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Transition form factors and mixing of pseudoscalar mesons from anomaly sum rule

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Transition form factors of π^0, η, η' mesons are investigated by means of the anomaly sum rule—an exact nonperturbative relation which follows from the dispersive representation of the axial anomaly. Considering the problem of contributions of operators originated from a non-(local) operator product expansion, we found that they are required by the available set of experimental data, including the most recent data from the Belle Collaboration (which, if taken alone, can be described without such contributions, although are compatible with them). In this approach, we analyzed the experimental data on η and η' meson transition form factors and obtained the constraints on the decay constants and mixing parameters.

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I. INTRODUCTION

The phenomenon of the axial anomaly [1,2] plays an important role in nonperturbative QCD and hadronic physics. The axial anomaly is known to govern the two-photon decays of the π^0, η, and η' mesons and is usually considered for a case of real photons. However, the dispersive form of it [3] can be considered for virtual photons also [4–6], leading to a number of interesting applications.

One of the consequences of the dispersive approach to the axial anomaly is a so-called anomaly sum rule (ASR) [5]. It gives, in particular, a complementary way to describe the π^0 [7] and the η, η' [8,9] transition form factors (later developed also in Refs. [10,11]) at all Q^2, even beyond the QCD factorization. This is especially important in view of the recent experimental studies of the $\gamma\gamma^* \to \pi^0(\eta, \eta')$ transitions [12–14]. In particular, the pion transition form factor, measured by the *BABAR* Collaboration [12], revealed unexpectedly large values in the range of $Q^2 = 10$–$35\ \mathrm{GeV}^2$, resulting in an excess of the pQCD predicted limit [15] $Q^2F_{\pi\gamma} \to \sqrt{2}f_\pi$, $f_\pi = 0.1307$ GeV. This striking result attracted a lot of interest and motivated extensive theoretical investigations. As a result, the transition form factors were (re)investigated using the framework of light cone sum rules [16–20], including the flatlike modifications of the distribution amplitude [21–23], the light cone holography approach [24,25], and various model approaches, like a chiral quark model [26] (see also Refs. [27–30]), a vector meson dominance model, and its modifications [31,32]. Some other approaches can be found in Refs. [33–42].

In this paper we extend and develop the anomaly sum rule approach [7,8] to the transition form factors with a systematic account of the effects of mixing of η, η' mesons and quark-hadron duality. Also, we performed a new analysis of the anomaly sum rule for π^0 using available experimental data, including the most recent ones from the Belle Collaboration [14].

The paper is organized as follows. In Sec. II we give an overview of the anomaly sum rule approach and apply it to analyze the pion (isovector channel of the ASR) and η, η' (octet channel of the ASR) transition form factors. We pay special attention to the seemingly controversial data from the *BABAR* [12] and Belle [14] collaborations. The analysis of different sets of data show that inclusion of the *BABAR* data requires a non-(local) operator product expansion (OPE) correction to the spectral density, while the Belle data alone neither require nor exclude it. In Sec. III we develop and reformulate the description of mixing, which plays a special role for the η, η' mesons, in a way which does not require the introduction of intermediate nonphysical states. The problem of compatibility of different mixing schemes is also discussed. In Sec. IV we perform a numerical analysis of the mixing parameters of the η-η' system based on the obtained sum rule for the transition form factors. The possibility of the non-OPE correction to the spectral density in the octet channel is investigated as well. The summary is presented in Sec. V.

II. ANOMALY SUM RULE APPROACH

The axial anomaly in QCD results in a nonvanishing divergence of axial current in the chiral limit. It is common to consider an octet of axial currents $J^{(a)}_{\mu5} = \sum_q \bar{q}\gamma_5\gamma_\mu \frac{\lambda^a}{\sqrt{2}}q$, ($a = 1, \dots 8$; the sum is over u, d, s flavors; λ^a are Gell-Mann matrices) and a singlet axial current $J^{(0)}_{\mu5} = \frac{1}{\sqrt{3}}(\bar{u}\gamma_\mu\gamma_5 u + \bar{d}\gamma_\mu\gamma_5 d + \bar{s}\gamma_\mu\gamma_5 s)$. The singlet axial current acquires both electromagnetic and gluonic anomalous terms:

$$\partial^\mu J^{(0)}_{\mu5} = \frac{1}{\sqrt{3}}(m_u\bar{u}\gamma_5 u + m_d\bar{d}\gamma_5 d + m_s\bar{s}\gamma_5 s) + \frac{\alpha_{\mathrm{em}}}{2\pi}C^{(0)}N_cF\tilde{F} + \frac{\sqrt{3}\alpha_s}{4\pi}N_cG\tilde{G}, \tag{1}$$

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where F and G are electromagnetic and gluonic strength tensors, respectively; $\tilde{F}$ and $\tilde{G}$ are their duals; and $N_c = 3$ is the number of colors. On the contrary, diagonal components of the octet of axial currents, i.e., $J^{(3)}_{\mu 5} = \frac{1}{\sqrt{2}}(\bar{u}\gamma_\mu\gamma_5 u - \bar{d}\gamma_\mu\gamma_5 d)$ and $J^{(8)}_{\mu 5} = \frac{1}{\sqrt{6}}(\bar{u}\gamma_\mu\gamma_5 u + \bar{d}\gamma_\mu\gamma_5 d - 2\bar{s}\gamma_\mu\gamma_5 s)$, acquire an electromagnetic anomalous term only:

$$\partial^\mu J^{(3)}_{\mu 5} = \frac{1}{\sqrt{2}}(m_u\bar{u}\gamma_5 u - m_d\bar{d}\gamma_5 d) + \frac{\alpha_{\rm em}}{2\pi}C^{(3)}N_c F\tilde{F}, \quad (2)$$

$$\partial^\mu J^{(8)}_{\mu 5} = \frac{1}{\sqrt{6}}(m_u\bar{u}\gamma_5 u + m_d\bar{d}\gamma_5 d - 2m_s\bar{s}\gamma_5 s) + \frac{\alpha_{\rm em}}{2\pi}C^{(8)}N_c F\tilde{F}. \quad (3)$$

The electromagnetic charge factors $C^{(a)}$ are

$$C^{(3)} = \frac{1}{\sqrt{2}}(e_u^2 - e_d^2) = \frac{1}{3\sqrt{2}},$$
$$C^{(8)} = \frac{1}{\sqrt{6}}(e_u^2 + e_d^2 - 2e_s^2) = \frac{1}{3\sqrt{6}},$$
$$C^{(0)} = \frac{1}{\sqrt{3}}(e_u^2 + e_d^2 + e_s^2) = \frac{2}{3\sqrt{3}}. \quad (4)$$

In short, in what follows, we call $J^{(3)}_{\mu 5}$ and $J^{(8)}_{\mu 5}$ the isovector and octet current, respectively.

