Skyrmions with vector mesons in the hidden local symmetry approach

Yong-Liang Ma,[1,*] Ghil-Seok Yang,[2,†] Yongseok Oh,[2,‡] and Masayasu Harada[1,§]

[1]*Department of Physics, Nagoya University, Nagoya 464-8602, Japan*
[2]*Department of Physics, Kyungpook National University, Daegu 702-701, Korea*

The roles of light ρ and ω vector mesons in the Skyrmion are investigated in a chiral Lagrangian derived from the hidden local symmetry (HLS) up to $O(p^4)$ including the homogeneous Wess-Zumino terms. We write a general ''master formula'' that allows us to determine the parameters of the HLS Lagrangian from a class of holographic QCD models valid at the large-N_c and -λ ('t Hooft constant) limit by integrating out the infinite towers of vector and axial-vector mesons other than the lowest ρ and ω mesons. Within this approach we find that the physical properties of the Skyrmion as the solitonic description of baryons are *independent* of the HLS parameter a. Therefore the only parameters of the model are the pion decay constant and the vector-meson mass. Once determined in the meson sector, we have a totally parameter-free theory that allows us to study unequivocally the role of light vector mesons in the Skyrmion structure. We find, as suggested by Sutcliffe, that the inclusion of the ρ meson reduces the soliton mass, which makes the Skyrmion come closer to the Bogomol'nyi-Prasad-Sommerfield soliton, but the role of the ω meson is found to increase the soliton mass. In stark contrast, the Δ-N mass difference, which is determined by the moment of inertia in the adiabatic collective quantization of the Skyrmion, is increased by the ρ vector meson, while it is reduced by the inclusion of the ω meson. All these observations show the importance of the ω meson in the properties of the nucleon and nuclear matter in the Skyrme model.

I. INTRODUCTION

In accessing dense baryonic matter, one possible approach that unifies both the elementary baryons and multibaryon systems was proposed in Refs. [1,2]. In this approach, starting with a chiral Lagrangian, the single baryon is generated as a Skyrmion, and multi-Skyrmions are put on a crystal lattice to simulate many-baryon systems and dense matter.

However, the previous works in this approach suffer from the modeling of the effective chiral Lagrangian and the determination of the low-energy constants (LECs). This problem becomes serious when one considers a higher-order chiral Lagrangian or introduces more mesonic degrees of freedom, which prevents systematic studies for Skyrmion properties. On the other hand, when one starts with holographic QCD (hQCD) models and integrates out infinite towers of mesons such as vector and axial-vector mesons except a few low-lying mesons, as done in Refs. [3,4], it leads to a chiral Lagrangian with the values of all the LECs fixed by only a few phenomenological inputs. Therefore, the number of parameters is drastically reduced, which allows for the study of Skyrmions in a systematic way. In the present article, we use the general master formula derived in this way to determine the LECs of the $O(p^4)$ terms of the hidden local symmetry (HLS) Lagrangian, which leaves undetermined by

*ylma@hken.phys.nagoya-u.ac.jp
†ghsyang@gmail.com
‡yohphy@knu.ac.kr
§harada@hken.phys.nagoya-u.ac.jp

theory the pion decay constant, vector meson mass, and the HLS parameter a. As will be shown, all physical observables that we will consider are independent of a. So by fixing the pion decay constant and the vector-meson mass from the meson sector, we are able to study the Skyrmion properties in a totally parameter-independent way, a feat that—as far as we know—has not been achieved before. In addition, we work with the chiral Lagrangian including the homogeneous Wess-Zumino (hWZ) terms for studying the role of the ω vector meson in the properties of a single Skyrmion. This is our first step towards more complete studies on Skyrmions for nucleon structure and baryonic matter. The main results of this work were quoted in Ref. [5], and here we provide the details of the calculations in this model as well as more detailed analyses on the Skyrmion mass and size.

The Skyrme model [6,7] is the nonlinear sigma model stabilized by the Skyrme term, a four-derivative term, where the baryons emerge as stable field configurations with a nontrivial geometrical structure. It is well accepted that the nonlinear sigma model captures the physics of QCD at very low energy scales, and as the energy scales increase vector mesons should be excited. An elegant way to describe the vector meson physics is the hidden local symmetry [8–10] in which the vector mesons emerge as the gauge bosons of the HLS. Furthermore, as the energy scale goes up, an infinite number of local symmetries appear and the corresponding gauge fields are identified with the infinite vector and axial-vector mesons. This infinite number of hidden gauge vector fields together with the pion field in four dimensions (4D) can be dimensionally

deconstructed to five-dimensional (5D) Yang-Mills (YM) action in curved space [11] with the fifth dimension being the energy scale.

A similar situation exits in the gravity sector (referred to as the "bulk" sector) of gravity/gauge (holographic) duality that comes from string theory. (See, for example, a recent review addressed to hadron physicists in Ref. [12] and references therein.) When the Kaluza-Klein (KK) modes are decomposed to 4D and all the KK modes except the lowest-lying vector mesons and pseudoscalar mesons are integrated out in a way consistent with hidden local symmetry from the bulk sector, this model is dual to the HLS up to $O(p^4)$ [3,4]. This dual (bulk-sector) model is justified in the large number of colors N_c, the large 't Hooft coupling $\lambda \equiv g_{YM}^2 N_c$ limit, and the chiral limit where the quark masses vanish. Furthermore, baryons can be described as solitons in the holographic QCD [13–17]. The hQCD model includes only two parameters, λ and the KK mass M_{KK}, which can be fixed from meson physics. This, therefore, enables a parameter-free calculation of the Skyrmion properties with vector mesons and provides a way to perform a systematic study of the role of vector mesons in the Skyrmion structure. Here, the five-dimensional Chern-Simons (CS) term that is responsible for the anomalous part of the HLS Lagrangian is a topological quantity and, therefore, is free from the warping of the space-time. As we shall see below, the CS term is very important for understanding the role of the ω meson in the soliton structure.

