are the topological properties of this solution? At times $t > 0$, let us assume that (i) the spatial section of the solution is simply connected. The spatial curvature is negative, since $E(r) > 0$ over the region of $r > 0$ studied (Figs. 2 and 11 of Ref. [12]). Let us extend the solution by assuming that (ii) $E(r) > 0 \,\forall\; r > 0$. Thus, spatial sections at $t > 0$ are the 3-manifold H^3, with nonconstant curvature.

Figures 3 and 12 of Ref. [12] show that when -2 Gyr $\lesssim t < 0$, a spatial section of the universe has a hole in the center, where space has not yet emerged from the initial singularity. For example, consider a spatial section at

$t = -1$ Gyr in Fig. 3 of Ref. [12]. In comoving coordinates, the closed three-dimensional ball

$$\mathcal{V} = \{(r, \theta, \phi)\colon r \le r_{\rm inf}(t = -1\ \text{Gyr})\}, \tag{8}$$

where

$$r_{\rm inf}(t) := \inf\{r\colon t - t_{\rm B}(r) > 0\} \tag{9}$$

and $r_{\rm inf}(t = -1\ \text{Gyr}) \approx 1.7$ Gpc consists of the initial singularity $\partial\mathcal{V}$ and a region of coordinate space beyond (earlier than) the singularity. The metric is only Lorentzian for $t > t_B(r)$, i.e., $r > r_{\rm inf}(t)$, so the universe at $t = -1$ Gyr is $H^3\backslash\mathcal{V}$, i.e., a 3-manifold with a hole created by removing $\mathcal{V}$ from H^3 (Fig. 1).

Thus, this universe model evolves from $H^3\backslash\mathcal{V}$ to H^3 at early times. What is the areal radius $R(t, r)$ on the boundary $\partial\mathcal{V}$? This is given by (4), (7), and (2) [and (6) for $r = 0$] of Ref. [12]. As $\eta(t, r) \to 0^+$, we have $\phi(t, r) \to 0^+$ and $\xi(t, r) \to 0^+$, and thus $R(t, r) \to 0^+$, and $t - t_{\rm B}(r) \to 0^+$, since $E(r) > 0$ and $M(r)$ in Figs. 2 and 4–6, are nonzero (for $r > 0$) functions of r only. Thus, the spatial volume of a shell at r shrinks to zero as $t \to t_{\rm B}(r)^+$ for fixed r, or as $r \to r_{\rm inf}(t)^+$ at a fixed t. Within the spatial section $H^3\backslash\mathcal{V}$ at t, the boundary $\partial\mathcal{V}$ appears metrically as a single missing point. In coordinate space imagined intuitively (Fig. 1) with, for example, a Euclidean metric, $\partial\mathcal{V}$ would have an area of $4\pi r^2$, but this is not physical; the metrical area of $\partial\mathcal{V}$ is $4\pi R^2 = 0$.

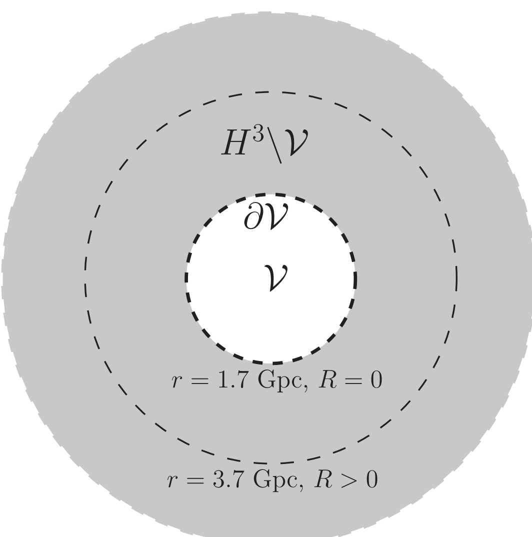

FIG. 1. Coordinate-space illustration of the spatial, comoving section $H^3\backslash\mathcal{V}$ in the empirical solution in Ref. [12] at $t = -1$ Gyr, discussed in Sec. II. The closed three-dimensional ball $\mathcal{V}$ (8) consists of coordinate space that is not part of the physically defined, spatial 3-manifold. The boundary $\partial\mathcal{V}$ has zero metrical area $4\pi R^2$ and corresponds to a spatial section through the initial singularity, i.e., $\partial\mathcal{V} \subset \mathcal{V} \Rightarrow \partial\mathcal{V} \subset H^3\backslash\mathcal{V}$. The $r = 3.7$ Gpc 2-sphere shows the limit of the authors' fit to observations. We extrapolate this to arbitrarily large r. The physically defined spatial section $H^3\backslash\mathcal{V}$ is shaded in grey up to an arbitrary cutoff radius.

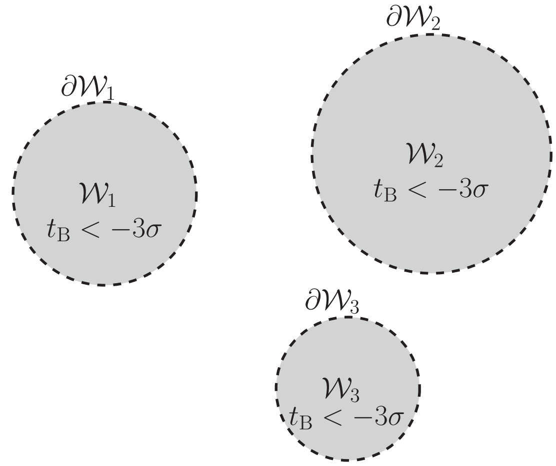

FIG. 2. Coordinate-space illustration of part of the spatial, comoving section at $t = -3\sigma$ of the spacetime solution suggested in Sec. III to be more general than the empirical solution in Ref. [12]. The physical (metrical) spatial section at coordinate time $t = -3\sigma$ is the spatially disconnected space $\cup\{\mathcal{W}_i\}$, shown here for $1 \le i \le 3$. The boundary $\cup\{\partial\mathcal{W}_i\}$ has zero metrical area and corresponds to a spatial section through the initial singularity.