The calculation of the matrix elements of exact operator equations (2) and (3), associated with the photon-meson transitions, leads to the triangle graph amplitude, composed of the axial current $J_{\alpha 5}$ with momentum $p = k + q$ and two vector currents with momenta k and q [vector-vector-axial (VVA) amplitude]

$$T_{\alpha\mu\nu}(k, q) = \int d^4x d^4y e^{(ikx+iqy)}\langle 0|T\{J_{\alpha 5}(0)J_\mu(x)J_\nu(y)\}|0\rangle. \quad (5)$$

This amplitude can be decomposed [43] (see also Refs. [44,45]) as

$$T_{\alpha\mu\nu}(k, q) = F_1\varepsilon_{\alpha\mu\nu\rho}k^\rho + F_2\varepsilon_{\alpha\mu\nu\rho}q^\rho + F_3k_\nu\varepsilon_{\alpha\mu\rho\sigma}k^\rho q^\sigma + F_4q_\nu\varepsilon_{\alpha\mu\rho\sigma}k^\rho q^\sigma + F_5k_\mu\varepsilon_{\alpha\nu\rho\sigma}k^\rho q^\sigma + F_6q_\mu\varepsilon_{\alpha\nu\rho\sigma}k^\rho q^\sigma, \quad (6)$$

where the coefficients $F_j = F_j(p^2, k^2, q^2; m^2)$, $j = 1, \ldots, 6$ are the corresponding Lorentz invariant amplitudes constrained by current conservation and Bose symmetry. Note that the latter includes the interchange $\mu \leftrightarrow \nu$, $k \leftrightarrow q$ in the tensor structures and $k^2 \leftrightarrow q^2$ in the arguments of the scalar functions F_j.

In what follows, we consider the case when one of the photons is real ($k^2 = 0$) while the other is real or virtual ($Q^2 = -q^2 \geq 0$).

For the isovector and octet currents, using the dispersive treatment of the axial anomaly [3], one can derive the ASR [5]:

$$\int_{4m^2}^{\infty} A_3^{(a)}(s, Q^2; m^2)ds = \frac{1}{2\pi}N_c C^{(a)}, \qquad a = 3, 8, \quad (7)$$

where $A_3 = \frac{1}{2}\,{\rm Im}_{p^2}(F_3 - F_6)$ and m is a quark mass.

The ASR (7) has a remarkable property—both perturbative and nonperturbative corrections to the integral are absent [46]. The perturbative corrections are excluded because of the Adler-Bardeen theorem [47], while the nonperturbative corrections are also absent, as is expected from 't Hooft's principle. 't Hooft's principle in its original form [48] implies that the anomalies of the fundamental fields are reproduced on the hadron level. In the dispersive approach, this means the absence of the corrections to the dispersive sum rules.

Let us stress that the spectral density $A_3^{(a)}(s, Q^2; m^2)$ can have both perturbative and nonperturbative corrections (however, the first-order correction $\propto \alpha_s$ is zero in the massless limit [49]), while the integral $\int_{4m^2}^{\infty} A_3^{(a)}(s, Q^2; m^2)ds$ equals exactly $\frac{1}{2\pi}N_c C^{(a)}$.

Saturating the lhs of the three-point correlation function (5) with the resonances and singling out their contributions to the ASR (7), we get (an infinite [7]) sum of resonances with appropriate quantum numbers

$$\pi\sum f_M^a F_{M\gamma} = \int_{4m^2}^{\infty} A_3^{(a)}(s, Q^2; m^2)ds = \frac{1}{2\pi}N_c C^{(a)}. \quad (8)$$

Here the projections of the axial currents $J^{(a)}_{5\alpha}$ onto one-meson states $M(= \pi^0, \eta, \eta')$ define the coupling (decay) constants f_M^a

$$\langle 0|J^{(a)}_{\alpha 5}(0)|M(p)\rangle = ip_\alpha f_M^a, \quad (9)$$

while the form factors $F_{M\gamma}$ of the transitions $\gamma\gamma^* \to M$ are defined by the matrix elements

$$\int d^4x e^{ikx}\langle M(p)|T\{J_\mu(x)J_\nu(0)\}|0\rangle = \epsilon_{\mu\nu\rho\sigma}k^\rho q^\sigma F_{M\gamma}. \quad (10)$$

The relation (8) expresses the global duality between hadrons and quarks.

A. Isovector channel (π^0)

For a case of the *isovector channel*, the first contribution is given by π^0, while the higher contributions can be absorbed by the "continuum" contribution $\int_{s_0^{(3)}}^{\infty} A_3^{(3)}(s, Q^2; m^2)$, so the ASR (8) takes the form

$$\pi f_\pi F_{\pi\gamma}(Q^2; m^2) = \frac{1}{2\pi}N_c C^{(3)} - \int_{s_0^{(3)}}^{\infty} A_3^{(3)}(s, Q^2; m^2)ds, \quad (11)$$

where we assume for simplicity $m = m_u = m_d$.

The lower limit $s_0^{(3)}$ of the integral we will refer to as a "continuum threshold," bearing in mind that in a local quark-hadron duality hypothesis it means the interval of the duality of a pion. Also, it can be determined directly from the ASR, as we will later demonstrate.

The contribution to the spectral density $A_3^{(3)}(s, Q^2; m^2)$ for a given flavor q can be calculated from the VVA triangle diagram [5],

$$A_3^{(q)}(s, Q^2; m_q^2) = \frac{e_q^2}{2\pi} \frac{1}{(Q^2 + s)^2} \left(Q^2 R + 2m_q^2 \ln \frac{1+R}{1-R} \right), \tag{12}$$

where $R(s, m_q^2) = \sqrt{1 - \frac{4m_q^2}{s}}$.

From (11) and (12) a straightforward calculation gives an expression for the pion transition form factor,

$$F_{\pi\gamma}(Q^2; m^2) = \frac{1}{2\sqrt{2}\pi^2 f_\pi} \frac{s_0^{(3)}}{s_0^{(3)} + Q^2} \times \left[1 - \frac{2m^2}{s_0^{(3)}} \left(\frac{2}{R_0 + 1} + \ln \frac{1 + R_0}{1 - R_0} \right) \right], \tag{13}$$

where $R_0 = R(s_0^{(3)}, m^2)$. This expression (to our best knowledge, for the first time) takes into account the contribution of quark mass.

Let us note that the quark mass term in (13) (for $m \simeq 7$ MeV, and $s_0 \simeq 0.7$ GeV^2) gives only $\simeq 0.15\%$ contribution and can be neglected.

In the massless limit, the spectral density (12) is proportional to $\delta(s)$ at $Q^2 = 0$, so the continuum term in the ASR (11) goes to zero. This corresponds to the fact that contributions of axial states are zero at $Q^2 = 0$, and contributions of higher pseudoscalar states should be suppressed in order for the axial current to conserve in the chiral limit.

Relying on the local quark-hadron duality hypothesis, the analysis of the two-point correlation function gives the value for the continuum threshold $s_0^{(3)} = 0.75$ GeV^2 [50]. Actually, $s_0^{(3)}$ can be determined directly from the high-Q^2 asymptotic of ASR, where the QCD factorization predicts the value of the transition form factor $Q^2 F_{\pi\gamma}^{\rm as} = \sqrt{2} f_\pi$ [15]. The high-Q^2 limit of (13) ($m = 0$) immediately leads to $s_0^{(3)} = 4\pi^2 f_\pi^2 = 0.67$ GeV^2. This expression, substituted in (13) with $m = 0$, gives

$$F_{\pi\gamma}(Q^2; 0) = \frac{1}{2\sqrt{2}\pi^2 f_\pi} \frac{4\pi^2 f_\pi^2}{4\pi^2 f_\pi^2 + Q^2}, \tag{14}$$

so it proves the Brodsky-Lepage interpolation formula for the pion transition form factor [51], which was later confirmed by Radyushkin [52] in the approach of local quark-hadron duality. Let us stress that in this way we found that it is a direct consequence of the anomaly sum rule (which is an exact nonperturbative QCD relation).