In the literature, Skyrmions have been studied based on the $O(p^2)$ HLS Lagrangian, as in Refs. [18–21]. This model has three parameters, f_π, g, and a, where f_π is the pion decay constant, g is the HLS gauge coupling constant, and a is a free parameter in the HLS. (See Sec. II for the definition of these parameters.) The HLS parameter a is normally taken to be $1 \lesssim a \lesssim 2$ [8–10]. In free space $a \simeq 2$ is preferred, but in a hadronic medium at high temperature and/or density, one gets $a \simeq 1$ [10]. The dependence of a on circumstances hinders the systematic investigation of the properties of a single Skyrmion and baryonic matter. For example, the soliton mass reported in Ref. [19] within a ρ-meson stabilized model is

$$M_{\text{sol}} = (667\text{–}1575) \text{ MeV} \tag{1}$$

for $1 \leq a \leq 4$ with $m_\pi = 0$, and the pion-mass effect is found to be small in the soliton mass. This shows that the ambiguity in the value of a results in a large uncertainty in the soliton mass.

Furthermore, the description of baryons as Skyrmions is supported by the large-N_c limit [22]. In the HLS, the higher-order terms such as the $O(p^4)$ terms are at $O(N_c)$ like the $O(p^2)$ terms.[1] As a result, in the N_c counting, these

---

[1] The loop corrections from the $O(p^2)$ Lagrangian to the $O(p^4)$ terms are $O(N_c^0)$ and therefore are subdominant. We do not consider the $O(p^6)$ Lagrangian in the present work.

higher-order terms should be taken into account. However, including the higher-order terms inevitably calls for a more complicated form of the Lagrangian and an uncontrollably large number of low-energy constants. In this paper, these constants will be determined in a controllable way by using a master formula.

This paper is organized as follows. In Sec. II, we introduce the HLS Lagrangian up to $O(p^4)$ including all the hWZ terms. The soliton wave functions are constructed and the collective quantization method is also briefly explained. We show a general master formula to determine the parameters of the HLS Lagrangian induced from a class of 5D gauge models, including hQCD models in Sec. III. In Sec. IV, we present our results on the Skyrmion mass and size as well as the moment of inertia calculated in the present work. Here, we consider two models for the parameters, which include the HLS induced from the Sakai-Sugimoto (SS) model and the HLS induced from the Bogomol'nyi-Prasad-Sommerfield (BPS) model. The results from these two parameter sets are discussed and compared. Section V contains a summary and discussion. The complete explicit expression for the soliton mass and the equations of motion for the static fields are given in Appendix A. The moment of inertia and the associated equations of motion for the excited fields are collected in Appendix B.

II. SKYRMIONS FROM THE HIDDEN LOCAL SYMMETRY

In order to study the role of vector mesons in Skyrmions, we first briefly introduce the chiral effective Lagrangian with vector mesons, referring for the details to Refs. [10,23]. Here, we consider both the isoscalar ω meson and the isovector ρ meson, as well as the chiral field, as explicit degrees of freedom in the theory. These vector mesons are introduced as the gauge bosons of the HLS of the nonlinear sigma model [8–10].

The full symmetry group of our effective Lagrangian is $G_{\text{full}} = [\text{SU}(2)_L \times \text{SU}(2)_R]_{\text{chiral}} \times [\text{U}(2)]_{\text{HLS}}$, with $[\text{U}(2)]_{\text{HLS}}$ being the HLS. In the absence of the external sources, the HLS Lagrangian can be constructed by making use of the two 1-forms $\hat{\alpha}_{\parallel\mu}$ and $\hat{\alpha}_{\perp\mu}$ defined by

$$\hat{\alpha}_{\parallel\mu} = \frac{1}{2i}(D_\mu \xi_R \xi_R^\dagger + D_\mu \xi_L \xi_L^\dagger),$$
$$\hat{\alpha}_{\perp\mu} = \frac{1}{2i}(D_\mu \xi_R \xi_R^\dagger - D_\mu \xi_L \xi_L^\dagger), \tag{2}$$

with the chiral fields ξ_L and ξ_R, which are written in the unitary gauge as

$$\xi_L^\dagger = \xi_R \equiv \xi = e^{i\pi/2f_\pi} \quad \text{with} \quad \pi = \boldsymbol{\pi} \cdot \boldsymbol{\tau}, \tag{3}$$

where $\boldsymbol{\tau}$ are the Pauli matrices. The covariant derivative is defined as

$$D_\mu \xi_{R,L} = (\partial_\mu - iV_\mu)\xi_{R,L}, \tag{4}$$

with V_μ being the gauge boson of the HLS. This is the way to introduce vector mesons in the HLS, where the vector meson field V_μ is [8–10]

$$V_\mu = \frac{g}{2}(\omega_\mu + \rho_\mu), \qquad (5)$$

with

$$\rho_\mu = \boldsymbol{\rho}_\mu \cdot \boldsymbol{\tau} = \begin{pmatrix} \rho_\mu^0 & \sqrt{2}\rho_\mu^+ \\ \sqrt{2}\rho_\mu^- & -\rho_\mu^0 \end{pmatrix}. \qquad (6)$$

Then one can construct the chiral Lagrangian up to $O(p^4)$ as

$$\mathcal{L}_{\mathrm{HLS}} = \mathcal{L}_{(2)} + \mathcal{L}_{(4)} + \mathcal{L}_{\mathrm{anom}}, \qquad (7)$$

which is our working Lagrangian. Here,

$$\mathcal{L}_{(2)} = f_\pi^2 \, \mathrm{Tr}(\hat{a}_{\perp\mu}\hat{a}_\perp^\mu) + af_\pi^2 \, \mathrm{Tr}(\hat{a}_{\parallel\mu}\hat{a}_\parallel^\mu)$$
$$- \frac{1}{2g^2} \, \mathrm{Tr}(V_{\mu\nu}V^{\mu\nu}), \qquad (8)$$

where f_π is the pion decay constant, a is the parameter of the HLS, g is the vector meson coupling constant, and the field-strength tensor of the vector meson is

$$V_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu - i[V_\mu, V_\nu]. \qquad (9)$$