Relativistically, there is no problem with this solution. The high-r universe is born first, with the coordinate-space shell at $r_{\rm inf}(t)$, i.e., the 3-manifold boundary point at $r_{\rm inf}(t)$, representing the unfinished early big bang process. The flexibility of the areal radius $R(t, r)$ in LTB solutions allows comoving space to continuously be born from this singularity, which moves to successively lower values of $r_{\rm inf}(t)$ as t increases up to $t = 0$. At $t = 0$, we have $r_{\rm inf}(0) = 0$ and the singularity is replaced by ordinary spacetime points at ($t > 0$, $r = 0$). We can summarize these properties of the authors' solution in Ref. [12] as follows:

$$\begin{gathered} r_{\rm inf}(t) > 0, \quad \forall\ t < 0, \qquad {\rm d}r_{\rm inf}(t)/{\rm d}t < 0, \quad \forall\ t < 0, \\ \mathcal{V}(t) := \mathcal{V}[r \le r_{\rm inf}(t)], \quad \forall\ t \le 0, \\ \int_{\partial\mathcal{V}(t)} {\rm d}\Omega = \lim_{\hat{r}\to r_{\rm inf}} 4\pi R^2(t, \hat{r}) = 0, \quad \forall\ t \le 0, \end{gathered} \tag{10}$$

where the pre-big-bang universe $\mathcal{V}\backslash\partial\mathcal{V}$ is considered to be nonphysical, the metric is given in (1) of Ref. [12], and ${\rm d}\Omega$ is the metric area element. Comoving space is continuously born from the singularity $\partial\mathcal{V}$ until $t = 0$ when the singularity disappears in the same way that it disappeared in parts of comoving space that were born earlier. Thus, this universe model evolves from $H^3\backslash\{0\} = S^2 \times \mathbb{R}^+$ [31] at $t \le 0$ to H^3 at $t > 0$. This is a topology change, i.e., a change in π_2 homotopy classes for this comoving, synchronous spacetime foliation. Some 2-spheres cannot be continuously shrunk to a point at $t \le 0$, but all 2-spheres can be continuously shrunk to a point at $t > 0$. Dropping the simplifying assumptions (i) and (ii) above does not make it possible to avoid the topology change in this interpretation of the authors' solution in Ref. [12], since it just replaces H^3 by a more generic 3-manifold $\mathcal{M}$.

III. GAUSSIAN $t_B(r, \theta, \phi)$ DISTRIBUTION

The LTB solution presented in Refs. [12,24]) is intended to demonstrate an example solution that fits key cosmological observations but is not intended as a definitive replacement for the FLRW model with concordance model metric parameter values. Moreover, the LTB model is not a generic inhomogeneous model. An LTB solution constrained by more observational data could be expected to have a more complicated nonconstant t_B function (unless this is imposed by assumption). A more realistic solution using the inverse method would result from using the observational data to infer a more generic, inhomogeneous, dust solution. This could also reasonably be expected to have a nonconstant t_B function, as a function of three spatial variables rather than just one, i.e.,

$$t_B = t_B(r, \theta, \phi). \tag{11}$$

The solution [12] has only one (continuous) comoving spatial region where $t_B < \max t_B$. This simplicity is unlikely to be a requirement either of LTB models or of more general cosmological (comoving dust) solutions of the Einstein equations expressed in comoving, synchronous coordinate systems.

In solution [12], lower density ρ tends to correlate with older regions of the universe, i.e., more negative t_B (cf. Fig. 3 of Ref. [12] and the solid curves in Fig. 10 of Ref. [12]). A qualitative way to interpret this in terms of FLRW models is that for a fixed Hubble constant H_0, a lower matter density Ω_m universe is older than a higher matter density universe [32]. This is only a qualitative guide to the LTB case, since both density and any typically defined equivalent of the Hubble parameter vary with t and r differently to the FLRW case. The same Figs. 3 and 10 in Ref. [12] show that this qualitative inference does not always hold: lower ρ does not always correlate with more negative t_B.

In order to consider a more general solution than that of Ref. [12], let us suppose that a comoving dust solution to the Einstein equations expressed in synchronous, comoving, spherically symmetric coordinates has $t_B(r, \theta, \phi)$ drawn from a Gaussian distribution $G(0, \sigma)$, i.e., of mean zero and standard deviation σ when smoothed on a length scale Δx. Gaussian density fluctuations on an FLRW background are a standard ingredient of modern cosmology, so even if it is unlikely that a given solution has a t_B–ρ relation that is a function $t_B(\rho)$ (let alone a monotonic function), a Gaussian t_B distribution is a heuristically reasonable hypothesis. Now consider an approximately flat, cubical, small region of side length $3\Delta x$ of which the central $(\Delta x)^3$ small cube contains a region with $t_B < -3\sigma$, i.e., born unusually early. The probability that this small cube is connected—in coordinate space—to another small cube with $t_B < -3\sigma$, i.e., that it is not isolated by iso-bang (constant t_B) contours, is the complement of the probability that the 26 small cubes around it all have $t_B \geq -3\sigma$, i.e., $P = 1 - (\frac{1}{2}[1 + \mathrm{erf}(3/\sqrt{2})])^{26} \approx 3\%$ [33]. The chances that the second $t_B < -3\sigma$ cube touches a third cube outside of the original $(3\Delta x)^3$ region, and that the $(n > 2)$th cube touches another small cube yet further away for $n \geq 3$, rapidly decrease with increasing n.

Thus, $t_B = -3\sigma$ iso-bang contours in coordinate space will tend to form isolated 2-surfaces. That is, with a fixed smoothing scale Δx and in a large enough region of comoving coordinate space, a Gaussian distribution in t_B implies that there will tend to (statistically) exist a set of many regions (3-volumes) $\{\mathcal{W}_i\}$ with $t_B < -3\sigma$ that are spatially isolated from one another in coordinate space, and thus also consist of isolated regions of the (metrically defined) 3-manifold. At $t \gg 0$, we label the latter $\mathcal{M}$. Let us assume that $\mathcal{M}$ is connected and that its volume is $\gg (\Delta x)^3$.