It is interesting to note that by extending the expression for the pion transition form factor (13) into the timelike region, one immediately gets a pole at $Q^2 = s_0^{(3)}$ which is

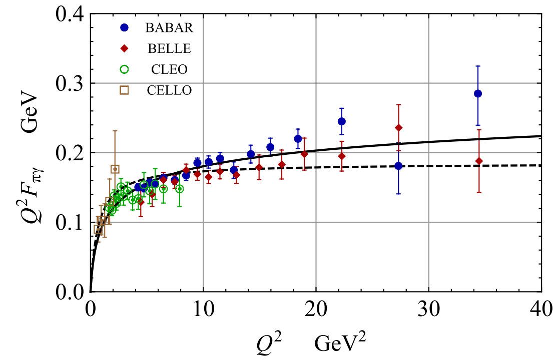

FIG. 1 (color online). Pion transition form factor (multiplied by Q^2) with correction [Eq. (20), solid curve] and without correction [Eq. (14), dashed curve] as a function of Q^2 compared with experimental data.

numerically close to the mass of a ρ meson. This can be a kind of interplay between anomaly and vector dominance.

Currently the data from the CELLO [53], CLEO [54], *BABAR* [12], and Belle [14] collaborations cover the region of $Q^2 = 0.735$ GeV^2 (see Fig. 1). While at $Q^2 < 10$ GeV^2 they are consistent, at larger virtualities the *BABAR* and newly released Belle data are quite different. In this situation we will consider two sets of data: CELLO + CLEO + Belle (I) and CELLO + CLEO + *BABAR* (II).

When compared to the experimental data set (I), Eq. (14) gives a reasonable description consistent with the data (χ^2/d.o.f. = 1.01, d.o.f. = 35; see the dashed line in Fig. 1). For the data set (II), the description is worse (χ^2/d.o.f. = 2.29, d.o.f. = 37). So, if the data of the *BABAR* Collaboration are correct, we come to a violation of the ASR-based expression for $F_{\pi\gamma}$ (14).

This means that the spectral density (12) must have a substantial correction δA_3, which results in corrections to the continuum $\delta I_{\rm cont} = \int_{s_0^{(3)}}^\infty \delta A_3 ds$ and pion $\delta I_\pi = \int_{4m^2}^{s_0^{(3)}} \delta A_3 ds$ contributions. At the same time, as the *full* integral remains constant, the corrections can be related:

$$\delta I_\pi + \delta I_{\rm cont} = 0. \tag{15}$$

It is important that the main terms of the continuum $I_{\rm cont}$ and the pion I_π contributions have essentially different Q^2 behavior

$$I_{\rm cont} = \int_{s_0^{(3)}}^\infty A_3^{(3)}(s, Q^2) ds = \frac{1}{2\sqrt{2}\pi} \frac{Q^2}{s_0^{(3)} + Q^2}, \tag{16}$$

$$I_\pi = \int_0^{s_0^{(3)}} A_3^{(3)}(s, Q^2) ds = \frac{1}{2\sqrt{2}\pi} \frac{s_0^{(3)}}{s_0^{(3)} + Q^2}, \tag{17}$$

so the $1/Q^2$ power correction to the continuum contribution is of the order of the main term of the pion contribution.

TABLE I. χ^2/d.o.f. obtained for Eq. (14) ($\delta I_\pi = 0$) and the best fits of Eq. (20) ($\delta I_\pi \neq 0$) to different data sets.

	$\delta I_\pi = 0$: χ^2/d.o.f.	$\delta I_\pi \neq 0$: χ^2/d.o.f.	λ	σ
CELLO + CLEO + *BABAR* + Belle	1.86	0.91	0.12	−2.50
CELLO + CLEO + Belle (set I)	1.01	0.46	0.07	−3.03
CELLO + CLEO + *BABAR* (set II)	2.29	0.94	0.14	−2.36
BABAR	3.61	0.99	0.20	−2.39
Belle	0.80	0.40	0.14	−2.86

Let us now discuss the sources of possible corrections to the spectral densities which in our approach are the counterparts of the nonlocal operator and higher twist corrections in the accurate pQCD fits [18]. Note that one-loop corrections to the spectral densities of all structures in the VVA correlator in the massless case are zero, which can be easily deduced from the results of Shifman *et al.* [49]. If this nullification is due to conformal invariance [55,56], one may expect the two-loop and higher corrections to be nonzero due to the beta-function effects, which have recently been observed in the soft-photon approximation [57]. Nevertheless, these higher α_s corrections, as well as the local OPE-induced corrections, are small enough to produce an enhancement in the pion transition form factor shown by *BABAR*: $\delta F_{\pi\gamma} \sim \log(Q^2)/Q^2$. Clearly, from dimensional arguments, such a term cannot appear from the local OPE [7]. Let us note also that even if some larger effective quark mass, instead of its current value, is taken in Eq. (13), the mass term worsens the experimental data description because of its negative sign. Thus, to comply with the ASR, the correction should be of a non-(local) OPE origin, simulating the contribution of the operator of dimension 2. Among the possible sources of such correction are nonlocal condensates, instantons, and short strings [58].

Although the exact form of such a correction is not yet known, we can construct the simplest form of it relying on general requirements. Namely, the correction should vanish at $s_0^{(3)} \to \infty$ (the continuum contribution vanishes), at $s_0^{(3)} \to 0$ (the full integral has no corrections), at $Q^2 \to \infty$ (the perturbative theory works at large Q^2) and at $Q^2 \to 0$ (the anomaly perfectly describes the pion decay width). Therefore, the correction satisfying those limits can be written as

$$\delta I_\pi = \frac{s_0^{(3)} Q^2}{(s_0^{(3)} + Q^2)^2} f\left(\frac{Q^2}{s_0}\right), \tag{18}$$

where f is a dimensionless function of Q^2, s_0, and some parameters. Expecting the $\log(Q^2)/Q^2$ behavior, we can suggest the simplest (although not unique) form of the correction

$$\delta I_\pi = \frac{1}{2\sqrt{2}\pi} \frac{\lambda s_0^{(3)} Q^2}{(s_0^{(3)} + Q^2)^2} \left(\ln\frac{Q^2}{s_0^{(3)}} + \sigma\right), \tag{19}$$

where λ and σ are dimensionless parameters. Then the pion transition form factor with this correction reads

$$F_{\pi\gamma}(Q^2) = \frac{1}{\pi f_\pi}(I_\pi + \delta I_\pi) = \frac{1}{2\sqrt{2}\pi^2 f_\pi} \frac{s_0^{(3)}}{s_0^{(3)} + Q^2}\left[1 + \frac{\lambda Q^2}{s_0^{(3)} + Q^2}\left(\ln\frac{Q^2}{s_0^{(3)}} + \sigma\right)\right]. \tag{20}$$

For the continuum threshold we will use $s_0^{(3)} = 4\pi^2 f_\pi^2 = 0.67\ \text{GeV}^2$, which implies that at very high Q^2 the factorization restores. Nevertheless, if the factorization is violated at all Q^2, one can use a different value for $s_0^{(3)}$. Moreover, one can even consider the dependence of $s_0^{(3)}$ on Q^2 [38], which can lead to an effective change of σ.

The fit of (20) to the data set (II) gives $\lambda = 0.14$, $\sigma = -2.36$ with $\chi^2/\text{d.o.f.} = 0.94$ (d.o.f. = 35). The plot of $Q^2 F_{\pi\gamma}$ for these parameters is depicted in Fig. 1 as a solid curve. Note that the proposed correction changes its sign at $Q^2 \simeq 8\ \text{GeV}^2$, improving the description in both regions: at $Q^2 \lesssim 8\ \text{GeV}^2$ and at $Q^2 \gtrsim 8\ \text{GeV}^2$. Also, $F_{\pi\gamma}$ (20) with these parameters λ, σ [obtained from the fit to the data (II)] describes well also the data set (I) with $\chi^2/\text{d.o.f.} = 0.84$ (d.o.f. = 35).

