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Shear and bulk viscosities of a weakly coupled quark gluon plasma with finite chemical potential and temperature: Leading-log results

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We calculate the shear (η) and bulk (ζ) viscosities of a weakly coupled quark gluon plasma at the leading-log order with finite temperature T and quark chemical potential μ. We find that the shear viscosity to entropy density ratio η/s increases monotonically with μ and eventually scales as $(\mu/T)^2$ at large μ. In contrast, ζ/s is insensitive to μ. Both η/s and ζ/s are monotonically decreasing functions of the quark flavor number N_f when $N_f \geq 2$. This property is also observed in pion gas systems. Our perturbative calculation suggests that QCD becomes the most perfect (i.e. with the smallest η/s) at $\mu = 0$ and $N_f = 16$ (the maximum N_f with asymptotic freedom). It would be interesting to test whether the currently smallest η/s computed close to the phase transition with $\mu = 0$ and $N_f = 0$ can be further reduced by increasing N_f.

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I. INTRODUCTION

Viscosity, diffusivity, and conductivity are transport coefficients which characterize the dynamics of long wavelength and low frequency fluctuations in a medium. The quantity shear viscosity (η) per entropy density (s) has attracted a lot of attention because of the intriguing conjecture that η/s has a minimum bound of $1/4\pi$ for all systems [1]. This conjecture is inspired by the anti-de Sitter space/conformal field theory correspondence (AdS/CFT) [2–4] which is rooted in string theory. Surprisingly, the hot and dense matter produced at RHIC [5–8] (for reviews, see e.g. Refs. [9–12]) just above the phase transition temperature (T_c) has $\eta/s = 0.1 \pm 0.1(\text{theory}) \pm 0.08(\text{experiment})$ [13], a value close to the conjectured bound. A robust limit of $1/(4\pi) \leq \eta/s \leq 2.5/(4\pi)$ at $T_c \leq T \leq 2T_c$ was recently extracted from a VISHNU hybrid model [14] and a lattice computation of gluon plasma yields $\eta/s = 0.102(56)$ at temperature $T = 1.24T_c$ [15].

The QCD transport coefficients have also been studied in other temperatures. When $T \gg T_c$, the η of a weakly interacting quark gluon plasma is inversely proportional to the scattering rate, $\eta \propto 1/\Gamma \propto 1/\alpha_s^2 \ln\alpha_s^{-1}$ [16], where α_s is the strong coupling constant. The bulk viscosity ζ is suppressed by an additional factor of $(T^\mu_\mu)^2$, arising from the response of the trace of the energy momentum tensor (T^μ_μ) to a uniform expansion. Thus, ζ vanishes when the system is "conformal" or scale invariant. For a gluon plasma, the running of the coupling constant breaks the scale invariance. Thus, $T^\mu_\mu \propto \beta(\alpha_s) \propto \alpha_s^2$, $\zeta \propto \alpha_s^2/\ln\alpha_s^{-1}$ [17]. In the perturbative region, $\zeta/\eta \propto \alpha_s^4 \ll 1$. When $T \ll T_c$, the effective degrees of freedom are pions. In the chiral limit (u and d quarks are massless), $\eta/s \propto f_\pi^4/T^4$ [18] and $\zeta/s \propto T^4/f_\pi^4$ [19] where f_π is the pion decay constant. A compilation of perturbative QCD calculations of η and ζ can be found, e.g., in Refs. [20,21]. Most of these calculations are performed with finite T but zero quark chemical potential μ.

The purpose of this work is to extend the previous perturbative QCD calculation of η and ζ to finite μ at the leading-log (LL) order. At this order, we find $\eta \propto 1/\alpha_s^2 \ln\alpha_s^{-1}$ and $\zeta \propto \alpha_s^2/\ln\alpha_s^{-1}$, the same as in the limit of $\mu = 0$, which give parametrically the dominant contribution at asymptotically large (compared with the Λ_{QCD} scale where QCD becomes nonperturbative) T or μ. The vacuum in our calculation has no spontaneous symmetry breaking; thus, it cannot be applied to the color superconducting phase in the $\mu/T \to \infty$ limit. In the context of finding the minimal η/s and hence the most "perfect" fluid, we explore whether η/s can be further reduced by varying μ and the quark flavor number N_f in the hope that our perturbative calculation can shed light on the nonperturbative region near T_c where QCD is found to be the most perfect matter ever produced in nature.

II. EFFECTIVE KINETIC THEORY

η and ζ can be calculated through the linearized response function of thermal equilibrium states using the Kubo formula. In the leading order (LO) expansion in the coupling constant, the computation involves an infinite number of diagrams [22–25]. However, it has been shown

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that the summation of the LO diagrams in a weakly coupled ϕ^4 theory [22,26] or in hot QED [27] is equivalent to solving the linearized Boltzmann equation with temperature-dependent particle masses and scattering amplitudes. This conclusion is expected to hold in perturbative QCD as well.

The Boltzmann equation of a quark gluon plasma describes the evolution of the color and spin averaged distribution function $\tilde{f}_p^a(x)$ for particle a:

$$\frac{d\tilde{f}_p^a(x)}{dt} = \mathcal{C}_a, \tag{1}$$

where $\tilde{f}_p^a(x)$ is a function of space-time $x^\mu = (t, \mathbf{x})$ and momentum $p^\mu = (E_p, \mathbf{p})$.

The LL contribution comes from two-particle scattering processes denoted as $ab \leftrightarrow cd$. Near forward scattering, the singularity similar to that of Rutherford scattering is regularized by the Debye and dynamical screenings which give thermal masses to particles. This yields a logarithmic enhancement factor $\ln(q_{\max}^2/m^2)$ where $q_{\max}$ is the size of the maximum momentum transferred in the t-channel process and m is the thermal mass. $q_{\max}$ is set by T or μ, whichever is bigger, and m has the expression shown in Eqs. (8) and (9). Thus $\ln(q_{\max}^2/m^2)$ scales as $\ln(1/\alpha_s)$.