Now consider the coordinate-space spatial section at $t = -3\sigma$. The boundaries of the regions $\{\mathcal{W}_i\}$ defined by $t_B < -3\sigma$, i.e., $\{\partial\mathcal{W}_i\}$, are 2-spatial iso-bang contours. The regions $\{\mathcal{W}_i\}$ have already emerged from the initial singularity, with $t - t_B = -3\sigma - t_B > 0$. Since the $\{\mathcal{W}_i\}$ are isolated from one another, they constitute a set of disconnected 3-manifolds. Hence, the universe at $t = -3\sigma$ consists of the *spatially disconnected* 3-manifold $\cup\{\mathcal{W}_i\}$, shown in Fig. 2.

The choice of -3σ is for illustration only. Any reasonably high $x \gtrsim 3$ will (statistically) give a spatially disconnected universe at $t = -x\sigma$, given a large enough spatial volume and a Gaussian distribution of t_B as stated above. At the same time t, the parts of "future" comoving space $\mathcal{M}\setminus\cup\{\mathcal{W}_i\}$ have not yet emerged from the initial singularity and only exist in coordinate space. If we follow the spatial section back in time from $t = -3\sigma$, then the boundaries $\partial\mathcal{W}_i$ correspond to $t = -x\sigma$ for increasing x, i.e., they shrink smoothly, possibly subdividing further, eventually vanishing into the singularity. For $t \leq \min t_B(r, \theta, \phi)$, the global bang time, no more $\mathcal{W}_i$ exist.

Moving forward in time, how do the $\mathcal{W}_i$ merge together? The boundary $\partial\mathcal{W}_i$ for the ith disconnected region has zero 2-surface area, as in the case of $R(t, r_{\mathrm{inf}}) = 0$ in the solution [12]. That is, the boundary of $\mathcal{W}_i$ is S^2 in coordinate space with zero 2-surface area, i.e., metrically it is a pointlike singularity. Thus, $\mathcal{W}_i$ can be thought of metrically as a 3-manifold with one point excluded. For intuitive purposes, it can be useful to think of an azimuthal equidistant projection of the Earth's surface, centred at an arbitrary geographical location, with the antipode corresponding to the big bang initial singularity. The antipode can be thought of either as a large, coordinate-space, zero-circumference circle that bounds the 2-manifold from the "outside" in the projected map, or metrically as a single missing point "on" our usual intuition of the Earth's surface.

Again, as in the solution [12], comoving space is born from this singularity, so that there is *comoving* growth of the spatial region $\mathcal{W}_i$. As t increases, the t_B threshold for the iso-bang contours increases ($dt > 0$), so that the $\mathcal{W}_i$ eventually touch and pairs (or n-tuples) of $\mathcal{W}_i$ merge together. Again writing $t = -x\sigma$, as $-x$ becomes more positive, t reaches a high enough $-x\sigma \gg 0$ such that the probability for an isolated $t_B = -x\sigma$ region to exist becomes negligible and the universe becomes fully connected.

For a Gaussian t_B distribution, it could be expected that the zero 2-surface area of the boundary of an isolated region $\mathcal{W}_i$ at $t = -x\sigma$ will tend to topologically be S^2 in coordinate space, so that $\mathcal{W}_i$ is $S^3 \backslash \partial \mathcal{W}_i = S^3 \backslash \{0\}$ topologically. Other 2-manifolds for the coordinate space representation of $\partial \mathcal{W}_i$ could also be possible.

Thus, we find that if a universe described by the Einstein equations is t_B-inhomogeneous, then, even with several simplifying assumptions (Gaussian distribution of t_B at a given smoothing length, $\mathcal{M}$ connected and simply connected for $t \gg 0$), there is a very high probability that it emerged from the (spacetime-smooth) mergers of comoving spatial sections that were spatially disconnected from each other prior to their mergers. The temporal sense of "merged" refers here to the comoving, synchronous spacetime coordinate system. Foliations of the same spacetime according to which there is no 3-spatial topology evolution are likely to exist but are unlikely to provide a model as intuitively simple as the comoving, synchronous foliation. The early-epoch disconnectivity in the comoving, synchronous foliation is distinct from questions of causal connectivity. Interpretations of this topology evolution are discussed in Sec. V after first proposing a generalization and verifying that some examples of the proposed subclasses of solutions exist.

IV. RELATIVISTIC, POSTQUANTUM-EPOCH TOPOLOGY EVOLUTION

Let us formalize the meaning and existence of spacetimes that solve the Einstein equations and yet have "postquantum-epoch" spatial topology change.

Definition 1 Let us define the generic class A where $\{\mathbf{g}^-\} \in A$ if $\mathbf{g}^-$ is a (regular) extension to $t_B < t < t_0$, using a synchronous coordinate system, of a dust (pressureless) metric $\mathbf{g}|_{t_0}$, i.e., $\mathbf{g}^-$ solves the Einstein equations over $t_B < t < t_0$, where $\mathbf{g}|_{t_0}$ on a comoving spatial section (3-manifold) at $\sim t_0$, which we call $\mathcal{M}$,

(1) solves the Einstein equations,

(2) is regular, and

(3) has an approximately homogeneous density ρ.

Here, t_0 is the age of the Universe at the location of our Galaxy, and t_B is a function of comoving spatial position defined by the initial big bang singularity. Two distinct subclasses of A are A^d ("disconnected") and A^m ("multiply connected") as follows, using coordinate time t.

(i) A^d, in which the universe is born from an initial singularity at $t \to (\min t_B)^+$ as two or more spatially disconnected regions (3-manifolds) $\mathcal{W}_i$, each of which is bounded by at least one singularity (of zero spatial volume) from which comoving space emerges continuously. The $\mathcal{W}_i$ successively merge together to form the connected 3-manifold $\mathcal{M}$ at t where $t > \max t_B > \min t_B$. The $\mathcal{W}_i$ themselves are born, in general, at different times, and their enumeration changes as a function of t, because of their mergers.

(ii) A^m, in which the universe is born from an initial singularity at $t \to (\min t_B)^+$ as a connected, simply connected region $\mathcal{W}_1$ bounded by at least two singularities during $\min t_B < t < \min t_B + \delta t$ for some $\delta t > 0$. Comoving space is continuously born from the singularities, which join together smoothly in pairs (or n-tuples, with $n > 2$), so that the spatial section at $t > \max t_B$ is a connected, multiply connected 3-manifold $\mathcal{M}$.