The summary of the fitting results for $F_{\pi\gamma}$ with [Eq. (20)] and without [Eq. (14)] correction for different sets of data is shown in Table I. One can see that the data sets involving the *BABAR* data require taking into account the correction. The data sets which do not involve the *BABAR* data may be described without such a correction, although the correction may improve the description of the data.

Let us emphasize that the correction (19) requires a $\log Q^2$ term in δA_3 itself. This form of the correction is different from the one proposed in Ref. [10] and could match it only if the prelogarithmic factor in (19) did not depend on $s_0^{(3)}$. However, such a factor would violate the above mentioned requirement of nullification of the correction [in the limit $s_0^{(3)} \to 0$].

Also, there is a clear distinction with the natural emergence of the $\log Q^2$ term in the triangle amplitude (which was used for the description of the *BABAR* data), where the triangle amplitude itself is used as a model for the pion transition form factor [26]. Such an approach applies the partially conserved axial current relation for the matrix elements involving large virtualities. In our opinion this procedure is not justified to the same degree of rigor as for the soft processes. In our approach, the log Q^2 term appears in the spectral density which is translated to a transition form factor by an integral relation. Nevertheless, we currently also cannot justify it by strict theoretical arguments.

B. Octet channel (η, η')

In this subsection we consider the ASR for the octet channel, where, like for the isovector case, only the electromagnetic anomaly gives a contribution and the gluonic anomaly is absent. However, in comparing to the isovector channel, here we have some differences.

First, due to significant mixing in the η-η' system, the η' meson contributes to the octet channel. Since η' decays into two real photons, it should be taken into account explicitly along with the η meson.

Second, the spectral density in the octet channel $A_3^{(8)} = \frac{1}{\sqrt{6}}(A_3^{(u)} + A_3^{(d)} - 2A_3^{(s)})$ gets a more significant (in comparison to the isovector case) mass contribution due to the strange quark. Also, there can be direct instanton contributions to the spectral density, which, however, should vanish in the massless limit as the singlet-octet transition is forbidden in the exact $SU(3)$ limit [59]. This is in agreement with the consideration of the instanton contributions to the two-point correlators [60], where such contributions are $\propto m_s^2$ in the singlet-octet correlator and are absent in the octet-octet correlator.

In this paper we restrict ourselves to the leading approximation, where the quark mass corrections [both from triangle diagram (12) and direct instantons] to the spectral density $A_3^{(8)}$ are neglected. Then treating the ASR in the same way as for the isovector channel gives

$$f_\eta^8 F_{\eta\gamma}(Q^2) + f_{\eta'}^8 F_{\eta'\gamma}(Q^2) = \frac{1}{2\sqrt{6}\pi^2}\frac{s_0^{(8)}}{s_0^{(8)} + Q^2}, \tag{21}$$

where f_η^8, $f_{\eta'}^8$ are the decay constants defined in (9), and $s_0^{(8)}$ is a continuum threshold in the octet channel.

As soon as this approximation provides a reasonable description of the experimental data (see Sec. IV), this may possibly indicate a partial cancellation of the instanton and mass effects in the VVA correlation function.

Note also that the discrepancy with the two-photon decay width of the η meson, considered in Ref. [61] as a possible signal of instantons, is in fact eliminated when the mixing is taken into account [62], leading to specific values of f_η^8, $f_{\eta'}^8$. This may also be interpreted as a partial absorption of the instanton contributions to the two-point correlation function by the values of f_η^8, $f_{\eta'}^8$.

At the same time, the reliable estimation of $s_0^{(8)}$ from such a two-point correlator by the usual QCD sum rule method meets difficulties (see, e.g., discussion in Ref. [9]). Fortunately, the ASR approach allows us to determine $s_0^{(8)}$ in the octet channel from the high-Q^2 asymptotic, just the same way as in the isovector channel. Generalization of the pion case gives the asymptotes for the η, η' transition form factors [63,64] ($M = \eta$, η'):

$$Q^2 F_{M\gamma}^{\mathrm{as}} = 6(C^{(8)} f_M^8 + C^{(0)} f_M^0). \tag{22}$$

So, the $Q^2 \to \infty$ limit of the ASR (21) gives

$$s_0^{(8)} = 4\pi^2((f_\eta^8)^2 + (f_{\eta'}^8)^2 + 2\sqrt{2}[f_\eta^8 f_\eta^0 + f_{\eta'}^8 f_{\eta'}^0]). \tag{23}$$

In the $Q^2 = 0$ limit the transition form factors are expressed in terms of the two-photon decay widths of mesons, so the ASR (21) takes the form

$$f_\eta^8 F_{\eta\gamma}(0) + f_{\eta'}^8 F_{\eta'\gamma}(0) = \frac{1}{2\sqrt{6}\pi^2}, \tag{24}$$

where

$$F_{M\gamma}(0) = \sqrt{\frac{4\Gamma_{M\to\gamma\gamma}}{\pi\alpha^2 m_M^3}}.$$

Solving Eqs. (23) and (24) with respect to $s_0^{(8)}$ and one of the decay constants f_M^a and substituting them into general ASR (21), we can relate the transition form factors $F_{\eta'\gamma}(Q^2)$ and the decay constants f_M^a.

Actually, the four decay constants f_M^a can be related based on a particular mixing scheme. In Ref. [8] the ASR was analyzed for several sets of parameters and mixing schemes. The more general consideration of mixing schemes and extraction of the mixing parameters are performed in the next sections.

III. MIXING

The problem of mixing in the η-η' system is usually addressed either in the octet-singlet [$SU(3)$] or quark-flavor mixing scheme (see, e.g., Refs. [65,66] and references therein). Basically, meson mixing implies that the "nondiagonal" decay constants f_M, $M = \eta$, η' in Eq. (9) are nonzero. The π^0 and the isovector current can be decoupled from the η-η' system and octet and singlet currents because of a very small mixing.

Let us recall the common approach to the mixing, when physical states are represented as a linear combination of states with definite $SU(3)_f$ quantum numbers or quark-flavor content. But, since these states do not have definite masses, one cannot write the analogue of Eq. (9) with these states instead of physical states η, η'. Indeed, if a state has a definite momentum p^μ it also has a definite mass $m^2 = p_\mu p^\mu$.

One can avoid this problem by formulating the mixing in terms of the fields ϕ_i related to the physical states $|i\rangle$ by the matrix elements $\langle 0|\phi_i|j\rangle = \delta_{ij}$.