In the most general form of the $O(p^4)$ Lagrangian there are several terms that include two traces in the flavor space such as the y_{10}–y_{18} terms listed in Ref. [10].[2] These terms are suppressed by N_c compared to the other terms in the Lagrangian and are not considered in the present work. Then the $O(p^4)$ Lagrangian which we study in this paper is given by

$$\mathcal{L}_{(4)} = \mathcal{L}_{(4)y} + \mathcal{L}_{(4)z}, \qquad (10)$$

where

$$\mathcal{L}_{(4)y} = y_1 \, \mathrm{Tr}[\hat{a}_{\perp\mu}\hat{a}_\perp^\mu\hat{a}_{\perp\nu}\hat{a}_\perp^\nu] + y_2 \, \mathrm{Tr}[\hat{a}_{\perp\mu}\hat{a}_{\perp\nu}\hat{a}_\perp^\mu\hat{a}_\perp^\nu]$$
$$+ y_3 \, \mathrm{Tr}[\hat{a}_{\parallel\mu}\hat{a}_\parallel^\mu\hat{a}_{\parallel\nu}\hat{a}_\parallel^\nu] + y_4 \, \mathrm{Tr}[\hat{a}_{\parallel\mu}\hat{a}_{\parallel\nu}\hat{a}_\parallel^\mu\hat{a}_\parallel^\nu]$$
$$+ y_5 \, \mathrm{Tr}[\hat{a}_{\perp\mu}\hat{a}_\perp^\mu\hat{a}_{\parallel\nu}\hat{a}_\parallel^\nu] + y_6 \, \mathrm{Tr}[\hat{a}_{\perp\mu}\hat{a}_{\perp\nu}\hat{a}_\parallel^\mu\hat{a}_\parallel^\nu]$$
$$+ y_7 \, \mathrm{Tr}[\hat{a}_{\perp\mu}\hat{a}_{\perp\nu}\hat{a}_\parallel^\nu\hat{a}_\parallel^\mu] + y_8\{\mathrm{Tr}[\hat{a}_{\perp\mu}\hat{a}_\parallel^\mu\hat{a}_{\perp\nu}\hat{a}_\parallel^\nu]$$
$$+ \mathrm{Tr}[\hat{a}_{\perp\mu}\hat{a}_{\parallel\nu}\hat{a}_\perp^\nu\hat{a}_\parallel^\mu]\} + y_9 \, \mathrm{Tr}[\hat{a}_{\perp\mu}\hat{a}_{\parallel\nu}\hat{a}_\perp^\mu\hat{a}_\parallel^\nu], \qquad (11)$$

$$\mathcal{L}_{(4)z} = iz_4 \, \mathrm{Tr}[V_{\mu\nu}\hat{a}_\perp^\mu\hat{a}_\perp^\nu] + iz_5 \, \mathrm{Tr}[V_{\mu\nu}\hat{a}_\parallel^\mu\hat{a}_\parallel^\nu]. \qquad (12)$$

In the present work, we also consider the anomalous parity hWZ terms that are written as

---

[2]Another example of this kind is $\mathrm{Tr}[\hat{a}_\parallel^\mu]\mathrm{Tr}[\hat{a}_{\parallel\mu}]$, which generates the mass difference between the ρ and ω mesons.

$$\mathcal{L}_{\mathrm{anom}} = \frac{N_c}{16\pi^2} \sum_{i=1}^{3} c_i \mathcal{L}_i, \qquad (13)$$

where

$$\mathcal{L}_1 = i \, \mathrm{Tr}[\hat{\alpha}_L^3 \hat{\alpha}_R - \hat{\alpha}_R^3 \hat{\alpha}_L], \qquad (14a)$$
$$\mathcal{L}_2 = i \, \mathrm{Tr}[\hat{\alpha}_L \hat{\alpha}_R \hat{\alpha}_L \hat{\alpha}_R], \qquad (14b)$$
$$\mathcal{L}_3 = \mathrm{Tr}[F_V(\hat{\alpha}_L \hat{\alpha}_R - \hat{\alpha}_R \hat{\alpha}_L)], \qquad (14c)$$

in the 1-form notation with

$$\hat{\alpha}_L = \hat{\alpha}_\parallel - \hat{\alpha}_\perp, \quad \hat{\alpha}_R = \hat{\alpha}_\parallel + \hat{\alpha}_\perp, \quad F_V = dV - iV^2. \qquad (15)$$

In order to study the properties of the soliton obtained from the Lagrangian (7), we take the standard parametrization for the soliton configuration. For the pion field, we use the standard hedgehog configuration,

$$\xi(\boldsymbol{r}) = \exp\left[i\boldsymbol{\tau} \cdot \hat{\boldsymbol{r}} \frac{F(r)}{2}\right]. \qquad (16)$$

The configuration of the vector mesons are written as [20]

$$\omega_\mu = W(r)\delta_{0\mu}, \qquad \rho_0 = 0, \qquad \boldsymbol{\rho} = \frac{G(r)}{gr}(\hat{\boldsymbol{r}} \times \boldsymbol{\tau}). \qquad (17)$$

For the baryon number $B = 1$ solution, these wave functions satisfy the following boundary conditions:

$$F(0) = \pi, \qquad F(\infty) = 0, \qquad G(0) = -2,$$
$$G(\infty) = 0, \qquad W'(0) = 0, \qquad W(\infty) = 0. \qquad (18)$$

Given the Lagrangian and the wave functions, it is now straightforward to derive the soliton mass M_{sol}. The explicit expression for the soliton mass is given in Appendix A. Minimizing the soliton mass then gives the coupled equations of motion for the wave functions $F(r)$, $W(r)$, and $G(r)$. These are also given in Appendix A.