For the LL calculation, we only need to consider two-particle scattering with the collision terms of the form

$$C_{ab\leftrightarrow cd} \equiv \int_{k_1k_2k_3} d\Gamma_{ab\to cd}[\tilde{f}_{k_1}^a\tilde{f}_{k_2}^b\tilde{F}_p^c\tilde{F}_{k_3}^d - \tilde{F}_{k_1}^a\tilde{F}_{k_2}^b\tilde{f}_p^c\tilde{f}_{k_3}^d], \tag{2}$$

where $\tilde{F}^g = 1 + \tilde{f}^g$ and $\tilde{F}^{q(\bar{q})} = 1 - \tilde{f}^{q(\bar{q})}$ and

$$d\Gamma_{ab\to cd} = \frac{1}{2E_p}|M_{ab\to cd}|^2 \prod_{i=1}^{3} \frac{d^3k_i}{(2\pi)^3 2E_{k_i}} \times (2\pi)^4\delta^{(4)}(k_1 + k_2 - k_3 - p), \tag{3}$$

where $|M_{ab\to cd}|^2$ is the matrix element squared with all colors and helicities of the initial and final states summed over. They are tabulated in Table I in the Appendix. Then the collision term for a quark of flavor a is

$$\begin{aligned} N_q\mathcal{C}_{q_a} = {}& \frac{1}{2}C_{q_aq_a\leftrightarrow q_aq_a} + C_{q_a\bar{q}_a\leftrightarrow q_a\bar{q}_a} + \frac{1}{2}C_{gg\leftrightarrow q_a\bar{q}_a} \\ & + C_{q_ag\leftrightarrow q_ag} + \sum_{b,b\neq a}(C_{q_aq_b\leftrightarrow q_aq_b} + C_{q_a\bar{q}_b\leftrightarrow q_a\bar{q}_b} \\ & + C_{q_b\bar{q}_b\leftrightarrow q_a\bar{q}_a}), \end{aligned} \tag{4}$$

where $N_q = 2\times 3 = 6$ is the quark helicity and color degeneracy factor and the factor $1/2$ is included when the initial state is formed by two identical particles. Similarly,

$$N_g\mathcal{C}_g = \frac{1}{2}C_{gg\leftrightarrow gg} + \sum_a(C_{gq_a\leftrightarrow gq_a} + C_{g\bar{q}_a\leftrightarrow g\bar{q}_a} + C_{q_a\bar{q}_a\leftrightarrow gg}), \tag{5}$$

where $N_g = 2\times 8 = 16$ is the gluon helicity and color degeneracy factor. In equilibrium, the distributions are denoted as $f^{q(\bar{q})}$ and f^g, with

$$f_p^g = \frac{1}{e^{u\cdot p/T} - 1}, \tag{6}$$

TABLE I. Matrix elements squared for two-particle scattering processes in QCD. The helicities and colors of all initial and final state particles are summed over. q_1 and q_2 represent quarks of distinct flavors, $\bar{q}_1$ and $\bar{q}_2$ are the associated antiquarks, and g represents a gluon. d_F and d_A denote the dimensions of the fundamental and adjoint representations of the $SU_c(N)$ gauge group while C_F and C_A are the corresponding quadratic Casimirs. In a $SU_c(3)$ theory with fundamental representation fermions, $d_F = C_A = 3$, $C_F = 4/3$, and $d_A = 8$.

$ab \to cd$	$\lvert M_{a(k_1)b(k_2)\to c(k_3)d(k_4)}\rvert^2$
$q_1q_2 \to q_1q_2$ $\bar{q}_1q_2 \to \bar{q}_1q_2$ $q_1\bar{q}_2 \to q_1\bar{q}_2$ $\bar{q}_1\bar{q}_2 \to \bar{q}_1\bar{q}_2$	$8g^4\frac{d_F^2C_F^2}{d_A}(\frac{s^2+u^2}{t^2})$
$q_1q_1 \to q_1q_1$ $\bar{q}_1\bar{q}_1 \to \bar{q}_1\bar{q}_1$	$8g^4\frac{d_F^2C_F^2}{d_A}(\frac{s^2+u^2}{t^2} + \frac{s^2+t^2}{u^2}) + 16g^4d_FC_F(C_F - C_A/2)\frac{s^2}{tu}$
$q_1\bar{q}_1 \to q_1\bar{q}_1$	$8g^4\frac{d_F^2C_F^2}{d_A}(\frac{s^2+u^2}{t^2} + \frac{t^2+u^2}{s^2}) + 16g^4d_FC_F(C_F - C_A/2)\frac{u^2}{st}$
$q_1\bar{q}_1 \to q_2\bar{q}_2$	$8g^4\frac{d_F^2C_F^2}{d_A}\frac{t^2+u^2}{s^2}$
$q_1\bar{q}_1 \to gg$	$8g^4d_FC_F^2(\frac{u}{t} + \frac{t}{u}) - 8g^4d_FC_FC_A(\frac{t^2+u^2}{s^2})$
$q_1g \to q_1g$ $\bar{q}_1g \to \bar{q}_1g$	$-8g^4d_FC_F^2(\frac{u}{s} + \frac{s}{u}) + 8g^4d_FC_FC_A(\frac{s^2+u^2}{t^2})$
$gg \to gg$	$16g^4d_AC_A^2(3 - \frac{su}{t^2} - \frac{st}{u^2} - \frac{tu}{s^2})$

$$f_p^{q(\bar{q})} = \frac{1}{e^{(u\cdot p\mp\mu)/T}+1}, \tag{7}$$

where T is the temperature, u is the fluid four-velocity and μ is the quark chemical potential. They are all space-time dependent.

The thermal masses of gluons and quarks or antiquarks for external states (the asymptotic masses) can be computed via [28,29]

$$m_g^2 = \frac{2g^2}{d_A}\int\frac{d^3p}{(2\pi)^3 2E_p}[N_g C_A f_p^g + N_f N_q C_F(f_p^q + f_p^{\bar{q}})], \tag{8}$$

$$m_q^2 = m_{\bar{q}}^2 = 2C_F g^2\int\frac{d^3p}{(2\pi)^3 2E_p}(2f_p^g + f_p^q + f_p^{\bar{q}}), \tag{9}$$

where $d_A = 8$, $C_A = 3$, and $C_F = 4/3$. This yields

$$m_g^2 = \frac{C_A}{6}g^2T^2 + \frac{N_f C_F}{16}g^2\Big(T^2 + \frac{3}{\pi^2}\mu^2\Big), \tag{10}$$

$$m_q^2 = \frac{1}{4}C_F g^2\Big(T^2 + \frac{\mu^2}{\pi^2}\Big), \tag{11}$$

where we have set $E_p = |\mathbf{p}|$ in the integrals on the right-hand sides of Eqs. (8) and (9). The difference from nonvanishing masses is of higher order.