It is well known that the field of the pion ϕ_π is defined from the divergence of the isovector component of the axial channel as a partially conserved axial current relation,

$$\partial_\mu J^{(3)}_{\mu 5} = f^{(3)}_\pi m^2_\pi \phi_\pi. \quad (25)$$

To consider the η and η' mixing, one can write down a straightforward generalization of this relation,

$$\partial_\mu \mathbf{J}_{\mu 5} = \mathbf{FM\Phi}, \quad (26)$$

where we introduced the matrix notations:

$$\mathbf{J}_{\mu 5} \equiv \begin{pmatrix} J^\alpha_{\mu 5} \\ J^\beta_{\mu 5} \end{pmatrix}, \qquad \mathbf{F} \equiv \begin{pmatrix} f^\alpha_\eta & f^\alpha_{\eta'} & f^\alpha_G & \dots \\ f^\beta_\eta & f^\beta_{\eta'} & f^\beta_G & \dots \end{pmatrix},$$

$$\mathbf{\Phi} \equiv \begin{pmatrix} \phi_\eta \\ \phi_{\eta'} \\ \phi_G \\ \vdots \end{pmatrix}, \qquad \mathbf{M} \equiv \mathrm{diag}(m^2_\eta, m^2_{\eta'}, m^2_G, \dots). \quad (27)$$

The vector $\mathbf{J}_{\mu 5}$ in the lhs consists of the components of the axial current of definite $SU(3)$ symmetry, a so-called octet-singlet basis ($\alpha = 8$, $\beta = 0$), or of the components of the axial current with the decoupled light and strange quark composition, a so-called quark-flavor basis ($\alpha = q$, $\beta = s$):

$$J^q_{\mu 5} = \frac{1}{\sqrt{2}}(\bar{u}\gamma_\mu\gamma_5 u + \bar{d}\gamma_\mu\gamma_5 d), \quad J^s_{\mu 5} = \bar{s}\gamma_\mu\gamma_5 s. \quad (28)$$

The elements of matrix $\mathbf{F}$ are the meson decay constants defined in (9). The vector $\mathbf{\Phi}$ of physical fields contains the fields of η and η' mesons ϕ_η and $\phi_{\eta'}$ and the fields of higher mass states, which we denote as $\phi_G, \dots$.

It is possible and common to get an additional model constraint for the matrix $\mathbf{F}$ which is fulfilled by applying the respective mixing scheme. Let us first introduce the new vector of fields $\tilde{\mathbf{\Phi}}$ relating the $SU(3)$ symmetry property or the quark-flavor contents of currents and meson fields. The first two components of $\tilde{\mathbf{\Phi}}$ are labeled with the same indices α and β as the currents and correspond to the same symmetry ($\alpha = 8$, $\beta = 0$) or quark content ($\alpha = q$, $\beta = s$). The relation between $\mathbf{\Phi}$ and $\tilde{\mathbf{\Phi}}$ is provided by the orthogonal transformation $\mathbf{U}$

$$\tilde{\mathbf{\Phi}} = \mathbf{U\Phi}, \qquad \tilde{\mathbf{\Phi}} \equiv \begin{pmatrix} \tilde\phi_\alpha \\ \tilde\phi_\beta \\ \tilde\phi_G \\ \vdots \end{pmatrix}. \quad (29)$$

In terms of these fields, Eq. (26) can be rewritten as

$$\partial_\mu \mathbf{J}_{\mu 5} = \tilde{\mathbf{F}}\tilde{\mathbf{M}}\tilde{\mathbf{\Phi}}, \quad (30)$$

where $\tilde{\mathbf{F}} = \mathbf{FU}$, $\tilde{\mathbf{M}} = \mathbf{U^T M U}$.

In our notations the octet-singlet (quark-flavor) mixing scheme implies that the matrix $\tilde{\mathbf{F}}$ has a (rectangular) diagonal form in the respective octet-singlet (quark-flavor) basis,

$$\tilde{\mathbf{F}} = \begin{pmatrix} f_\alpha & 0 & 0 & \dots \\ 0 & f_\beta & 0 & \dots \end{pmatrix}. \quad (31)$$

This relation can be obtained from the effective Lagrangian $\mathcal{L}$ which contains an interaction term $\Delta\mathcal{L}_{\mathrm{int}} = \frac{1}{2}\tilde{\mathbf{\Phi}}^{\mathbf{T}}\tilde{\mathbf{M}}\tilde{\mathbf{\Phi}} = \frac{1}{2}\sum_{i,j}\tilde{m}^2_{ij}\tilde\phi_i\tilde\phi_j$:

$$\partial_\mu J^a_{\mu 5} = f_a \frac{\delta\mathcal{L}}{\delta\tilde\phi_a} = f_a \sum_k \tilde{m}^2_{ak}\tilde\phi_k, \qquad a = \alpha, \beta. \quad (32)$$

Note that from the requirement that matrix $\mathbf{FU}$ has a (rectangular) diagonal form (31) immediately follows that $\mathbf{FF^T}$ is a diagonal matrix. So, imposing the mixing scheme is equivalent to imposing the constraint for the decay constants:

$$f^\alpha_\eta f^\beta_\eta + f^\alpha_{\eta'} f^\beta_{\eta'} + f^\alpha_G f^\beta_G + \dots = 0. \quad (33)$$

Here the sum is over all physical meson states included in the vector $\mathbf{\Phi}$.

If we restrict ourselves to consideration of the η and η' mesons only, then the decay constants form a 2×2 matrix and in the octet-singlet and quark-flavor bases satisfy the respective diagonality constraints,

$$f^8_\eta f^0_\eta + f^8_{\eta'} f^0_{\eta'} = 0, \quad (34)$$

$$f^q_\eta f^s_\eta + f^q_{\eta'} f^s_{\eta'} = 0. \quad (35)$$

For instance, in the case of the octet-singlet mixing scheme (34), the matrix of decay constants can be expressed in terms of one mixing angle θ and two parameters f_8, f_0, forming the well-known one-angle mixing scheme:

$$\mathbf{F_{80}} = \begin{pmatrix} f^8_\eta & f^8_{\eta'} \\ f^0_\eta & f^0_{\eta'} \end{pmatrix} = \begin{pmatrix} f_8\cos\theta & f_8\sin\theta \\ -f_0\sin\theta & f_0\cos\theta \end{pmatrix}. \quad (36)$$

Similarly, if the quark-flavor mixing scheme restriction (35) is applied, then it is common to express the decay constants in terms of parameters ϕ, f_8, f_0,

$$\mathbf{F_{qs}} = \begin{pmatrix} f^q_\eta & f^q_{\eta'} \\ f^s_\eta & f^s_{\eta'} \end{pmatrix} = \begin{pmatrix} f_q\cos\phi & f_q\sin\phi \\ -f_s\sin\phi & f_s\cos\phi \end{pmatrix}. \quad (37)$$

While either of the mixing schemes (octet-singlet or quark-flavor) is self-consistent, they are incompatible [9,65]. Indeed, octet-singlet and quark-flavor bases of axial currents are related by means of a rotation matrix:

$$\begin{pmatrix} J^8_{\mu 5} \\ J^0_{\mu 5} \end{pmatrix} = \mathbf{V}(\alpha)\begin{pmatrix} J^q_{\mu 5} \\ J^s_{\mu 5} \end{pmatrix}, \qquad \mathbf{V}(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}, \quad (38)$$

where $\tan\alpha = \sqrt{2}$. Then, as follows from (26), the matrices of decay constants $\mathbf{F}_{\alpha\beta} \equiv \mathbf{F}$ (27) in the octet-singlet (36) and quark-flavor (37) bases are related as

$$\mathbf{F_{80}} = \mathbf{V}(\alpha)\mathbf{F_{qs}}, \quad (39)$$

and so

$$\mathbf{F_{80}}\mathbf{F_{80}^T} = \mathbf{V}(\alpha)\mathbf{F_{qs}}\mathbf{F_{qs}^T}\mathbf{V}(\alpha)^\mathbf{T}. \quad (40)$$

We see that in the general case, the decay constants cannot follow the octet-singlet and quark-flavor mixing scheme simultaneously (since the matrices $\mathbf{F_{80}}\mathbf{F_{80}^T}$ and $\mathbf{F_{qs}}\mathbf{F_{qs}^T}$ cannot be diagonal simultaneously). The bases are compatible only if $f_8 = f_0$ ($f_q = f_s$), i.e., in case of the exact $SU(3)_f$ symmetry.