The classical configuration of the soliton obtained above should be quantized to describe physical baryons of definite spin and isospin. Here, we follow the standard collective quantization method [24], which transforms the chiral field and the vector meson field as

$$\xi(\boldsymbol{r}) \rightarrow \xi(\boldsymbol{r}, t) = A(t)\xi(\boldsymbol{r})A^\dagger(t),$$
$$V_\mu(\boldsymbol{r}) \rightarrow V_\mu(\boldsymbol{r}, t) = A(t)V_\mu(\boldsymbol{r})A^\dagger(t), \qquad (19)$$

where $A(t)$ is a time-dependent SU(2) matrix. We define the angular velocity $\boldsymbol{\Omega}$ of the collective coordinate rotation as

$$i\boldsymbol{\tau} \cdot \boldsymbol{\Omega} \equiv A^\dagger(t)\partial_0 A(t). \qquad (20)$$

Under the rotation (19), the space component of the ω field and the time component of the ρ field, i.e., ω^i and ρ^0, get excited. The most general forms for the vector-meson excitations are written as [20]

$$\rho^0(\boldsymbol{r}, t) = A(t)\frac{2}{g}[\boldsymbol{\tau} \cdot \boldsymbol{\Omega}\xi_1(r) + \hat{\boldsymbol{\tau}}\hat{\boldsymbol{r}} \cdot \boldsymbol{\Omega} \cdot \hat{\boldsymbol{r}}\xi_2(r)]A^\dagger(t),$$

$$\omega^i(\boldsymbol{r}, t) = \frac{\varphi(r)}{r}(\boldsymbol{\Omega} \times \hat{\boldsymbol{r}})^i. \tag{21}$$

With these wave functions the moment of inertia can be calculated, and its explicit expression is given in Appendix B. It is then straightforward to obtain the Euler-Lagrange equations for the wave functions $\xi_1(r)$, $\xi_2(r)$, and $\varphi(r)$ by minimizing the moment of inertia, and the results are also given in Appendix B. The boundary conditions imposed on the excited fields are

$$\xi_1'(0) = \xi_1(\infty) = 0,$$
$$\xi_2'(0) = \xi_2(\infty) = 0, \tag{22}$$
$$\varphi(0) = \varphi(\infty) = 0,$$

and $\xi_1(r)$ and $\xi_2(r)$ at $r = 0$ satisfy the constraint

$$2\xi_1(0) + \xi_2(0) = 2. \tag{23}$$

In the adiabatic collective quantization scheme, the baryon mass is given by

$$M = M_{\text{sol}} + \frac{i(i + 1)}{2\mathcal{I}} = M_{\text{sol}} + \frac{j(j + 1)}{2\mathcal{I}}, \tag{24}$$

where i and j are the isospin and the spin of the baryon, respectively. Then the Δ-N mass difference reads

$$\Delta_M \equiv M_\Delta - M_N = \frac{3}{2\mathcal{I}}. \tag{25}$$

The baryonic size of a baryon should be computed by the baryon-number current of the Skyrmion. However, in order to intuitively see the effects of the vector mesons on the Skyrmion size in a simple way, here we consider the winding number and energy root-mean-square radii. The root-mean-square (rms) radius of the winding-number current is defined by

$$\langle r^2 \rangle_W^{1/2} = \left[\int_0^\infty d^3 r r^2 B^0(r) \right]^{1/2}, \tag{26}$$

where $B^0(r)$ is the time component of the winding-number current that is explicitly written as

$$B^0 = -\frac{1}{2\pi^2 r^2} F' \sin^2 F. \tag{27}$$

We define the energy root-mean-square radius $\langle r^2 \rangle_E^{1/2}$ as

$$\langle r^2 \rangle_E^{1/2} = \left[\frac{1}{M_{\text{sol}}} \int_0^\infty d^3 r r^2 M_{\text{sol}}(r) \right]^{1/2}, \tag{28}$$

where $M_{\text{sol}}(r)$ is the soliton mass (energy) density given in Appendix A.

III. HIDDEN LOCAL SYMMETRY INDUCED FROM HOLOGRAPHIC QCD

A. Master formula

In this section, following Refs. [3,4], we provide a general master formula to determine the parameters of the HLS Lagrangian by integrating out the infinite towers of vector and axial-vector mesons in a class of hQCD models expressed by the following general 5D action:

$$S_5 = S_5^{\text{DBI}} + S_5^{\text{CS}}, \tag{29}$$

where the 5D Dirac-Born-Infeld (DBI) part S_5^{DBI} and the Chern-Simons part S_5^{CS} are expressed as

$$S_5^{\text{DBI}} = N_c G_{\text{YM}} \int d^4x dz \Big\{ -\frac{1}{2} K_1(z) \text{Tr}[\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}]$$
$$+ K_2(z) M_{KK}^2 \text{Tr}[\mathcal{F}_{\mu z}\mathcal{F}^{\mu z}] \Big\}, \tag{30}$$

$$S_5^{\text{CS}} = \frac{N_c}{24\pi^2} \int_{M^4 \times R} w_5(A). \tag{31}$$

where the rescaled 't Hooft coupling constant is defined as $G_{\text{YM}} \equiv \lambda/(108\pi^3)$ and the field strength of the 5D gauge field[3] $\mathcal{A}_M$ is $\mathcal{F}_{MN} = \partial_M \mathcal{A}_N - \partial_N \mathcal{A}_M - i[\mathcal{A}_M, \mathcal{A}_N]$. Here, $K_{1,2}(z)$ are the metric functions of z constrained by the gauge/gravity duality. The gravity enters in the z dependence of the YM coupling, giving rise to the warping of the space. In Eq. (31), M^4 and R stand for the 4D Minkowski space-time and z-coordinate space, respectively, and $w_5(A)$ is the CS 5-form, written as

$$w_5(A) = \text{Tr}\Big[\mathcal{A}\mathcal{F}^2 + \frac{i}{2}\mathcal{A}^3\mathcal{F} - \frac{1}{10}\mathcal{A}^5 \Big]. \tag{32}$$

Here, $\mathcal{F} = d\mathcal{A} + i\mathcal{A}\mathcal{A}$ is the field strength of the 5D gauge field $\mathcal{A} = \mathcal{A}_\mu dx^\mu + \mathcal{A}_z dz$. It should be noted that the DBI part is of $O(\lambda^1)$ while the CS term is of $O(\lambda^0)$ with the 't Hooft coupling constant λ.