A. Linearized Boltzmann equation

To extract transport coefficients, it is sufficient to consider infinitesimal perturbations away from equilibrium which have infinite wave lengths. Using the Chapman-Enskog expansion we linearize the Boltzmann equation to the first order in the derivative expansion in x. Thus, we only need the thermal equilibrium distributions f^g and $f^{q,\bar{q}}$ on the left-hand side of the Boltzmann equation. We can also make use of the zeroth-order energy-momentum conservation relation, $\partial_\mu T^{(0)\mu\nu} = 0$, to replace the time derivatives $\partial T/\partial t$ and $\partial\mu/\partial t$ with spatial gradients:

$$\begin{aligned}\frac{\partial T}{\partial t} &= -T\Big(\frac{\partial P}{\partial\epsilon}\Big)_n\nabla\cdot\mathbf{u},\\ \frac{\partial\mu}{\partial t} &= -\Big[\mu\Big(\frac{\partial P}{\partial\epsilon}\Big)_n + \Big(\frac{\partial P}{\partial n}\Big)_\epsilon\Big]\nabla\cdot\mathbf{u},\end{aligned} \tag{12}$$

where $n \equiv n_q - n_{\bar{q}}$ is the baryon number density. We work in the local rest frame of the fluid element where $u = (1, 0, 0, 0)$ which implies $\partial_\mu u^0 = 0$ after taking a derivative on $u^\mu(x)u_\mu(x) = 1$. For the right-hand side of the Boltzmann equation, we expand the distribution function of particle a as a local equilibrium distribution plus a correction

$$\tilde{f}^a \simeq f^a[1 - \chi^a(1 \pm f^a)], \tag{13}$$

and χ^a can be parametrized as

$$\chi^a(x,p) = \Big[\frac{A^a(p)}{T}\nabla_i u_i + \frac{B^a(p)}{T}\hat{p}_{[i}\hat{p}_{j]}\nabla_{[i}u_{j]}\Big], \tag{14}$$

where $i, j = 1, 2, 3$ and $[\ldots]$ means the enclosed indices are made symmetric and traceless.

Applying Eq. (13) to the right-hand side of the Boltzmann equation and equating it to the left-hand side, we get linear equations for $B^a(p)$ and $A^a(p)$. For $\tilde{f}^g$, we obtain

$$p_{[i}p_{j]} = \frac{E_g}{f^g F^g}\frac{1}{N_g}\Big[\frac{1}{2}B^{ij}_{gg\leftrightarrow gg} + \sum_{a=1}^{N_f}(B^{ij}_{gq_a\leftrightarrow gq_a} + B^{ij}_{g\bar{q}_a\leftrightarrow g\bar{q}_a} + B^{ij}_{q_a\bar{q}_a\leftrightarrow gg})\Big], \tag{15}$$

where

$$B^{ij}_{ab\leftrightarrow cd}(p) \equiv \int_{k_1k_2k_3} d\Gamma_{ab\to cd} f^a_{k_1} f^b_{k_2} F^c_p F^d_{k_3}[-B^a_{ij}(k_1) - B^b_{ij}(k_2) + B^c_{ij}(p) + B^d_{ij}(k_3)] \tag{16}$$

with $B^a_{ij}(p) = B^a(p)\hat{p}_{[i}\hat{p}_{j]}$ and we have suppressed the p dependence in B^a_{ij} and B^a when there is no ambiguity. Similarly, for $\tilde{f}^q$, we obtain

$$p_{[i}p_{j]} = \frac{E_q}{f^q F^q}\frac{1}{N_q}\Big[\frac{1}{2}B^{ij}_{q_1q_1\leftrightarrow q_1q_1} + B^{ij}_{q_1\bar{q}_1\leftrightarrow q_1\bar{q}_1} + \frac{1}{2}B^{ij}_{gg\leftrightarrow q_1\bar{q}_1} + B^{ij}_{q_1g\leftrightarrow q_1g} + \sum_{a=2}^{N_f}(B^{ij}_{q_1q_a\leftrightarrow q_1q_a} + B^{ij}_{q_1\bar{q}_a\leftrightarrow q_1\bar{q}_a} + B^{ij}_{q_a\bar{q}_a\leftrightarrow q_1\bar{q}_1})\Big]. \tag{17}$$

The corresponding equation for $\tilde{f}^{\bar{q}}$ can be obtained from the above equation by interchanging $q \leftrightarrow \bar{q}$. The linear equations for $B^a(p)$ are relevant to the shear viscosity computation. They can be written in a compact form

$$|S_\eta\rangle = \mathcal{C}_\eta|B\rangle, \tag{18}$$

with $p_{[i}p_{j]}$ taken as the source $|S_\eta\rangle$ for the shear viscosity.

Similarly, the linear equations for $A^a(p)$ for the quark and gluon are given by

$$\begin{aligned}&\frac{p^2}{3} - \Big(E_g^2 - T^2\frac{\partial m_g^2}{\partial T^2} - \mu^2\frac{\partial m_g^2}{\partial\mu^2}\Big)\Big(\frac{\partial P}{\partial\epsilon}\Big)_n + E_g\Big(\frac{\partial E_g}{\partial\mu} - a_g\Big)\Big(\frac{\partial P}{\partial n}\Big)_\epsilon\\ &= \frac{E_g}{f^g F^g}\frac{1}{N_g}\Big[\frac{1}{2}A_{gg\leftrightarrow gg} + \sum_{a=1}^{N_f}(A_{gq_a\leftrightarrow gq_a} + A_{g\bar{q}_a\leftrightarrow g\bar{q}_a} + A_{q_a\bar{q}_a\leftrightarrow gg})\Big],\end{aligned} \tag{19}$$