Although the octet-singlet mixing scheme is more convenient for the anomaly sum rule relations, which are exact in the cases of isovector and octet channels, there are arguments from chiral perturbation theory against it [67,68] (see also Refs. [65,69]).

The mixing scheme-independent extraction of the decay constants from the experimental data can finally tell us which basis is more adequate for describing the mixing in the η-η' system. This problem will be addressed in the next section.

IV. OCTET CHANNEL: NUMERICAL ANALYSIS

In Sec. II, as a consequence of the ASR in the octet channel, we obtained the relation between transition form factors and decay constants of η and η' mesons. In this section we use this relation to analyze the decay constants in different mixing schemes, described in Sec. III.

First, let us consider the *octet-singlet mixing scheme*. In order to determine the mixing parameters of this scheme, we employ the mixing scheme constraint (34) and the ASR relations (21), (23), and (24). In terms of the parameters (36), Eq. (23) reads $s_0^{(8)} = 4\pi^2 f_8^2$. Then, the regions in f_8, θ parameter space which are constrained by the fit of Eq. (21) (χ^2/d.o.f. < 1) to the *BABAR* data [13] and by Eq. (24) [the experimental errors of $\Gamma_{\eta(\eta')\to 2\gamma}$ are taken into account] are shown in Fig. 2. The yellow intersection determines the parameters, which can be estimated as $f_8 = (0.88 \pm 0.04)f_\pi$, $\theta = -(14.2 \pm 0.7)°$.

In order to determine the constant f_0 [which does not enter Eqs. (21), (23), and (24) in the case of the octet-singlet mixing scheme], we need an additional constraint.

As an additional constraint, it is convenient to use the ratio of radiative decays of J/Ψ for which the more cumbersome contribution of the gluonic anomaly is under control. Indeed, according to Novikov *et al.* [70], the radiative decays $J/\Psi \to \eta(\eta')\gamma$ are dominated by the nonperturbative gluonic matrix elements, and the ratio of the decay rates $R_{J/\Psi} = (\Gamma(J/\Psi) \to \eta'\gamma)/(\Gamma(J/\Psi) \to \eta\gamma)$ is given by

$$R_{J/\Psi} = \left|\frac{\langle 0 \mid G\tilde{G} \mid \eta'\rangle}{\langle 0 \mid G\tilde{G} \mid \eta\rangle}\right|^2 \left(\frac{p_{\eta'}}{p_\eta}\right)^3, \quad (41)$$

where $p_{\eta(\eta')} = M_{J/\Psi}(1 - m^2_{\eta(\eta')}/M^2_{J/\Psi})/2$.

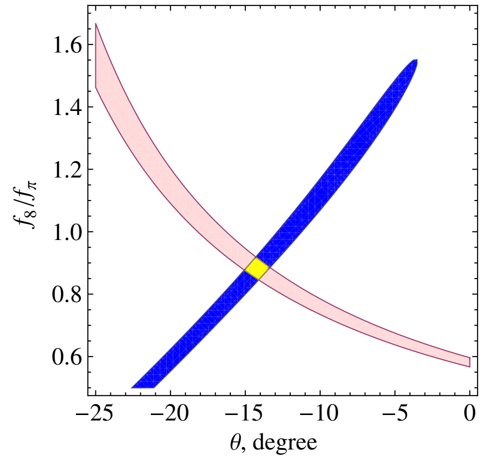

FIG. 2 (color online). Octet-singlet mixing scheme parameters f_8, θ. Dark blue region: Constraint of Eq. (21) (χ^2/d.o.f. < 1). Light red region: Constraint of Eq. (24). (Experimental uncertainties are taken into account).

Taking matrix elements of the divergencies of the singlet (1) and octet (3) currents between vacuum and $\eta(\eta')$ states and neglecting the u, d quark masses and electromagnetic anomaly term, the ratio (41) can be expressed [71] in terms of the decay constants (9) as follows:

$$R_{J/\Psi} = \left(\frac{f^8_{\eta'} + \sqrt{2}f^0_{\eta'}}{f^8_\eta + \sqrt{2}f^0_\eta}\right)^2 \left(\frac{m_{\eta'}}{m_\eta}\right)^4 \left(\frac{p_{\eta'}}{p_\eta}\right)^3. \quad (42)$$

The current experimental value of this ratio is $R_{J/\Psi} = 4.67 \pm 0.15$ [72].

Employing this ratio for the octet-singlet mixing scheme and taking into account Eqs. (21) and (24) one can determine the singlet constant: $f_0 = (0.81 \pm 0.07)f_\pi$. So, the full set of constants of the octet-singlet scheme is

$$f_8 = (0.88 \pm 0.04)f_\pi, \qquad f_0 = (0.81 \pm 0.07)f_\pi,$$
$$\theta = -(14.2 \pm 0.7)°. \quad (43)$$

For the *quark-flavor mixing scheme* we can perform a similar analysis, using the constraint of the scheme (35); Eqs. (23), (24), and (42); and fitting the ASR (21) to the *BABAR* data [13]. The decay constants of the quark-flavor basis $f^{q,s}_{\eta,\eta'}$ are expressed in terms of those of the octet-singlet basis $f^{8,0}_{\eta,\eta'}$ by means of Eq. (39). In terms of the mixing parameters f_q, f_s, ϕ (37), Eqs. (23) and (42) read

$$s_0^{(8)} = (4/3)\pi^2(5f_q^2 - 2f_s^2), \quad (44)$$

$$R_{J/\Psi} = (\tan\phi)^2 \left(\frac{m_{\eta'}}{m_\eta}\right)^4 \left(\frac{p_{\eta'}}{p_\eta}\right)^3. \quad (45)$$

Equation (45) determines the parameter $\phi = (38.1 \pm 0.5)°$. Then the other two parameters f_s, f_q can be estimated from Eqs. (24) and (21). The plot of the regions

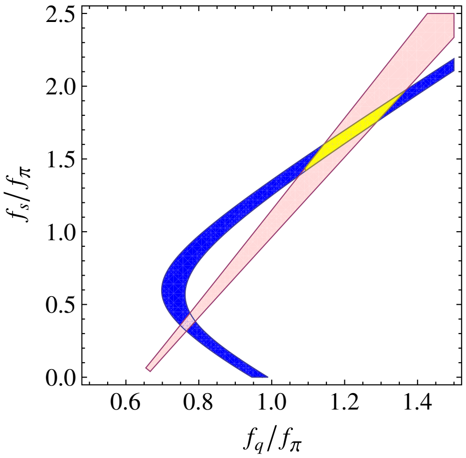

FIG. 3 (color online). Quark-flavor mixing scheme parameters f_q, f_s. Dark blue region: Constraint of Eq. (21) (χ^2/d.o.f. < 1). Light red region: Constraint of Eq. (24). (Experimental uncertainties are taken into account).