Theorem 1 (i) The subclass A^d is nonempty. (ii) The subclass A^m is nonempty.

The heuristic Gaussian t_B discussion (Sec. III) suggests that (i) of Theorem 1 is correct, without establishing it rigorously in an exact solution of the Einstein equations. Neither (i) nor (ii) of Theorem 1 are relativistically problematic. However, both (i) and (ii), if they are correct, are contrary to common intuition, since, if the subclasses A^d and/or A^m are "common" according to a measure over the class of possible universes, then early, comoving, synchronous topology evolution is likely to occur at time slices where any given time slice at t includes both very early and very late universe *ages* $t - t_B(r, \theta, \phi)$. within a single time slice at t.

Examples of members of A^d and A^m are given in Secs. IV A and IV B, proving Theorem 1. This provides the basis for hypothesising that solutions with nonconstant t_B are more common than those with constant t_B, in which case formal hypotheses about measure spaces are needed.

Conjecture 1 For a measure μ on A that is physically motivated at late epochs (and that does not contradict early disconnectedness), the measure of solutions that are not primordially disconnected (in terms of comoving, synchronous coordinate time t) is small, i.e., $\mu(A \backslash A^d) \ll \mu(A)$.

Corollary 1 If

(i) conjecture 1 is correct, and

(ii) $\mathbf{g}^-$ for our real Universe is chosen randomly from A, and

(iii) the standard deviation of the t_B time scale is $\gg 10^{-60}$ times that estimated empirically in Ref. [12], then spatial disconnectedness occurred at early epochs t, in the sense that

$$1 \text{ s} \gg \max_{r,\theta,\phi}\{t - t_B(r, \theta, \phi)\}, \tag{12}$$

is satisfied on the spatial section at t, but that section also includes significantly postquantum regions, i.e.,

$$\max_{r,\theta,\phi}\{t - t_{\rm B}(r, \theta, \phi)\} \gg 10^{-43}\ \text{s}, \tag{13}$$

on the same spatial section, hereafter, a "mixed-epoch" spatial section or time slice.

A. Spatially disconnected sections that merge

We show that class $A^{\rm d}$, i.e., (i) in Definition 1, is nonempty, using an explicit example of the "string of beads" LTB solution [22] (see also Ref. [34]). This is a positively curved solution. This class of solution requires the radial metric component g_{rr} to be defined as a limit, because of behavior at what (in the FLRW case) is the model's equator (e.g., Ref. [28]). This particular example has a $t_{\rm B}$ function with sinusoidal behavior, with all the minima and maxima occurring at a single pair of values, $\min t_{\rm B}$ and $\max t_{\rm B}$, respectively. This is not a general requirement, it is just a characteristic of this particularly simple solution.

Using the LTB metric (1) and Eqs. (2)–(6) we consider an example of the

$$E(r) < 0 \quad \forall\ r, t, \tag{14}$$

subcase. Following Sec. 8 of Ref. [22], we define

$$\begin{aligned} E(r) &:= -\frac{1}{2}[1 - E_1 \sin^2(r)],\\ M(r) &:= M_0(1 + M_1 \cos r),\\ t_{\rm B}(r) &:= \frac{-GM}{(-2E)^{3/2}} + GM_0(1 - M_1),\\ M_0 &:= \frac{\Omega_{\rm m}}{2GH_0(\Omega_{\rm m} - 1)^{1.5}}, \end{aligned} \tag{15}$$

where the FLRW dimensionless matter density parameter $\Omega_{\rm m} := 8\pi G\rho_0/(3H_0^2)$ is set to $\Omega_{\rm m} = 2$, ρ_0 is the present matter density, H_0 is the FLRW Hubble constant, and M_0 is chosen to get a time scale roughly comparable to that of an FLRW positively curved model with zero cosmological constant. In order to avoid $R_{,r}$ having zeroes where $M_{,r}$ does not have zeroes (see Sec. XIV.B of Ref. [35]), the second derivative of $(1 + M_1 \cos r)/(1 - E_1 \sin^2 r)$ at 0 must be negative, i.e., $E_1 < 0.5M_1/(1 + M_1)$. Thus, to obtain a sub-Gyr time scale of variation in $t_{\rm B}$, i.e., comparable to (but more conservative than) the solution [12], the parameters are set at

$$M_1 = 5 \times 10^{-5}, \qquad E_1 = 3 \times 10^{-6}. \tag{16}$$

Figures 3–8 show this solution at early epochs and the evolution of some key properties. The 3-Ricci scalar is

$${}^3R = -4\left(\frac{E_{,r}}{RR_{,r}} + \frac{E}{R^2}\right), \tag{17}$$

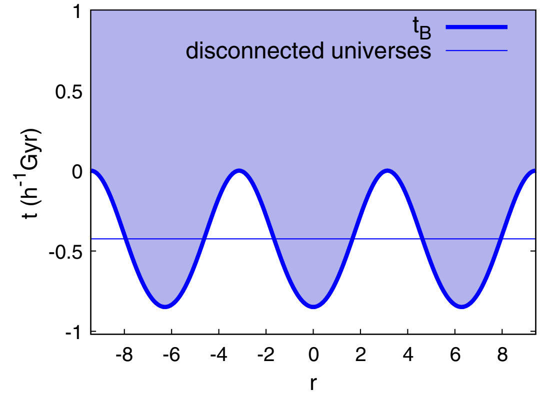

FIG. 3 (color online). Example of $t_{\rm B}$-inhomogeneous positively curved LTB solution (15) and (16) that is born as disconnected spatial sections that smoothly merge together via the initial singularity (cf. Fig. 7(a) of Ref. [22]), showing a finite part of the comoving spatial section, which is of infinite length in the r direction. The Universe exists (has emerged from the big bang singularity) in the shaded region (excluding the singularity itself, appearing as a sinusoid here). The thin horizontal line shows a spatial section of the Universe at $t = -0.42h^{-1}$ Gyr, during which some parts of the Universe exist, and others do not yet exist.

and following [35], a Hubble-like expansion parameter is defined (14), (29) of Ref. [35]

$$H := \frac{1}{3}\left(\frac{2R_{,t}}{R} + \frac{R_{,rt}}{R_{,r}}\right). \tag{18}$$