and

$$\frac{p^2}{3}-\left(E_q^2-T^2\frac{\partial m_q^2}{\partial T^2}-\mu^2\frac{\partial m_q^2}{\partial\mu^2}\right)\left(\frac{\partial P}{\partial\epsilon}\right)_n+E_q\left(\frac{\partial E_q}{\partial\mu}-a_q\right)\left(\frac{\partial P}{\partial n}\right)_\epsilon=\frac{E_q}{f^qF^q}\frac{1}{N_q}\left[\frac{1}{2}A_{q_1q_1\leftrightarrow q_1q_1}+A_{q_1\bar{q}_1\leftrightarrow q_1\bar{q}_1}+\frac{1}{2}A_{gg\leftrightarrow q_1\bar{q}_1}+A_{q_1g\leftrightarrow q_1g}+\sum_{a=2}^{N_f}(A_{q_1q_a\leftrightarrow q_1q_a}+A_{q_1\bar{q}_a\leftrightarrow q_1\bar{q}_a}+A_{q_a\bar{q}_a\leftrightarrow q_1\bar{q}_1})\right], \tag{20}$$

where $a_g=0$, $a_{q/\bar{q}}=\pm 1$ and

$$A_{ab\leftrightarrow cd}(p)\equiv\int_{k_1k_2k_3}d\Gamma_{ab\to cd}f^a_{k_1}f^b_{k_2}F^c_{k_3}F^d_p[-A^a(k_1)-A^b(k_2)+A^c(k_3)+A^d(p)]. \tag{21}$$

These equations are relevant for the bulk viscosity computation. They can be written compactly as

$$|S_\zeta\rangle=\mathcal{C}_\zeta|A\rangle. \tag{22}$$

Using the expression of the thermal mass in Eq. (8), we can evaluate the prefactor of $(\partial P/\partial\epsilon)_n$ in Eq. (19) as

$$E_g^2-T^2\frac{\partial m_g^2}{\partial T^2}-\mu^2\frac{\partial m_g^2}{\partial\mu^2}=\mathbf{p}^2-\beta(g^2)\left[\frac{C_A}{6}T^2+\frac{N_ft_F}{6}\left(T^2+\frac{3}{\pi^2}\mu^2\right)\right]\equiv\mathbf{p}^2+\tilde{m}_g^2, \tag{23}$$

where $\alpha_s=g^2/4\pi$ and the QCD beta function for $g(\tilde{\mu}\equiv\sqrt{T^2+\mu^2})$ is

$$\beta(g^2)=\frac{\tilde{\mu}^2dg^2}{d\tilde{\mu}^2}=\frac{g^4}{16\pi^2}\left(\frac{2N_f-33}{3}\right). \tag{24}$$

Similarly we obtain the prefactor of $(\partial P/\partial\epsilon)_n$ in Eq. (20) as

$$E_q^2-T^2\frac{\partial m_q^2}{\partial T^2}-\mu^2\frac{\partial m_q^2}{\partial\mu^2}=\mathbf{p}^2-\beta(g^2)\left[\frac{1}{4}C_F\left(T^2+\frac{\mu^2}{\pi^2}\right)\right]\equiv\mathbf{p}^2+\tilde{m}_q^2. \tag{25}$$

As will be shown later, $(\partial P/\partial\epsilon)_n-1/3\propto\beta(g^2)$ and $(\partial P/\partial n)_\epsilon\propto\beta(g^2)$; thus the source of the bulk viscosity $|S_\zeta\rangle\propto\beta(g^2)$. This means $|S_\zeta\rangle$ is, as expected, proportional to the conformal symmetry breaking.

B. Energy-momentum tensor and quark number density

The energy-momentum tensor of the kinetic theory can be written as [22] (note the sign difference in metric with [22])

$$T^{\mu\nu}=\sum_a N_a\int\frac{d^3p}{(2\pi)^3}\frac{f^a(p,x)}{E_a}\left[p^\mu p^\nu-\frac{1}{4}m_a^2(x)g^{\mu\nu}\right], \tag{26}$$

where a sums over the gluons and N_f flavors of quarks and antiquarks. This equation expresses the total energy-momentum tensor of the system as the sum of individual quasiparticles. There is no $u^\mu u^\nu$ term on the right-hand side because energy-momentum conservation cannot be satisfied unless this term vanishes. In principle, one expects the form of Eq. (26) (and kinetic theory itself) will be no longer valid at higher orders in the expansion of the coupling constant; however, Eq. (26) does reproduce all the thermal dynamical quantities of QCD at $\mathcal{O}(\alpha_s)$ correctly [30].

Expanding $T^{\mu\nu}$ to the first order of χ^i, we have the fairly simple expression

$$\delta T^{\mu\nu}=-\sum_a N_a\int\frac{d^3p}{(2\pi)^3E_a}f^aF^a\chi^a\times\left(p^\mu p^\nu-u^\mu u^\nu T^2\frac{\partial E_a^2}{\partial T^2}-u^\mu u^\nu\mu^2\frac{\partial E_a^2}{\partial\mu^2}\right). \tag{27}$$

In deriving the above equation one needs to carefully keep track of the implicit distribution function dependence in E_a through the thermal mass (9). This expression can then be matched to hydrodynamics.

In the local rest frame of the fluid element with $\mathbf{u}(x)=0$, the most general form of T^{ij} at the order of one space-time derivative (assuming parity and time reversal symmetry) can be decomposed into shear and bulk viscosity terms

$$\delta T^{ij}=-2\eta\left(\frac{\nabla_iu_j+\nabla_ju_i}{2}-\frac{1}{3}\delta_{ij}\nabla\cdot\mathbf{u}\right)-\xi\delta_{ij}\nabla\cdot\mathbf{u}. \tag{28}$$

[Recall that $\partial T/\partial t$ and $\partial\mu/\partial t$ can be replaced by $\nabla\cdot\mathbf{u}$ as shown in Eq. (12).] Then using Eq. (14) for χ and comparing Eqs. (27) and (28), we obtain

$$\eta=\frac{1}{10T}\sum_a N_a\int\frac{d^3p}{(2\pi)^3E_a}f^aF^aB^a_{jk}p_{[j}p_{k]}, \tag{29}$$

and

$$\zeta=\frac{1}{T}\sum_a\int\frac{d^3p}{(2\pi)^3E_a}\frac{p^2}{3}f^aF^aA^a. \tag{30}$$

They can be written schematically as

$$\eta=\langle B|S_\eta\rangle,\qquad\zeta=\langle A|S'_\zeta\rangle. \tag{31}$$

Note that $|S'_\zeta\rangle\neq|S_\zeta\rangle$, but we will show $\zeta=\langle A|S'_\zeta\rangle=\langle A|S_\zeta\rangle$ later.