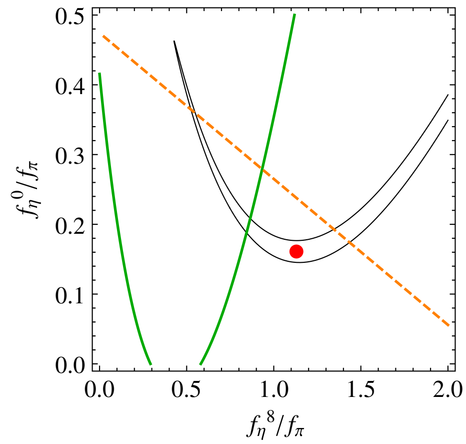

FIG. 4 (color online). Independent (of mixing scheme) estimation of decay constants. The black thin curve is a χ^2/d.o.f. $= 1$ level; the red dot is a minimum of χ^2 of Eq. (21). The green solid and orange dashed lines indicate the constraints of the octet-singlet (34) and quark-flavor (35) mixing schemes, respectively.

constrained by these equations is shown in Fig. 3. The light red band indicates the constraint of Eq. (24) [experimental errors of $R_{J/\Psi}$, $\Gamma_{\eta(\eta')\to 2\gamma}$ are taken into account] and the dark blue band indicates the fit of Eq. (21) to the *BABAR* data at the χ^2/d.o.f. < 1 level. One can observe two regions where both equations are compatible. We have chosen the physically motivated one, where $f_q, f_s > f_\pi$. So, the yellow intersection in Fig. 3 determines the parameters for the quark-flavor mixing scheme, which give the following ranges for them:

$$f_q = (1.20 \pm 0.15) f_\pi, \qquad f_s = (1.65 \pm 0.25) f_\pi, \qquad \phi = (38.1 \pm 0.5)^\circ. \tag{46}$$

The obtained mixing parameters (46) are in agreement with those obtained in other approaches [65,73,74].

It is interesting also to get the *mixing scheme independent* constraints on the decay constants. Equations (24) and (42) allow us to exclude two of the four decay constants which enter the ASR (21). The continuum threshold $s_0^{(8)}$ is excluded using (23). In Fig. 4, the levels of the χ^2/d.o.f. function of Eq. (21) in the space of constants f_η^8, f_η^0 are shown [*BABAR* experimental data on $F_{\gamma\eta(\eta')}(Q^2)$ and mean values of $R_{J/\Psi}$, $\Gamma_{\eta(\eta')\to 2\gamma}$ are used]. One can see that the χ^2/d.o.f. < 1 requirement (black curve) allows a rather wide range of the parameters. However, the minimum of χ^2 is reached at the point $f_\eta^8 = 1.11 f_\pi$, $f_\eta^0 = 0.16 f_\pi$ (χ^2/d.o.f. $= 0.84$, indicated by a red dot). Then (24) and (42) allow us to determine the other two constants: $f_{\eta'}^8 = -0.42 f_\pi$, $f_{\eta'}^0 = 1.04 f_\pi$. Therefore, the full set of decay constants of the mixing-scheme-independent extraction is

$$\begin{pmatrix} f_\eta^8 & f_{\eta'}^8 \\ f_\eta^0 & f_{\eta'}^0 \end{pmatrix} = \begin{pmatrix} 1.11 & -0.42 \\ 0.16 & 1.04 \end{pmatrix} f_\pi. \tag{47}$$

The constraints of the octet-singlet (green solid curve) (34) and quark-flavor (orange dashed curve) (35) are also depicted in Fig. 4. We see that both considered mixing schemes are consistent with the scheme-independent analysis based on the ASR at the level of χ^2/d.o.f. < 1, even if other experimental errors are not taken into account. However, the least χ^2 is reached in the region lying outside of both mixing scheme curves. Further improvement of the experimental data can clear up the question of the validity of different schemes and give more precise values of the mixing parameters.

One can expect that the possible non-OPE correction to the spectral density, discussed for the isovector channel, will manifest in the same way in the octet channel also. Although the *BABAR* data [13] do not show such a sturdy growth of $Q^2 F_{\eta\gamma}$ and $Q^2 F_{\eta'\gamma}$ as they do for $Q^2 F_{\pi\gamma}$, the octet combination of the transition form factors $Q^2(f_\eta^8 F_{\eta\gamma} + f_{\eta'}^8 F_{\eta'\gamma})$ does reveal a possible growth [75].

Expecting the similarity of the correction to the spectral density in the isovector and octet channels, we suppose that the correction in the octet channel has the same form as (19),

$$\delta I_8 = -\int_{s_0^{(8)}}^{\infty} \delta A_3^{(8)} ds = \frac{1}{2\sqrt{6}\pi} \frac{\lambda s_0^{(8)} Q^2}{(s_0^{(8)} + Q^2)^2} \left(\ln \frac{Q^2}{s_0^{(8)}} + \sigma \right). \tag{48}$$

This correction results in an additional term $\delta I_8/\pi$ in the rhs of (21),

$$f_\eta^8 F_{\eta\gamma}(Q^2) + f_{\eta'}^8 F_{\eta'\gamma}(Q^2) = \frac{1}{2\sqrt{6}\pi^2} \frac{s_0^{(8)}}{s_0^{(8)} + Q^2} + \frac{1}{2\sqrt{6}\pi^2} \times \frac{\lambda s_0^{(8)} Q^2}{(s_0^{(8)} + Q^2)^2} \left(\ln \frac{Q^2}{s_0^{(8)}} + \sigma \right). \tag{49}$$

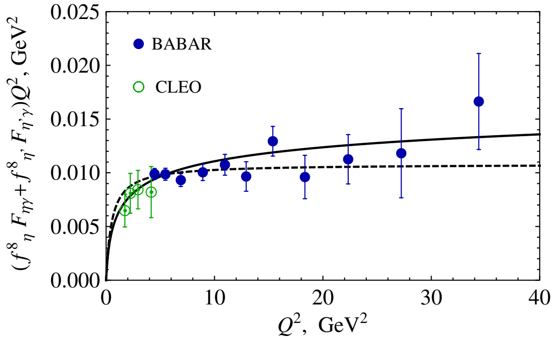

FIG. 5 (color online). ASR with correction [(49), solid line] and without correction [(21), dashed line] for the octet-singlet mixing scheme parameters (43) compared with experimental data.

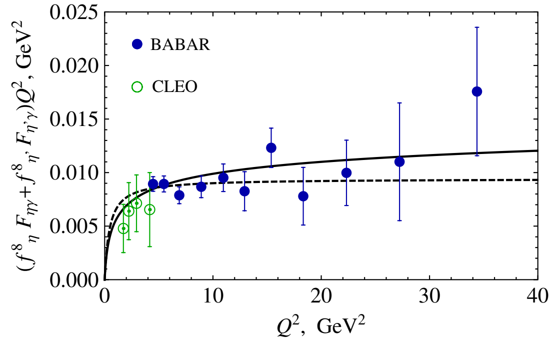

FIG. 7 (color online). ASR with correction [(49), solid line] and without correction [(21), dashed line] for the mixing-scheme-independent parameters (47) compared with experimental data.

The plots of Eq. (49) (solid line) and Eq. (21) (dashed line) for different mixing schemes are shown in Figs. 5–7. The parameters $\lambda = 0.14$, $\sigma = -2.36$ are taken to be the same as those obtained for the pion case (data set II); the decay constants (central values) for different mixing schemes are employed from (43), (46), and (47). We see that the current precision of the experimental data on the η and η' transition form factors could accommodate the same correction as in the pion case but does not require it.