We should stress here that, as noted in Ref. [4], the structure of the action (29) is shared both by the top-down Sakai-Sugimoto model and the bottom-up models, such as in Refs. [25,26] as well as the moose models in Ref. [11], with the difference appearing only in the warping factors. This allows us to write down a "master formula" which applies to all holographic models and moose constructions given appropriate warping factors.

Now, to induce the HLS Lagrangian from the action (30), we use the mode expansion of the 5D gauge field $\mathcal{A}_M(x, z)$ and integrate out all the modes except the pseudoscalar and the lowest-lying vector mesons, which reduces $\mathcal{A}_M(x, z)$ to $A_M^{\text{integ}}(x, z)$. In the $A_z(x, z) = 0$ gauge, this implies the substitution [3,4][4]

---

[3]We use the index $M = (\mu, z)$, with $\mu = 0, 1, 2, 3$.

[4]As emphasized in Ref. [4], the procedure of "integrating out" adopted here is different from the "naive truncation" that violates the chiral invariance.

$$A_\mu(x, z) \rightarrow A_\mu^{\text{integ}}(x, z)$$

$$= \hat{\alpha}_{\mu\perp}(x)\psi_0(z) + [\hat{\alpha}_{\mu\parallel}(x) + V_\mu(x)]$$

$$+ \hat{\alpha}_{\mu\parallel}(x)\psi_1(z), \qquad (33)$$

where $\{\psi_n\}$ are eigenfunctions satisfying the following eigenvalue equation obtained from the action (30):

$$-K_1^{-1}(z)\partial_z[K_2(z)\partial_z\psi_n(z)] = \lambda_n\psi_n(z), \qquad (34)$$

with λ_n being the nth eigenvalue ($\lambda_0 = 0$). By substituting Eq. (33) into the action in Eq. (30), the HLS Lagrangian up to $O(p^4)$ can be obtained. The explicit expressions for the LECs we need are derived as [3,4]

$$f_\pi^2 = N_c G_{\text{YM}} M_{\text{KK}}^2 \int dz K_2(z)[\dot{\psi}_0(z)]^2, \qquad af_\pi^2 = N_c G_{\text{YM}} M_{\text{KK}}^2 \lambda_1 \langle \psi_1^2 \rangle, \qquad \frac{1}{g^2} = N_c G_{\text{YM}} \langle \psi_1^2 \rangle,$$

$$y_1 = -y_2 = -N_c G_{\text{YM}} \langle (1 + \psi_1 - \psi_0^2)^2 \rangle, \qquad y_3 = -y_4 = -N_c G_{\text{YM}} \langle \psi_1^2(1 + \psi_1)^2 \rangle,$$

$$y_5 = 2y_8 = -y_9 = -2N_c G_{\text{YM}} \langle \psi_1^2 \psi_0^2 \rangle, \qquad y_6 = -(y_5 + y_7), \qquad y_7 = 2N_c G_{\text{YM}} \langle \psi_1(1 + \psi_1)(1 + \psi_1 - \psi_0^2) \rangle,$$

$$z_4 = 2N_c G_{\text{YM}} \langle \psi_1(1 + \psi_1 - \psi_0^2) \rangle, \qquad z_5 = -2N_c G_{\text{YM}} \langle \psi_1^2(1 + \psi_1) \rangle, \qquad c_1 = \left\langle\!\!\left\langle \dot{\psi}_0 \psi_1 \left(\frac{1}{2}\psi_0^2 + \frac{1}{6}\psi_1^2 - \frac{1}{2} \right) \right\rangle\!\!\right\rangle,$$

$$c_2 = \left\langle\!\!\left\langle \dot{\psi}_0 \psi_1 \left(-\frac{1}{2}\psi_0^2 + \frac{1}{6}\psi_1^2 + \frac{1}{2}\psi_1 + \frac{1}{2} \right) \right\rangle\!\!\right\rangle, \qquad c_3 = \left\langle\!\!\left\langle \frac{1}{2}\dot{\psi}_0 \psi_1^2 \right\rangle\!\!\right\rangle, \qquad (35)$$

where λ_1 is the smallest (nonzero) eigenvalue of the eigenvalue equation given in Eq. (34), and $\langle \rangle$ and $\langle\langle \rangle\rangle$ are defined as

$$\langle A \rangle \equiv \int_{-\infty}^{\infty} dz K_1(z)A(z), \qquad \langle\langle A \rangle\rangle \equiv \int_{-\infty}^{\infty} dz A(z) \quad (36)$$

for a function $A(z)$. Equation (35) provides the master formula for the LECs in the HLS Lagrangian induced from general hQCD models. Namely, one just needs to plug the warping factor and the eigenfunctions of Eq. (34) into Eq. (35) to obtain the values of the LECs. For example, $K_1(z) = K^{-1/3}(z)$ and $K_2(z) = K(z)$ with $K(z) = 1 + z^2$ correspond to the Sakai-Sugimoto model.

In addition to the general hQCD models, we also consider the BPS model studied in Refs. [27,28] which is characterized by the flat space-time. In this case, instead of solving the eigenvalue equation, the 5D gauge field is expanded in terms of the Hermite function ψ_n [27,28],

$$\psi_n(z) = \frac{(-1)^{n-1}}{\sqrt{(n-1)!2^{n-1}\sqrt{\pi}}} e^{-z^2/2} \frac{d^{n-1}}{dz^{n-1}} e^{-z^2}, \qquad (37)$$

where $n \geq 1$ and ψ_1 corresponds to the wave function of the lowest-lying vector meson. The wave function of the Nambu-Goldstone pseudoscalar boson is expressed in terms of the error function erf(z),

$$\psi_0(z) = \text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-\xi^2} d\xi. \qquad (38)$$

In the following calculation, we use the Hermite function and the error function as wave functions of the vector mode and pseudoscalar mode, respectively. Then, the LECs of the HLS Lagrangian are determined by using the above $\psi_0(z)$ and $\psi_1(z)$ in the master formulas in Eq. (35) with $K(z) = 1$.