The early epoch curve in Fig. 7, i.e., for $R(-0.42h^{-1}\ \text{Gyr}, r)$ shows numerically what can be seen in (4)–(6): provided that the factors that include E and M are well behaved, the one-sided limit $\eta \to 0^+ \Leftrightarrow \xi \to 0^+$

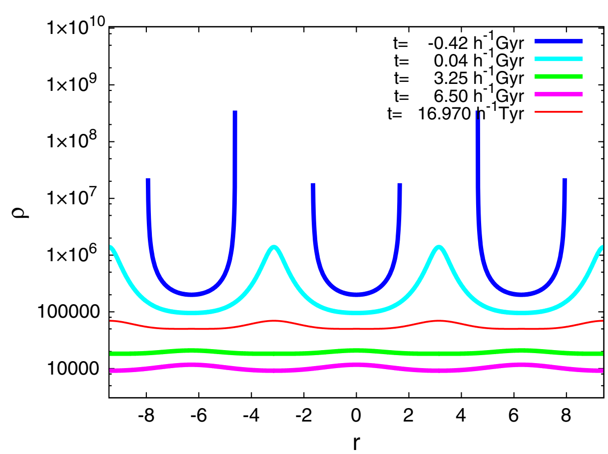

FIG. 4 (color online). Density ρ of the solution shown in Fig. 3. Values are plotted for the ranges of r where the Universe exists, i.e., $t > t_{\rm B}(r)$. Values increasing arbitrarily are shown to limits that depend on numerical implementation details and avoid obscuring the legends. A late epoch, near recollapse, is shown by a thinner curve in this figure and those following.

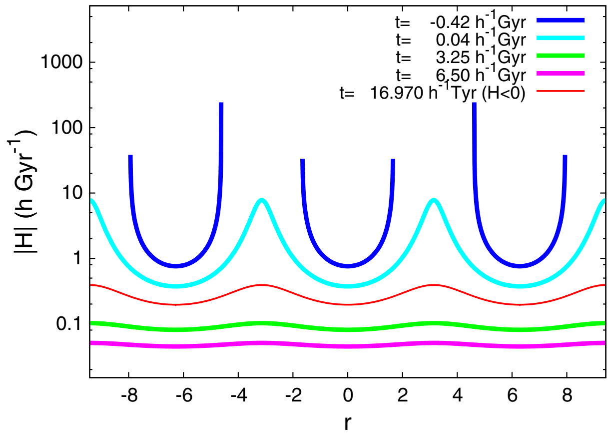

FIG. 5 (color online). As for Fig. 4, a Hubble-like parameter $|H|$ (18). The H parameter of the late epoch (near recollapse, thinner curve) has negative values of H.

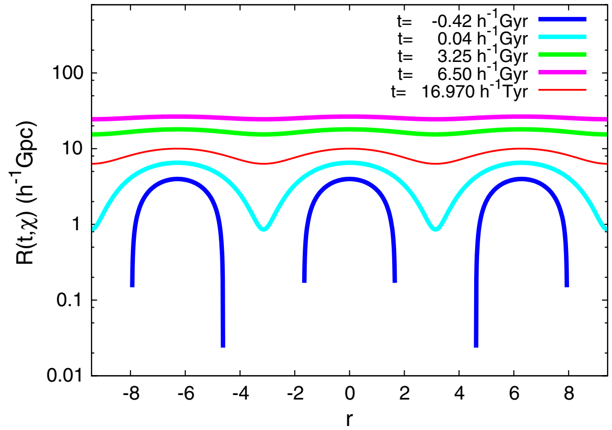

FIG. 7 (color online). As for Fig. 4, the areal radius $R(t, r)$.

corresponds to $R(t, r) \to 0^+$, and $t - t_B(r) \to 0^+$. Thus, as for the solution [12], zero-surface area 2-spheres, i.e., pointlike singularities, bound the post-big-bang parts of the universe model.

In coordinate space, it is clear that the spatial sections of the universe are disconnected at $t < 0$. What happens at and near the coordinate points $(t, r) = (0, (2n + 1)\pi)$, $n \in \mathbb{Z}$? Let us, without loss of generality, consider $(t, r) = (0, \pi)$. Since $R = 0$ at this point, the metric (1) has a non-Lorentzian signature: this is the initial big bang singularity from which the comoving spatial point $(t > 0, \pi)$ is born. Thus, at $t = 0$, the two parts of the spatial section $(0, -\pi < r < \pi)$ and $(0, \pi < r < 2\pi)$ are disconnected from one another by the point $(0, \pi)$ in coordinate space. While disconnection by a single missing point might seem trivial, since mathematically, adding a point to a manifold can remove a singularity, the physical significance would be nontrivial. The addition of a single point "at infinity" to infinite Euclidean 3-space $\mathbb{R}^3$ is enough to transform the latter into S^3, although physically, this would be absurd.

How does the radial component of the metric behave near the connection points $(0, (2n + 1)\pi))$? For $t > 0$, Figs. 7 and 8 show the anisotropic way in which the metric evolves. As $t \to 0^+$, R decreases (Fig. 7) but g_{rr} increases (Fig. 8). The latter increases without bound as $t \to 0^+$ at $(t, (2n + 1)\pi)$ and remains infinite at the r boundaries of the disconnected sections. However, the integrated proper length

$$d(t, r) := \int_0^r \sqrt{g_{rr}(t, \hat{r})} \mathrm{d}\hat{r}, \tag{19}$$

from the center of an initially disconnected section to its boundary, i.e., to the big-bang singularity, remains finite (Fig. 9).

Unless a pre-big-bang scenario is introduced, the comoving spatial sections of the universe during $\min t_B < t < 0$ consist of the disjoint union $\cup_{i \in \mathbb{Z}} S^2 \times (0, 1)$. This is not an issue of particle horizons within acausal spatial sections; the spatial sections are disconnected. With the parameters chosen, the delay before these grow and merge with the "rest" of the future-to-be-created spatial section is more than $100 h^{-1}$ Myr, i.e., long after nucleosynthesis.