We also need the total quark number density

$$n = N_f N_q \int \frac{d^3p}{(2\pi)^3}(f^q - f^{\bar{q}}). \tag{32}$$

Expanding n to the first order of χ^a, we have

$$\delta n = \sum_a N_a \int \frac{d^3p}{(2\pi)^3} f^a F^a \chi^a \left(\frac{\partial E_a}{\partial \mu} - a_a\right). \tag{33}$$

Equations (27) and (33) will be used in the computation of bulk viscosity.

C. Shear viscosity

It is well known that η should not be negative such that the second law of thermodynamics (entropy cannot decrease in time) is satisfied. This requirement is fulfilled by rewriting Eqs. (18) and (31) as

$$\eta = \langle B|\mathcal{C}_\eta|B\rangle \tag{34}$$

and showing that η is quadratic in $|B\rangle$ with a positive prefactor such that it is bounded from below by zero. Indeed, these conditions are satisfied in our expression:

$$\begin{aligned}\eta = {}& D^\eta_{gg\to gg} + \frac{N_f(N_f-1)}{2}\Big(4D^\eta_{q_aq_b\to q_aq_b} + 4D^\eta_{\bar{q}_a\bar{q}_b\to\bar{q}_a\bar{q}_b} \\ &+ 8D^\eta_{q_a\bar{q}_b\to q_a\bar{q}_b} + 8D^\eta_{q_a\bar{q}_a\to q_b\bar{q}_b}\Big)_{a\neq b} \\ &+ N_f\Big(D^\eta_{qq\to qq} + D^\eta_{\bar{q}\bar{q}\to\bar{q}\bar{q}} + 4D^\eta_{q\bar{q}\to q\bar{q}} \\ &+ 4D^\eta_{gg\to q\bar{q}} + 4D^\eta_{qg\to qg} + 4D^\eta_{\bar{q}g\to\bar{q}g}\Big),\end{aligned} \tag{35}$$

where there is no summation over a and b and we can just take $(a, b) = (1, 2)$ and where

$$\begin{aligned}D^\eta_{ab\to cd} \equiv {}& \frac{1}{80T}\int \frac{d^3p}{(2\pi)^3} d\Gamma_{ab\to cd} f^a_{k_1} f^b_{k_2} F^c_{k_3} F^d_p [B^a_{ij}(k_1) \\ &+ B^b_{ij}(k_2) - B^c_{ij}(k_3) - B^d_{ij}(p)]^2.\end{aligned} \tag{36}$$

Once η has the standard quadratic form in $|B\rangle$, and $\eta = \langle B|\mathcal{C}_\eta|B\rangle = \langle B|S_\eta\rangle$, we can use the standard algorithm to systematically approach η from below [20].

D. Bulk viscosity and the Landau-Lifshitz condition

For bulk viscosity, the collision kernel $\mathcal{C}_\zeta$ in Eq. (22) has two zero modes A_E and A_n which satisfy

$$\mathcal{C}_\zeta|A_E\rangle = \mathcal{C}_\zeta|A_n\rangle = 0. \tag{37}$$

A_E arises from energy conservation

$$A^a_E(p) = E_a, \qquad a = g, q, \bar{q}, \tag{38}$$

while A_n arises from quark number conservation

$$A^g_n(p) = 0, \qquad A^q_n(p) = 1, \qquad A^{\bar{q}}_n(p) = -1. \tag{39}$$

We can use the Landau-Lifshitz condition

$$\delta T^{00} = 0 \tag{40}$$

and

$$\delta n = 0 \tag{41}$$

to rewrite Eq. (30) by adding linear combinations of δT^{00} and δn:

$$\begin{aligned}\zeta = {}& \sum_a \frac{N_a}{T}\int \frac{d^3p}{(2\pi)^3 E_a} f^a F^a A^a \bigg[\frac{p^2}{3} - \left(\frac{\partial P}{\partial \epsilon}\right)_n \\ &\times \left(E_a^2 - T^2\frac{\partial E_a^2}{\partial T^2} - \mu^2\frac{\partial E_a^2}{\partial \mu^2}\right) \\ &+ \left(\frac{\partial P}{\partial n}\right)_\epsilon E_a\left(\frac{\partial E_a}{\partial \mu} - a_a\right)\bigg] \equiv \langle A|S_\zeta\rangle.\end{aligned} \tag{42}$$

Substituting $|S_\zeta\rangle$ with Eq. (22), ζ becomes quadratic in $|A\rangle$ with a positive prefactor:

$$\begin{aligned}\zeta &= \langle A|\mathcal{C}_\zeta|A\rangle \\ &= D^\zeta_{gg\to gg} + \frac{N_f(N_f-1)}{2}\Big(4D^\zeta_{q_aq_b\to q_aq_b} \\ &\quad + 4D^\zeta_{\bar{q}_a\bar{q}_b\to\bar{q}_a\bar{q}_b} + 8D^\zeta_{q_a\bar{q}_b\to q_a\bar{q}_b} + 8D^\zeta_{q_a\bar{q}_a\to q_b\bar{q}_b}\Big)_{a\neq b} \\ &\quad + N_f\Big(D^\zeta_{qq\to qq} + D^\zeta_{\bar{q}\bar{q}\to\bar{q}\bar{q}} + 4D^\zeta_{q\bar{q}\to q\bar{q}} + 4D^\zeta_{gg\to q\bar{q}} \\ &\quad + 4D^\zeta_{qg\to qg} + 4D^\zeta_{\bar{q}g\to\bar{q}g}\Big),\end{aligned} \tag{43}$$

where

$$\begin{aligned}D^\zeta_{ab\to cd} \equiv {}& \frac{1}{8T}\int \frac{d^3p}{(2\pi)^3} d\Gamma_{ab\to cd} f^a_{k_1} f^b_{k_2} F^c_{k_3} F^d_p [A^a(k_1) \\ &+ A^b(k_2) - A^c(k_3) - A^d(p)]^2.\end{aligned} \tag{44}$$