B. The a independence

In the phenomenological analysis, it is well known that the HLS parameter a plays an important role [8–10]. With the leading Lagrangian at $O(p^2)$, the choice of $a = 2$ reproduces the Kawarabayashi-Suzuki-Riazzudin-Fayyazudin relation and the ρ meson dominance in the pion electromagnetic form factor. At $O(p^4)$, it was shown that the quantum correction enhances the infrared value of a, and, therefore, a good description of low-energy phenomenology can be achieved with the bare value of a being ≤ 2 [10]. In the holographic approach, on the other hand, the parameter a is attributed to the normalization of the 5D wave function $\psi_1(z)$, which cannot be determined from the homogeneous eigenvalue equation (34). As a result, it turns out [4] that the physical quantities are independent of the parameter a as far as the leading order in N_c is concerned.[5]

In order to explicitly see that any physical quantities calculated with the HLS Lagrangian induced from hQCD models are actually independent of the parameter a, we start with Eq. (33). We first note that the vector-meson mass and the pion decay constant are related by the relation $m_\rho^2 = ag^2 f_\pi^2$, and m_ρ and f_π are fixed by their experimental values. Therefore, the HLS parameter a and the HLS gauge coupling g are connected through $ag^2 = m_\rho^2/f_\pi^2$. Therefore, the g independence of physical quantities is equivalent to their a independence.

To see the a independence explicitly, we define

$$\tilde{\psi}_1(z) = g\psi_1(z), \qquad (39)$$

so that the new function $\tilde{\psi}_1$ is normalized as

---

[5]When we include the loop corrections which can be regarded as a part of $1/N_c$ corrections, the a dependence becomes relevant to the physical quantities through the loop corrections.

$$N_c G_{YM} \int dz K_1(z)[\tilde{\psi}_1(z)]^2 = 1. \tag{40}$$

In terms of the normalized wave function $\tilde{\psi}_1(z)$ the 5D gauge field of Eq. (33) is written as

$$A_\mu^{\text{integ}}(x, z) = \hat{\alpha}_{\mu\perp}(x)\psi_0(z) + [\hat{\alpha}_{\mu\parallel}(x) + V_\mu(x)] + \tilde{\hat{\alpha}}_{\mu\parallel}(x)\tilde{\psi}_1(z), \tag{41}$$

where $\tilde{\hat{\alpha}}_{\mu\parallel} = (1/g)\hat{\alpha}_{\mu\parallel}$. In terms of the radial wave functions of the soliton, $F(r)$, $W(r)$, and $G(r)$, we have

$$\tilde{\alpha}_\parallel^0 = \frac{1}{g}\hat{\alpha}_\parallel^0 = \frac{1}{2}W(r), \tag{42}$$

$$\tilde{\alpha}_\parallel^i = \frac{1}{g}\hat{\alpha}_\parallel^i = \frac{1}{2}\tilde{G}(r)(\hat{r} \times \boldsymbol{\tau})^i, \tag{43}$$

where

$$\tilde{G}(r) = \frac{1}{gr}[G(r) + 1 - \cos F(r)]. \tag{44}$$

From the boundary conditions in Eq. (18), $W(r)$ and $\tilde{G}(r)$ satisfy the following boundary conditions:

$$W'(0) = 0, \qquad W(\infty) = 0, \\ \tilde{G}(0) = 0, \qquad \tilde{G}(\infty) = 0. \tag{45}$$

The coupled equations of motions for F, $\tilde{G}$, and W can be obtained by substituting the expression in Eq. (41) into the action (29), and then minimizing the resultant energy. This implies that there are no nontrivial boundary conditions to determine the absolute sizes of the vector-meson contributions $W(r)$ and $\tilde{G}(r)$. The normalization of the wave functions for the vector mesons $W(r)$ and $\tilde{G}(r)$ are fixed from the boundary condition of the pion contribution $F(r)$ through the coupled equations of motion of $F(r)$, $W(r)$, and $\tilde{G}(r)$, which are independent of the gauge coupling constant g since they are expressed in terms of the normalized $\tilde{\psi}_1$. This can be restated as follows: the LECs of the HLS Lagrangian are determined by substituting the expression in Eq. (41) into the action (29), and the LECs are expressed in terms of normalized wave functions $\tilde{\psi}_1$ and ψ_0. Then, all the expressions in Appendix A can be rewritten in terms of $F(r)$, $W(r)$, and $\tilde{G}(r)$ without the gauge coupling constant g. As a result, the soliton mass is free from the ambiguity of the normalization of ψ_1, so that it is independent of the parameter g, as we shall see in the next section.

Similar arguments also apply to the moment of inertia. Using Eq. (21), one has

$$\tilde{a}_{\parallel\mu} = A(t)(\tilde{a}'_{\parallel 0}, \tilde{a}'_{\parallel i})A^\dagger(t), \tag{46}$$

where

$$\tilde{a}_\parallel^{0\prime} = \tilde{a}_\parallel^0 + [\boldsymbol{\tau} \cdot \boldsymbol{\Omega}\tilde{\xi}_1(r) + \boldsymbol{\tau} \cdot \hat{r}\boldsymbol{\Omega} \cdot \hat{r}\tilde{\xi}_2(r)], \\ \tilde{a}_\parallel^{i\prime} = \tilde{a}_\parallel^i + \frac{\varphi(r)}{2r}(\boldsymbol{\Omega} \times \hat{r})^i, \tag{47}$$

with

$$\tilde{\xi}_1(r) = \frac{1}{g}[\xi_1(r) - 1 + \cos F(r)], \\ \tilde{\xi}_2(r) = \frac{1}{g}[\xi_2(r) + 1 - \cos F(r)]. \tag{48}$$

From Eqs. (18) and (22), the boundary conditions for $\tilde{\xi}_{1,2}$ read

$$\tilde{\xi}'_{1,2}(0) = 0, \qquad \tilde{\xi}_{1,2}(\infty) = 0. \tag{49}$$

Again, all the equations in Appendix B can be expressed in terms of $\tilde{\xi}_{1,2}$, φ, $\tilde{G}$, W, and F. A similar argument to that made following Eq. (45) yields that the moment of inertia should also be independent of the parameter a.