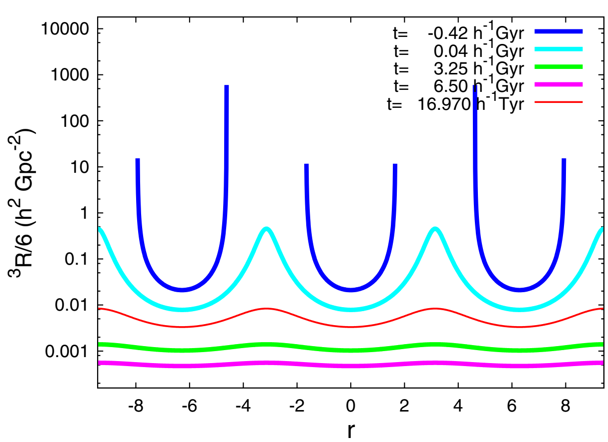

FIG. 6 (color online). As for Fig. 4, the Ricci scalar ${}^3R/6$ (17).

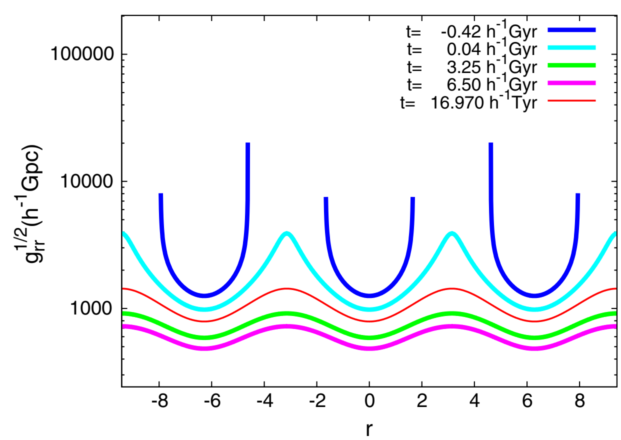

FIG. 8 (color online). As for Fig. 4, the radial metric component $\sqrt{g_{rr}(t, r)}$.

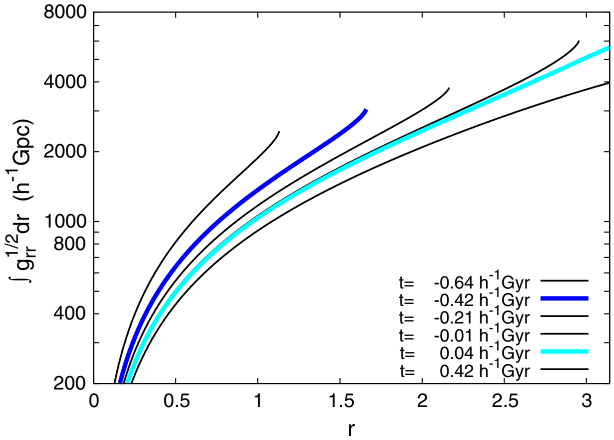

FIG. 9 (color online). As for Fig. 4, the radial proper length $d(t, r)$ (19) at several preconnection and postconnection epochs t. The two thick curves match epochs shown in the previous figures.

Thus, A^{d} of Definition 1 is a nonempty set. The age of the Universe t_0 used in this example is that for an FLRW model with $\Omega_{\mathrm{m}} = 1.015$, $\Omega_\Lambda = 0$; a $t_0 = 10h^{-1}$ Gyr model can be calculated trivially by modifying M_0.

B. A connected, simply connected section that becomes multiply connected

Taking the solution (15) and (16), we apply the holonomy

$$\gamma\colon (t, r, \theta, \phi) \mapsto (t, r + 2\pi, \theta, \phi). \tag{20}$$

The spatial sections for $t > 0$ are multiply connected, i.e., $S^2 \times S^1$. This is an exact, nonvacuum solution of the Einstein equations with a multiply connected spatial section, similar for $t > 0$ to the $S^2 \times S^1$ solution published earlier [36,37].

But at $t < 0$, the spatial section is $S^2 \times (0, 1)$, i.e., it is a single, connected, simply connected 3-manifold. Hence, a simply connected universe can smoothly become multiply connected at early (postquantum) epochs: the class of solutions A^{m} of Definition 1 is nonempty, establishing Theorem 1. Figure 10 shows the universal covering space of this solution.

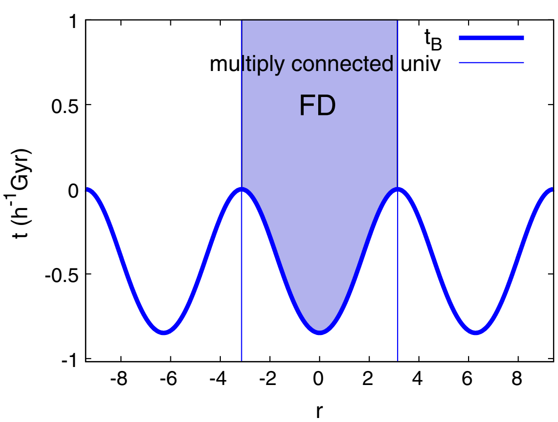

FIG. 10 (color online). As for Fig. 3, example of inhomogeneous positively curved LTB solution with nonsimultaneous (i.e., nonconstant) $t_{\mathrm{B}}(r)$ that is born with a simply connected spatial section that smoothly connects to itself via the initial singularity, becoming multiply connected ($S^2 \times S^1$), shown in the universal covering space. One copy of the fundamental domain (FD) lies between the two vertical lines.

V. DISCUSSION

A. Does relativistic, postquantum-epoch topology evolution require teleology?

Section IV establishes that universe models that evolve from being disconnected to being connected, and from being simply connected to being multiply connected, exist as classical, relativistic spacetimes. If Conjecture 1 is correct, i.e., if disconnected solutions are common, then Corollary 1 implies that the inverse method of using extragalactic, astronomical observations to extrapolate back towards the initial singularity, for example, numerically using the (3 + 1) formalism (e.g., Ref. [38]), would be likely to yield evidence of spatial disconnectedness in postquantum-epoch time slices that merge together as coordinate time t increases. This is intuitively surprising.