$(\partial P/\partial\epsilon)_n$ and $(\partial P/\partial n)_\epsilon$ in Eq. (42) can be obtained from thermodynamic relations

$$\begin{aligned}\left(\frac{\partial P}{\partial \epsilon}\right)_n &= \frac{sP_{,\mu,\mu} - nP_{,\mu,T}}{C_V P_{,\mu,\mu}}, \\ \left(\frac{\partial P}{\partial n}\right)_\epsilon &= \frac{nTP_{,T,T} + (n\mu - sT)P_{,\mu,T} - s\mu P_{,\mu,\mu}}{C_V P_{,\mu,\mu}},\end{aligned} \tag{45}$$

where the pressure $P(T, \mu)$ can be read off from Eq. (26) and where $P_{,X,Y} \equiv \partial^2 P/\partial X\partial Y$, $n = \partial P/\partial\mu$, $s = \partial P/\partial T$, $\epsilon = -P + Ts + \mu n$ and $C_V = T(P_{,T,T} - P^2_{,\mu,T}/P_{,\mu,\mu})$. We have also used

$$\left(\frac{\partial P}{\partial \epsilon}\right)_n = \left(\frac{\frac{\partial P}{\partial T}\delta T + \frac{\partial P}{\partial \mu}\delta\mu}{\frac{\partial \epsilon}{\partial T}\delta T + \frac{\partial \epsilon}{\partial \mu}\delta\mu}\right)_{\delta n = \frac{\partial n}{\partial T}\delta T + \frac{\partial n}{\partial \mu}\delta\mu = 0}. \tag{46}$$

Then it can be shown that

$$\langle A_E|S_\zeta\rangle = 0, \qquad \langle A_n|S_\zeta\rangle = 0. \tag{47}$$

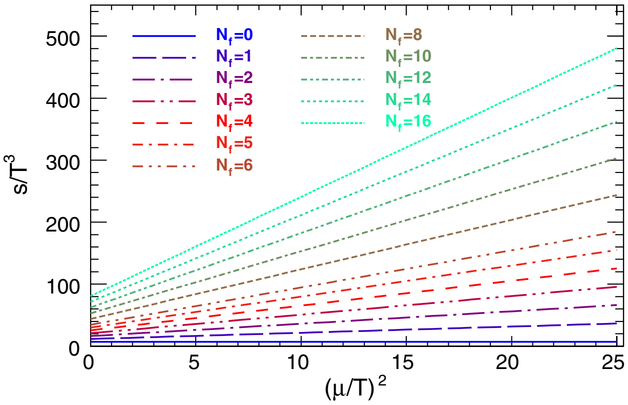

FIG. 1 (color online). s/T^3 as functions of $(\mu/T)^2$ for different N_f.

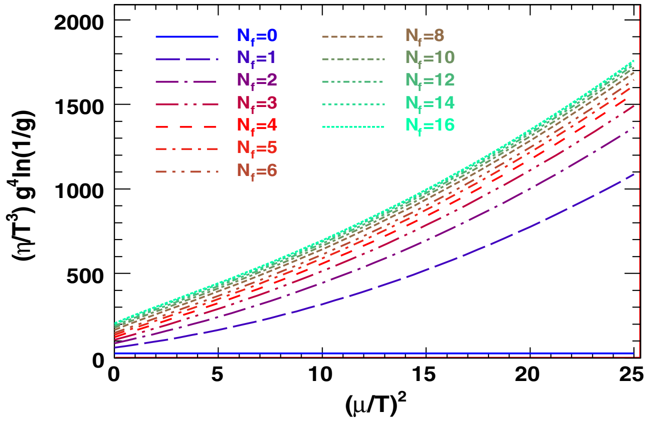

FIG. 2 (color online). $(\eta/T^3)g^4 \ln g^{-1}$ as functions of $(\mu/T)^2$ for different N_f.

Thus one can solve $\zeta = \langle A|\mathcal{C}_\zeta|A\rangle = \langle A|S_\zeta\rangle$ for ζ using the standard algorithm to systematically approach ζ from below [20].

E. Numerical results

As mentioned in the introduction, the LL result has $\eta \propto g^{-4}\ln^{-1}(1/g)$ and $\zeta \propto g^4\ln^{-1}(1/g)$. When $\mu/T \ll 1$, η, ζ, and s all scale as T^3 from dimensional analysis. For $T/\mu \ll 1$, η, ζ, and s should be even functions of μ because they should be invariant under $\mu \to -\mu$, i.e. the exchange of quarks and antiquarks.

We first show the result for the entropy density s. Only the leading order s for free particles is needed:

$$s = N_g s_g + N_f N_q (s_q + s_{\bar{q}}), \tag{48}$$

where

$$s_g = \int_p \left[\frac{\beta E_p}{e^{\beta E_p} - 1} - \ln(1 - e^{-\beta E_p})\right], \tag{49}$$

$$s_{q,\bar{q}} = \int_p \left[\frac{\beta(E_p \mp \mu)}{e^{\beta(E_p \mp \mu)} + 1} + \ln(1 + e^{-\beta(E_p \mp \mu)})\right]. \tag{50}$$

In Fig. 1, s/T^3 is shown as a function of $(\mu/T)^2$. When $\mu/T \ll 1$, s scales as T^3 and when $T/\mu \ll 1$, s scales as $\mu^2 T$. This agrees with the expectation that $s = 0$ when $T = 0$, and s increases with T for fixed μ. The entropy density s also increases monotonically with the number of flavors N_f. We stop at $N_f = 16$, just before the asymptotic freedom of QCD is lost when $N_f \geq 33/2$.