Finally, let us make the following remark. In this paper we developed the approach which does not rely on introduction of the nonphysical states which have no definite masses. At the same time, a hypothesis is widely discussed in the literature (see, e.g., Refs. [13,76]) that the transition form factor of the nonphysical state $|q\rangle \equiv \frac{1}{\sqrt{2}}(|\bar{u}u\rangle + |\bar{d}d\rangle)$ is related to the pion form factor as $F_{q\gamma}(Q^2) = (5/3)F_{\pi\gamma}(Q^2)$ [where the numerical factor comes from the quark charges $(e_u^2 + e_d^2)/(e_u^2 - e_d^2) = 5/3$]. The states $|q\rangle$ and $|s\rangle \equiv |\bar{s}s\rangle$ are assumed to be expressed in terms of the physical states $|\eta\rangle$, $|\eta'\rangle$ via the quark-flavor mixing scheme [77],

$$|q\rangle = \cos\phi|\eta\rangle + \sin\phi|\eta'\rangle,$$
$$|s\rangle = -\sin\phi|\eta\rangle + \sin\phi|\eta'\rangle. \qquad (50)$$

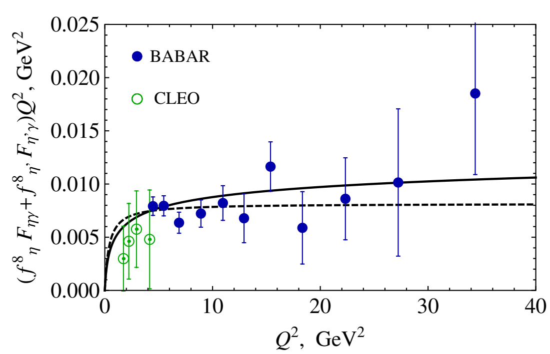

FIG. 6 (color online). ASR with correction [(49), solid line] and without correction [(21), dashed line] for the quark-flavor mixing scheme parameters (46) compared with experimental data.

Then one can relate the form factors:

$$\frac{5}{3}F_{\pi\gamma} = F_{\eta\gamma}\cos\phi + F_{\eta'\gamma}\sin\phi. \qquad (51)$$

Let us now try to incorporate this hypothesis into our approach. For this purpose, combining (21) and (51) and using (13) and (39), we can express the η, η' transition form factors in terms of the constants f_q, f_s, ϕ,

$$F_{\eta\gamma}(Q^2) = \frac{1}{4\pi^2 f_s f_\pi}\frac{s_0^{(3)}(\sqrt{2}f_s\cos\phi - f_q\sin\phi)}{s_0^{(3)} + Q^2} + \frac{1}{4\pi^2 f_s}\frac{s_0^{(8)}\sin\phi}{s_0^{(8)} + Q^2}, \qquad (52)$$

$$F_{\eta'\gamma}(Q^2) = \frac{1}{4\pi^2 f_s f_\pi}\frac{s_0^{(3)}(\sqrt{2}f_s\sin\phi + f_q\cos\phi)}{s_0^{(3)} + Q^2} + \frac{1}{4\pi^2 f_s}\frac{s_0^{(8)}\cos\phi}{s_0^{(8)} + Q^2}, \qquad (53)$$

where $s_0^{(3)} = 4\pi^2 f_\pi^2$, $s_0^{(8)} = (4/3)\pi^2(5f_q^2 - 2f_s^2)$.

The plot of Eqs. (52) and (53) with constants from our analysis (46) $f_q = 1.20f_\pi$, $f_s = 1.65f_\pi$, $\phi = 38.1°$ in comparison with experimental data is shown in Fig. 8.

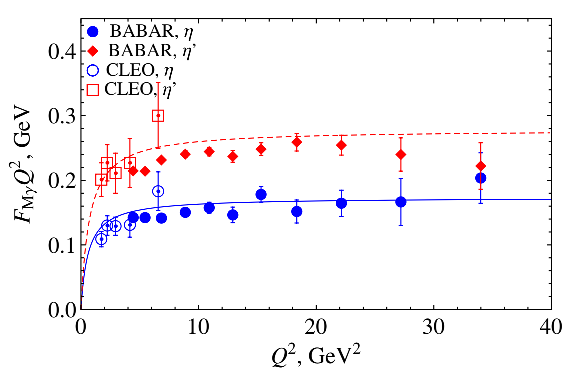

FIG. 8 (color online). Combinations $F_{\eta\gamma}Q^2$ (blue solid line) and $F_{\eta'\gamma}Q^2$ (red dashed line) from Eqs. (52) and (53), respectively, as functions of Q^2 compared with experimental data.

One can observe a reasonably good agreement with the experimental data. For the decay constants of Ref. [65], one also gets a good description.

The agreement with the experimental data may indicate that the effect of a strong anomaly for the $\frac{1}{\sqrt{2}}|\bar{u}u + \bar{d}d\rangle$ state is small and the strong anomaly predominantly appears in the $\bar{s}s$ channel. This statement can be rigorously checked by means of the anomaly sum rule for the singlet channel, which we postpone to future work.

V. SUMMARY AND DISCUSSION

The exact anomaly sum rule allows us to derive an expression for the pion transition form factor at arbitrary Q^2, giving the proof for the Brodsky-Lepage interpolation formula. At $Q^2 = 0$ it is related to the pion decay width, while at large Q^2, basing on the factorization approach, it allows us to define the pion interval of duality which numerically appears to be close to the value defined from the two-point correlator sum rule analysis. However, the proposed approach may be applied even when the factorization is broken. It was exactly the situation supported by *BABAR* data requiring a small nonperturbative correction [7] to the continuum spectral density. Having dimension 2, it cannot appear in (a local) OPE and should be attributed to, say, instantons or short strings.

In this work we included in our analysis the recent Belle Collaboration data. The main conclusions are the following. Although the Belle data themselves may be described without the mentioned correction, they do not also exclude its possibility. Unless the *BABAR* data is disproved, the need for the correction remains. This is supported by Table I, where the fits for various combinations of the data are shown.

The search for the discussed correction can be performed also by means of lattice simulations, which already provided evidence (see, e.g., Ref. [78] and references therein) for non-OPE vacuum condensates. In our case one may study the three-point VVA correlator on the lattice in a way similar to Ref. [79]. To be sensitive to the discussed correction, one should consider moderately large momentum transfer in one of the vector channels.

The corrections in the VVA correlator can be also studied analytically by generalization of the approach used in Ref. [58] to the case of the three-point correlation function. Some indications of dimension 2 corrections can also be obtained by the refined analysis [80] of e^+e^- annihilation data.

In the generalization of our approach to the η and η' mesons, the data may be described without such a correction. However, the possibility of the correction, similar to that discussed for the pion case, is not excluded by the current experimental data and is even supported by the slight growth of the octet combination of transition form factors. So we can conclude that the correction to the spectral density, first introduced in Ref. [7], seems to be universal for both isovector and octet channels. This conclusion is in agreement with a recent discussion in Ref. [11].

The mixing plays a special role in the octet channel. There are two mixing models on the market now: the octet-singlet and quark-flavor mixing schemes. We reformulate these models without introducing the nonphysical states with indefinite masses. Each scheme implies a certain constraint for the meson decay constants (34) and (35). Both mixing models are compatible only in the exact $SU(3)_f$ limit.

Using the data on the transition form factors of the η, η' mesons, the ASR allows us to extract the set of decay constants in the octet-singlet and quark-flavor schemes, as well as in the mixing-scheme-independent way, if we add an additional constraint of the ratio of radiation decays of the J/ψ meson (42). It is shown that the current data precision permits both the octet-singlet and quark-flavor mixing schemes. Future improvements to experimental data on transition form factors of η, η' mesons and the ratio $R_{J/\psi}$, expected from the Belle and BES-III collaborations, can determine which scheme is more suitable for the description of mixing in the η-η' system.

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