C. The CS term and the ω meson

Another important point which should be addressed here from the viewpoint of gauge/gravity duality is that the CS term is responsible for the role of the ω meson in the Skyrmion structure. This can be seen by decomposing the 5D gauge field $\mathcal{A}$ into the SU(2) and U(1) components as

$$\mathcal{A} = A_{\text{SU}(2)} + \frac{1}{2}\tilde{A}_{\text{U}(1)}. \tag{50}$$

Substituting this into the action of Eq. (30) leads to

$$S_5^{\text{DBI}} = N_c \int d^4x dz \left\{ -\frac{1}{2}K_1(z)[\text{Tr}(F_{\mu\nu}F^{\mu\nu}) + \text{Tr}(\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu})] + K_2(z)[\text{Tr}(F_{\mu z}F^{\mu z}) + \text{Tr}(\tilde{F}_{\mu z}\tilde{F}^{\mu z})] \right\}, \tag{51}$$

where F_{MN} is the field strength of the SU(2) gauge field $A_{\text{SU}(2)}$, and $\tilde{F}_{MN}$ stands for that of the U(1) gauge field $\tilde{A}_{\text{U}(1)}$. This explicitly shows that, without the CS term, when the 5D model is deconstructed from the 4D model, only the kinetic and mass terms of the isoscalar vector meson ω are allowed. This conclusion can be explicitly verified by using a specific hQCD model such as the SS model.

As can be read from Appendix A, the contribution from the kinetic and the mass terms of the ω meson to the soliton mass is

$$M_{\text{sol}}^\omega = 4\pi \int dr \left[-\frac{1}{2}r^2(ag^2f_\pi^2W^2 + W'^2) \right], \tag{52}$$

which gives the equation of motion of W as

$$W'' = ag^2f_\pi^2W - \frac{2}{r}W' \tag{53}$$

in the absence of the CS term. By making use of the partial integration with the boundary conditions given in Eq. (18), M_{sol}^ω can be calculated as

$$M_{\text{sol}}^{\omega} = 4\pi \int dr \left[-\frac{1}{2} r^2 \left(ag^2 f_{\pi}^2 W - \frac{2}{r} W' - W'' \right) W \right]$$

$$= 0, \tag{54}$$

because of the equation of motion of Eq. (53). Therefore, in the absence of the CS term, the ω field decouples from the other fields and M_{sol}^{ω} vanishes. This is consistent with the earlier studies on the Skyrmions stabilized by vector mesons [20]. As can be seen from the equation of motion of $W(r)$ given in Appendix A, the hWZ terms provide the source terms of the ω meson field. Therefore, in the absence of the hWZ terms, the ω field decouples and does not contribute to the soliton formation.

D. The effective Skyrme parameter e

Finally, we estimate the Skyrme parameter e in the original Skyrme model by integrating out the isovector ρ meson from the HLS. The original Skymre model Lagrangian reads

$$\mathcal{L}_{\text{Sk}} = \frac{f_{\pi}^2}{4} \text{Tr}(\partial_{\mu} U \partial^{\mu} U^{\dagger}) + \frac{1}{32e^2} \text{Tr}[\partial_{\mu} U U^{\dagger}, \partial_{\nu} U U^{\dagger}]^2, \tag{55}$$

where the chiral field U is $U = \xi^2$ and the first term is the nonlinear sigma model Lagrangian that can be written as $f_{\pi}^2 \text{Tr}(\alpha_{\perp\mu} \alpha_{\perp}^{\mu})$, where $\alpha_{\perp\mu}$ is defined as $\hat{\alpha}_{\perp\mu}$ without the vector field. In the earlier analyses [29] with the HLS Lagrangian up to $O(p^2)$, it was known that the Skyrme term can be obtained from the ρ-meson kinetic energy term in the limit of infinite ρ-meson mass. In this case, the Skyrme parameter e becomes the ρ-meson coupling, so that we have $e = g \simeq 6$, which is close to the empirical value $e = 5.45$ that is determined from the Δ-N mass difference. In the HLS Lagrangian up to $O(p^4)$, however, we have additional contributions from the pure $O(p^4)$ terms that lead to the Skyrme term. Explicitly, after integrating out the ρ meson, the effective Lagrangian is obtained as

$$\mathcal{L}_{\text{ChPT}} = f_{\pi}^2 \text{Tr}[\alpha_{\perp\mu} \alpha_{\perp}^{\mu}]$$

$$+ \left(\frac{1}{2g^2} - \frac{z_4}{2} - \frac{y_1 - y_2}{4} \right) \text{Tr}[\alpha_{\perp\mu}, \alpha_{\perp\nu}]^2$$

$$+ \frac{y_1 + y_2}{4} \text{Tr}\{\alpha_{\perp\mu}, \alpha_{\perp\nu}\}^2, \tag{56}$$

where [,] is the commutator and {, } is the anticommutator. The second term is the Skyrme term and we can read the effective Skyrme parameter e as

$$\frac{1}{2e^2} = \frac{1}{2g^2} - \frac{z_4}{2} - \frac{y_1 - y_2}{4}. \tag{57}$$

Since the gauge/gravity duality implies that $y_1 = -y_2$, the last term of Eq. (56) vanishes. Using Eq. (57) and the analytic expressions for the LECs given in Eq. (35), the Skyrme parameter is written as

$$\frac{1}{2e^2} = \frac{N_c G_{\text{YM}}}{2} \langle (1 - \psi_0^2)^2 \rangle. \tag{58}$$

With the experimental values of the two inputs f_{π} and m_{ρ}, we obtain the Skyrme parameter e as

$$e \simeq 7.31 \tag{59}$$

in the SS model, while in the flat-space-time case, i.e., for the BPS soliton model, we obtain

$$e \simeq 10.02. \tag{60}$$

These values are larger than the empirical value of the Skyrme parameter $e = 5.45$ because of the contributions from the y_1, y_2, and z_4 terms that are of $O(p^4)$.