In Fig. 2, we show the normalized shear viscosity $\tilde{\eta} \equiv (\eta/T^3)g^4 \ln g^{-1}$ as functions of $(\mu/T)^2$ for different N_f. When $\mu/T \to 0$, η scales as T^3 from dimensional analysis. When $T/\mu \to 0$, η cannot scale as μ^3 because it is an even function of μ. Instead, η scales as μ^4/T. Technically, this is because $f^q F^q \propto \delta((E-\mu)/T) = T\delta(E-\mu)$ as $T/\mu \to 0$ while the antiquark and gluon contributions vanish. Thus, Eqs. (29) and (35) are solved with $B_{jk} \propto 1/T$ and $\eta \propto 1/T$. Physically, this $1/T$ behavior emerges because η is inversely proportional to the collision rate which vanishes at $T = 0$. Also, η is monotonically increasing with N_f because the averaged coupling between gluons is stronger than those with quarks involved. Thus, the effective collision rate is smaller with higher μ and higher N_f.

Around $\mu = 0$, we make a Taylor expansion $\tilde{\eta} = a_\eta + b_\eta(\mu/T)^2 + \cdots$, where a_η is $\tilde{\eta}$ at zero quark chemical potential. The values of a_η and b_η for various N_f is tabulated in Table II. Our a_η is identical to AMY's to at least the second decimal place for all N_f computed in Ref. [16]. The agreement is better than 1%.

In Fig. 3, $(\eta/s)g^4 \ln g^{-1}$ is shown as functions of $(\mu/T)^2$. For a given coupling, the LL value of η/s is the smallest at $\mu = 0$, i.e. the fluid is the most perfect at $\mu = 0$. This perturbative QCD result at high T is consistent with the observation of Ref. [31] in the hadronic phase at low T. Thus, we speculate that this property might also be true near the phase transition temperature T_c such that QCD has its local minimum, perhaps its absolute minimum as well, at T_c with zero quark chemical potential.

If the coupling g is held fixed, our η/s is monotonically decreasing with N_f for $N_f \geq 2$ but not for $N_f = 0$ and 1

TABLE II. First two coefficients in the Taylor expansion $\tilde{\eta} = a_\eta + b_\eta(\mu/T)^2 + \cdots$ near $\mu = 0$. Our result is identical to AMY's [16] to at least the second decimal place.

N_f	a_η	b_η	N_f	a_η	b_η
0	27.125	0	9	172.564	50.381
1	60.808	16.619	10	178.839	51.301
2	86.472	27.281	11	184.389	52.028
3	106.664	34.454	12	189.333	52.608
4	122.957	39.459	13	193.764	53.074
5	136.380	43.055	14	197.760	53.450
6	147.627	45.703	15	201.380	53.755
7	157.187	47.690	16	204.675	54.003
8	165.412	49.207			

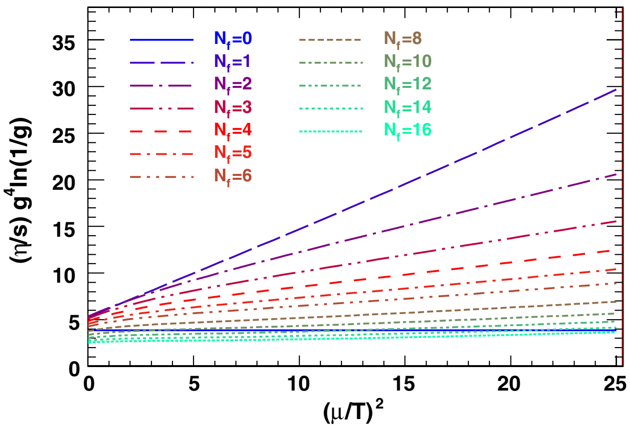

FIG. 3 (color online). $(\eta/s)g^4 \ln g^{-1}$ as functions of $(\mu/T)^2$ for different N_f.

TABLE III. Power expansion coefficients of $\tilde{\zeta}$ where $\tilde{\zeta} = a_\zeta + b_\zeta(\mu/T)^2 + \cdots$. Our result is identical to ADM's result [17] to at least the second decimal place.

N_f	a_ζ	b_ζ	N_f	a_ζ	b_ζ
0	0.4430	0	9	0.3847	0.0873
1	0.5816	0.0393	10	0.3113	0.0725
2	0.6379	0.0747	11	0.2389	0.0569
3	0.6568	0.0981	12	0.1706	0.0414
4	0.6495	0.1116	13	0.1096	0.0270
5	0.6218	0.1172	14	0.0592	0.0148
6	0.5778	0.1163	15	0.0225	0.0057
7	0.5213	0.1103	16	0.0026	0.0007
8	0.4558	0.1003			

(there is a crossing between the η/s of $N_f = 1$ and 2 at $\mu^2/T^2 \simeq 1.8$). This pattern looks random, but interestingly, it is qualitatively consistent with the pion gas result of Ref. [18] which has $\eta/s \propto f_\pi^4/N_f^2 T^4 \propto N_c^2/N_f^2$ (we have used $f_\pi^2 \propto N_c$). Again, this suggests that there is a natural connection between η/s above and below the phase transition. This pion gas analogy can also explain why $N_f = 0$ and 1 are special—there is no pion in these two cases.

We also observe that when $N_f \geq 8$, the fluid can be more perfect than that of $N_f = 0$. It would be interesting to test this in lattice QCD to find a more perfect fluid than the currently evaluated $N_f = 0$ case. It would also be interesting to investigate how the above qualitative N_f scaling changes due to the possible infrared fixed point for $N_f \gtrsim 12$ where chiral symmetry is not supposed to be broken and hence no pions exist anymore (see, e.g., [32] and references therein).

In Fig. 4 we show the normalized bulk viscosity $\tilde{\zeta} \equiv (\zeta/\alpha_s^2 T^3)\ln g^{-1}$ as functions of $(\mu/T)^2$ for different N_f. When $\mu/T \to 0$, ζ scales as T^3 from dimensional analysis. When $T/\mu \to 0$, ζ scales as $\mu^2 T$. Technically, this is because as $T \to 0$, $f^g = f^{\bar{q}} = 0$, the dominant contribution in Eq. (43) comes from the scattering between quarks. In quark scattering, the combination $K \equiv f_{k_1}^a f_{k_2}^b F_{k_3}^c F_p^d [A^a(k_1) + A^b(k_2) - A^c(k_3) - A^d(p)]^2 = \mathcal{O}(T^2)$ in Eq. (44). This is because when $T \to 0$, the scattering can only happen on the Fermi surface; otherwise it will be Pauli blocked (this is imposed by the vanishing of the prefactor $f_{k_1}^a f_{k_2}^b F_{k_3}^c F_p^d$). But scattering on the Fermi surface yields $A^a(k_1) = A^b(k_2) = A^c(k_3) = A^d(p)$ and thus $K = 0$. Therefore, the dimensionless combination K contributes at $\mathcal{O}(T^2/\mu^2)$ in Eq. (44), which leads to $\zeta \propto T$. [A similar argument can be applied to η. As $T \to 0$, $K' \equiv f_{k_1}^a f_{k_2}^b F_{k_3}^c F_p^d [B_{ij}^a(k_1) + B_{ij}^b(k_2) - B_{ij}^c(k_3) - B_{ij}^d(p)]^2 = \mathcal{O}(T^0)$ in Eq. (36), since the scattering on the Fermi surface does not have to be forward scattering; thus K' does not have to vanish. This leads to $\eta \propto 1/T$ in Eq. (36).] In contrast to η, ζ is not monotonically increasing with N_f because ζ is suppressed by an additional power