Is this a problem of teleology [39]? How is it possible for the singularities in spatially disconnected regions to "know" where other regions and their singularities are "located" in order for the singularities to join together by the "creation" of new comoving space? A coordinate system such as that used for LTB models is convenient to work with, but if there are spatial islands in the spatial part of the coordinate system, then the remaining "sea" of space consists of a purely fictional construct—useful for coordinate-based intuition—until comoving space is born there, converting it from fictive, coordinate space to physical (metric) space. If we only have a relativistic spacetime (with a Lorentzian metric everywhere), then individual $\mathcal{W}_i$ cannot "know" that they must be embedded in a future coordinate system that will unite them. The transition from simple to multiple connectedness is conceptually simpler, since the singularities exist in the same, connected initial manifold, but still appears to require spacelike physical interaction.

For the block universe interpretation of a Lorentzian spacetime (e.g., Ref. [40]), there is no problem of teleology: all of spacetime in the Lorentzian 4-manifold "just is." Lorentzian causality concerns the past and future time cones of a given spacetime event, not the time coordinate of a given spacetime foliation. The topology evolution of the four-dimensional spacetime viewed in terms of the comoving, synchronous representation of the metric is a property of the choice of foliation. A foliation defined by a time coordinate that makes the universe age constant within any given spatial section could be defined for the same spacetime (e.g., Sec. II.A. of Ref. [41]). For this refoliation of the model of Sec. IV A, there would be no topology evolution until epochs shortly before the big crunch (when the universe would become disconnected and the disconnected sections would end "simultaneously" in disconnected, individual big crunches). However, the coordinates would be noncomoving or asynchronous or both. The question of interpretation would then be, would it be reasonable to have initial conditions in a constant-t_B foliation whose later evolution (with constant 3-spatial topology) describes a four-dimensional spacetime that can equivalently be described with a simpler expression for the metric, i.e., in comoving, synchronous coordinates but with spatial topology evolution? The evolution from a complicated metric expression to a simpler one could be seen as teleological.

Similarly, for the multiply connected model of Sec. IV B, a constant-t_B refoliation would imply an interpretation that the universe is born multiply connected, with a noncomoving and/or asynchronous representation of the metric, and becomes simply connected when the big crunch appears as two individual spatial singularities into which all of comoving space disappears. Is this simpler than a universe that is born simply connected, becomes multiply connected, and later reverts to simple connectedness but has a metric representation that is comoving and synchronous at all times?

In both cases, there is a conflict in terms of Occam's razor and avoidance of teleology. What is the preferred model: a metric that can be expressed in a simple way with an evolving topology, or a simple (trivial) early topology evolution with a complicated metric expression? To help consider the former possibility, we speculate in Sec. V B on the minimal properties that an extension of general relativity could require.

B. Evolution of a connected 3-manifold

Let us consider a more conservative hypothesis than Conjecture 1, i.e., a hypothesis that rejects primordial disconnectedness as unlikely but does not force t_B to be constant.

Conjecture 2 For a measure μ on A that is physically motivated (and does not contradict the mergers of early epoch singularities),

(i) the measure of solutions that are disconnected is zero, i.e., $\mu(A^{\rm d}) = 0$, and

(ii) the measure of the class of solutions with constant t_B on comoving (always connected) spatial sections is small, i.e., $\mu(A \backslash A^{t_B}) \ll \mu(A^{t_B})$, where A^{t_B} is the class of solutions with nonconstant t_B and spatial sections that are always connected over $\min t_B < t < t_0$.

By the definition of $A^{\rm m}$ (Definition 1), $A^{\rm m} \subset A^{t_B}$.

If Conjecture 2 is correct, then a universe is most likely to be born connected at $\min t_B$, with at least one singularity that disappears later (as in Sec. II). If the universe is born with many singularities, then some may disappear individually (as in Sec. II), some may disappear in pairs (as in Sec. IV B), and others could, in principle, disappear in n-tuples with $n > 2$, even though it may be hard to find exact metrics as examples of regular mergers of $n > 2$ primordial singularities. Thus, if Conjecture 2 is correct and if the universe is born with many ($N \gg 1$) singularities, then an example of a minimal extension of general relativity that would describe the evolution of the universe would be a definition $\forall\ i, j \in \mathbb{Z}$: $i, j \le N$ of

(i) $P_1^i(\mathbf{g}(t), t)$, the probability that singularity i at time i disappears at time t in a way such that $\mathbf{g}$ is regular $\forall\ t' < t + \delta$ for some $\delta > 0$ over the whole spatial section, and

(ii) $\forall\ n$: $2 \le n \le N$, $P_n^{i_1, i_2, \ldots, i_n}(\mathbf{g}(t), t)$, the probability that the singularities $i_1, i_2, \ldots, i_n$ at time i merge together at time t in a way such that $\mathbf{g}$ is regular $\forall\ t' < t + \delta$ for some $\delta > 0$ over the whole spatial section.

Given the existence of the numerical solution in Sec. II and the analytical solution in Sec. IV B, and the requirement that in the latter case, the two premerger metrics must be postmerger compatible, it would seem reasonable that $P_1 \gg P_2$ and $i < j \Rightarrow P_i \gg P_j$, although this is only speculation. Two obvious classes of models would be those that define the probabilities P_1 and P_n, $n \ge 2$ to be independent of the (comoving) spatial locations of neighborhoods of the singularities, and those that define the probablities to be dependent on the spatial locations or on global properties (e.g., mean 3-Ricci scalar, topology) of the spatial section. The P_1 and P_n, $n \ge 2$ could also depend on the comoving spatial number density of the singularities.

A model of the functions P_n, $n \ge 1$ would provide a minimal extension of general relativity that could be used to calculate the probabilities that a universe evolved from a connected, simply connected spatial section to a connected, multiply connected spatial section and which topologies would be mostly likely to remain at $t > \max t_B$. If the P_2 and the number of spatial singularities are high enough, then evolution to a multiply connected spatial section would become more likely than evolution to a simply connected spatial section. If the standard deviation of the t_B time scale is $\gg 10^{-60}$ times that estimated empirically in Ref. [12], then this model would apply at significantly postquantum epochs. Nevertheless, the requirement of discreteness (since the singularities are discrete within comoving spatial sections) and the suggested probabilistic nature of the model suggest a quantum model.