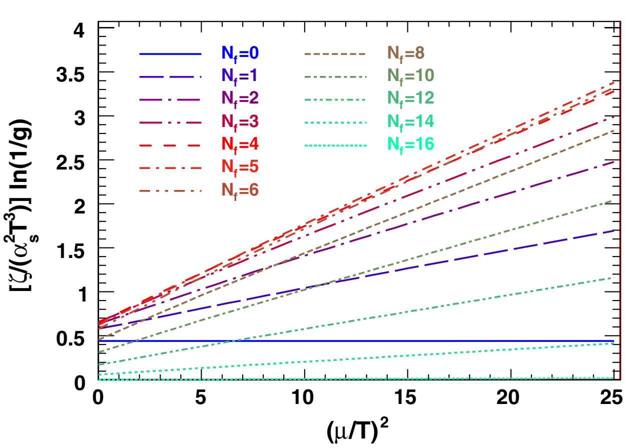

FIG. 4 (color online). $(\zeta/\alpha_s^2 T^3)\ln g^{-1}$ as functions of $(\mu/T)^2$ for different N_f.

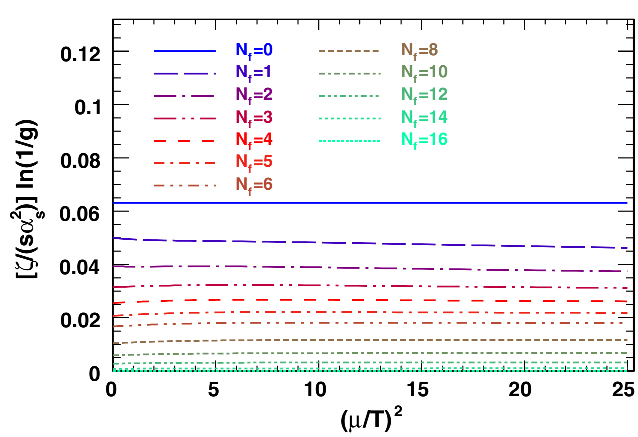

FIG. 5 (color online). $(\zeta/s\alpha_s^2)\ln g^{-1}$ as functions of $(\mu/T)^2$ for different N_f.

of $(T^{\mu}_{\mu})^2 \propto \beta^2(g^2) \propto (33 - 2N_f)^2$. Thus when N_f is small, ζ increases with N_f because quarks tend to make the averaged effective coupling weaker but at large N_f, the suppression factor $\beta^2(g^2)$ takes control to make ζ decrease with N_f. The maximum ζ happens when $N_f = 5$ or 6, depending on the value of μ/T.

Around $\mu = 0$, we make a Taylor expansion $\tilde{\zeta} = a_\zeta + b_\zeta(\mu/T)^2 + \cdots$, where a_ζ is $\tilde{\zeta}$ at zero quark chemical potential. The values of a_ζ and b_ζ for various N_f is tabulated in Table III. Our a_ζ is identical to ADM's at least to the second decimal place for all N_f computed in Ref. [17]. The agreement is better than 1%.

In Fig. 5, if the coupling g is fixed, our ζ/s is monotonically decreasing with N_f. This pattern is qualitatively consistent with the massless pion gas result of Ref. [19] which has $\zeta/s \propto T^4/N_f^2 f_\pi^4 \propto 1/N_c^2 N_f^2$ (only valid for $N_f \geq 2$ where pions exist).

III. CONCLUSION

We have calculated the shear and bulk viscosities of a weakly coupled quark gluon plasma at the leading-log order with finite temperature T and quark chemical potential μ. We have found that when normalized by the entropy density s, η/s increases monotonically with μ and eventually scales as $(\mu/T)^2$ at large μ. However ζ/s is insensitive to μ. Both η/s and ζ/s are monotonically decreasing functions of the quark flavor number N_f when $N_f \geq 2$. The same property is also observed in pion gas calculations. Our perturbative calculation suggests that QCD becomes the most perfect (with the smallest η/s) at $\mu = 0$ and $N_f = 16$ (the maximum N_f with asymptotic freedom). It would be interesting to test whether the currently smallest η/s computed close to the phase transition at $\mu = 0$ and $N_f = 0$ can be further reduced by increasing N_f.

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APPENDIX A: SCATTERING AMPLITUDES AND TAYLOR EXPANSION COEFFICIENTS OF VISCOSITIES

To describe the microscope scattering processes in a quark gluon plasma, we need scattering amplitudes between quarks and gluons. In a hot QCD plasma the infrared singularity in the amplitude can be regularized by a hard thermal loop dressed propagator. We use the same amplitude (shown in Table I) as Ref. [33].

Around $\mu = 0$, we make Taylor expansions $\tilde{\eta} \equiv (\eta/T^3) g^4 \ln g^{-1} = a_\eta + b_\eta(\mu/T)^2 + \cdots$ and $\tilde{\zeta} \equiv (\zeta/\alpha_s^2 T^3) \times \ln g^{-1} = a_\zeta + b_\zeta(\mu/T)^2 + \cdots$. The values of a_η, b_η, a_ζ and b_ζ for various N_f are tabulated in Tables II and III. Our a_η and a_ζ are identical to AMY's and ADM's to at least the second decimal place for all N_f computed in Refs. [16,17], respectively. The agreement is better than 1%.

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