C. Inferences from recent time cone observations

Let us now reconsider inferences from observations. Suppose that a given function t_B estimated from observations has a standard deviation $\sigma(t_{\rm B})$ over the three spatial coordinates that is $\sim 10^{20}$ times lower than $\sigma(t_{\rm B})$ for the empirically derived $t_{\rm B}$ in the solution [12], i.e., a much more conservative estimate by many orders of magnitude. The solution in Ref. [12] has $\max t_{\rm B} - \min t_{\rm B} \gtrsim 2$ Gyr over about 4 Gpc, suggesting $\sigma(t_{\rm B})$ of about the same order of magnitude. The study of LTB models with what are called "decaying modes" (in a perturbed FLRW context) in comparison with observations [41,42] indicates that $\sigma(t_{\rm B})$ is likely to be several orders of magnitude lower than that estimated for illustrative purposes by Ref. [12]. Let us also suppose that a comoving synchronous metric accurately describes the evolution of the observed recent Universe backwards towards the initial singularity, and that (for simplicity) we ignore the need to consider the change to a radiation-dominated epoch.

In this case, we would infer topology evolution of spatial sections that are significantly post-Planck but early, i.e., combining (13) and (12) as

$$1 \text{ s} \gg \max_{r,\theta,\phi}\{t - t_{\rm B}(r,\theta,\phi)\} \gg 10^{-43} \text{ s}. \qquad (21)$$

Thus, in terms of comoving synchronous coordinates, inhomogeneous models inferred from observations would, if Conjecture 1 is correct, typically find pairs of regions of the observed universe (e.g., the cosmic microwave background) to be spatially disconnected at early epochs, not just causally separated. If $\sigma(t_{\rm B})$ is only about 10^{10} times lower than the estimate in Ref. [12], then the spatial sections over which topology evolution occurs would include significantly postnucleosynthesis regions, i.e.,

$$10^6 \text{ s} \gtrsim \max_{r,\theta,\phi}\{t - t_{\rm B}(r,\theta,\phi)\} \gg 10^{-43} \text{ s}. \qquad (22)$$

VI. CONCLUSION

We have examined the early epoch topology evolution that corresponds to nonsimultaneous big bang times in nonempty, inhomogeneous dust models of the Universe using a recent empirical estimate and an older analytical exact solution. Even if $\sigma(t_{\rm B})$ estimated empirically is overestimated by several tens of orders of magnitude (in their introduction, the authors suggest that a more realistic time scale would be ~ 100 yr, i.e., about 10^7 times shorter than their empirical solution [12]), it is still postquantum unless $t_{\rm B}$ is constant to within the Planck time scale, i.e., $\sim 10^{-43}$ s. Other estimates of $\sigma(t_{\rm B})$ vary from Gyr (e.g., Figs. 6 and 8 [43]) to sub-Myr time scales [41,42], i.e., $\gg 10^{-43}$ s. Thus, the temporal evolution implied by $t_{\rm B}$-inhomogeneous models (ignoring the need to enter the radiation-dominated epoch) may imply 3-spatial topology evolution for a comoving, synchronous representation of the metric, either from disconnected spatial sections to connected spatial sections (Sec. IVA), or from multiply connected to simply connected spatial sections (Sec. IV B).

This surprising implication could be avoided by imposing $t_{\rm B} =$ constant as an assumption in cosmological modeling. One problem in assuming constant $t_{\rm B}$ is that for flat LTB solutions, generalizations beyond the FLRW model are rejected. That is, the combination of $E(r) = 0$ and $t_{\rm B}(r) = 0$ leaves no freedom to adjust the third "arbitrary" function $M(r)$; see VIII (63a), XIV.B in Ref. [35]. Section XIV.B of Ref. [35] also discusses the restrictions on LTB models implied by imposing $t_{\rm B}(r) = 0 \forall\ r$ in the positive and negative $E(r)$ cases. More importantly from a physical point of view, allowing $t_{\rm B}$ spatial dependence to be a result of comparison between models and observations rather than an assumption could potentially lead to evidence for early universe spatial sections that in comoving, synchronous coordinates undergo topology evolution. This evidence would be artificially suppressed if $t_{\rm B} = 0 \forall\ r$ were forced.

We have formalized some of the possible properties of subclasses of solutions of this type and of possible implications in Definition 1, Theorem 1, Conjecture 1, Corollary 1, and Conjecture 2. Conjecture 2 opens the way to calculations of the probabilities of a simply connected initial spatial section smoothly evolving into a multiply connected spatial section, based on a choice of functions $P_1^i(\mathbf{g}(t), t)$, $\forall\ n\colon 2 \le n \le N$, $P_n^{i_1,i_2,\ldots,i_n}(\mathbf{g}(t), t)$ as defined in the requirements of a physical theory suggested above (Sec. V B). Understanding how a multiply connected spatial section arises would have considerable observational interest (e.g., Refs. [44–52]), since it would offer an alternative to the topological acceleration effect [53–55] for theoretical understanding of the topology of the present-day (i.e., recent time-cone) Universe (see also Refs. [56,57]).

How was it possible that postquantum-epoch topology change without causality problems was overlooked in cosmic topology literature? It has generally been thought that if the spatial sections of the Universe are compact, then the topology of spatial sections of the Universe cannot have evolved at postquantum epochs, because this would imply the existence of closed timelike curves or a discontinuity in the choice of the forward light cone, as a consequence of Geroch's Theorem 2 [58], Sec. 9.4.1. of Refs. [59,60], and both are generally considered unphysical. Singularities make spacelike sections noncompact, so that the theorem no longer applies, but it is not immediately obvious that an astrophysically realistic black (let alone white) hole could change the large-scale, global topology of spatial sections in a way that leads to approximate homogeneity in the late-time Universe. What was overlooked was the fact that a nonconstant $t_{\rm B}$ provides (in general) an arbitrary number of singularities in early, postquantum comoving *spatial sections*, which in *spacetime* constitute just one singularity—the initial big bang singularity that is generally accepted as physical in relativistic, nonquantum cosmology. Moreover, as illustrated above, the density and curvature inhomogeneities near vanished singularities/connection points can become much weaker, i.e., enter a "decaying mode" in a perturbed FLRW context. The LTB models provide a useful tool for studying examples of characteristics that are counterintuitive for FLRW-like models.

Although in this work we only consider topology change implied by nonconstant t_B in classical relativity, see Ref. [60] for a quantum gravity approach using sums of histories and Morse theory.

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