Since the moment of inertia $\mathcal{I}$ is proportional to $1/e^3$ in the Skyrme model, with a larger value of e, we have a smaller moment of inertia, which results in a larger mass splitting between the Δ and the nucleon, as is verified numerically in the next section.

IV. NUMERICAL RESULTS FOR THE SKYRMION

In this section, we present the results of numerical calculations on the Skyrmion properties in the framework of the HLS discussed in the previous section. The HLS Lagrangian up to $O(p^4)$ in Eq. (7) that is considered in the present calculation contains 17 parameters, namely, f_{π}, a, g, y_i ($i = 1, \dots 9$), z_4, z_5, and $c_{1,2,3}$. We determine all these LECs through hQCD models, which are characterized by the warping factor $K(z)$ and the wave functions ψ_0 and ψ_1. Then all the LECs are obtained through the master formulas given in Eq. (35), which contain the mass scale M_{KK}, the 't Hooft coupling λ (or G_{YM}), and the integrals of the warping factor $K(z)$ and the wave functions ψ_0 and ψ_1. In the present work, we consider two hQCD models: the SS model and the BPS model.

In hQCD models, M_{KK} and G_{YM} are free parameters. In the present work, we fix them by using the empirical values of f_{π} and m_{ρ}:

$$m_{\rho} = 775.49 \text{ MeV}, \qquad f_{\pi} = 92.4 \text{ MeV}. \tag{61}$$

Then we have the necessary information to calculate all LECs of the HLS Lagrangian through the master formulas. Note, however, that the master formulas can determine only the product of $ag^2 = m_{\rho}^2/f_{\pi}^2$, and, therefore, a or g remains unfixed. However, as discussed in the previous section, the physical quantities are independent of a (or g). To be specific, we will first work with $a = 2$, which is widely used in the model of the $O(p^2)$ HLS Lagrangian [9,10], and then examine how each component of the soliton mass and the moment of inertia behaves as the value of the HLS parameter a is varied. This verifies numerically how the a independence comes about.

In the present work, we consider three versions of the HLS model induced from each hQCD model. The first version is the model that includes the pion, ρ meson, and

TABLE I.   Low-energy constants of the HLS Lagrangian at $O(p^4)$ with $a = 2$.

Model	y_1	y_3	y_5	y_6	z_4	z_5	c_1	c_2	c_3
SS model	-0.001096	-0.002830	-0.015917	$+0.013712$	0.010795	-0.007325	$+0.381653$	-0.129602	0.767374
BPS model	-0.071910	-0.153511	-0.012286	-0.196545	0.090338	-0.130778	-0.206992	$+3.031734$	1.470210

ω meson. The second one is the model without the hWZ terms, i.e., the model that includes the pion and the ρ meson. The third one is obtained by integrating out the ρ meson in the second version of the model. Therefore, this corresponds to the original Skyrme model but with the Skyrme parameter determined by the hQCD model. In this section, we will examine the three versions of the SS model and of the BPS model. The obtained results will be compared with those of the $O(p^2)$ models, such as the ρ-stabilized model of Ref. [19] and "the minimal model" of Ref. [20] that includes the ω meson in a minimal way.

A. Skyrmion in the HLS induced from the Sakai-Sugimoto model

In this subsection, we first consider the Sakai-Sugimoto model [30,31] to determine the LECs of the HLS Lagrangian. This model is characterized by the following warping factor:

$$K_1(z) = (1 + z^2)^{-1/3}, \qquad K_2(z) = 1 + z^2. \qquad (62)$$

Since M_{KK} and G_{YM} are determined by f_π and m_ρ, all LECs except a or g can be determined. This will be called the HLS_1 model [5]. As we discussed above, we will take the commonly used value $a = 2$ as a typical example and then we will test the results by varying the value of a. The values of the LECs obtained with $a = 2$ are given in the first row of Table I.

Equipped with the numerical values of the LECs given in Table I, the equations of motion for the soliton wave function and for the soliton excitations can be solved numerically, which allows us to calculate the soliton mass and the moment of inertia. The main results of the present work are summarized in the HLS_1 columns of Table II. The results of the two models with the HLS of $O(p^2)$ are also presented for comparison.

The obtained soliton wave functions for the $HLS_1(\pi, \rho, \omega)$ model are shown in Figs. 1 and 2. This

model results in the soliton mass $M_{sol} \approx 1184$ MeV and the moment of inertia $\mathcal{I} \approx 0.661$ fm, which leads to $\Delta_M \approx 448$ MeV. These numbers should be compared with the empirical values, $M_{sol} = 867$ MeV and $\Delta_M \approx 292$ MeV. Compared with the widely used "minimal model" of the HLS up to $O(p^2)$ [20,32,33], this shows that the $HLS_1(\pi, \rho, \omega)$ model improves the soliton mass.

We then consider the $HLS_1(\pi, \rho)$ model that is constructed from the HLS Lagrangian without the hWZ terms. In other words, we set $c_1 = c_2 = c_3 = 0$ and remove the ω-meson mass term and its kinetic energy term in the $HLS_1(\pi, \rho, \omega)$ model to obtain the $HLS_1(\pi, \rho)$ model. Therefore, this model is very similar to the model studied in Refs. [16,34]. In addition to the soliton mass, however, we also calculate the moment of inertia, which was not given in Refs. [16,34]. Finally, we then consider the $HLS_1(\pi)$ model that is defined with the Lagrangian (56) with the Skyrme parameter given in Eq. (59). All the results are summarized in Table II, and here are several comments made in order.

(1) As claimed in the literature [16,27,28,34], we found that the inclusion of the ρ meson reduces the soliton mass. In the present work, the soliton mass reduces from 922 MeV in the $HLS_1(\pi)$ to 834 MeV in the $HLS_1(\pi, \rho)$, which confirms the claim that the inclusion of the ρ meson makes the Skyrmion closer to the BPS soliton. However, when we include the ω meson, the soliton mass increases to 1184 MeV. This is in contrast to the naive expectation that including more vector mesons would decrease the soliton mass. Since the ω meson interacts with the other mesons through the hWZ terms, this observation shows the importance of the hWZ terms in the Skyrmion phenomenology. The role of the ω meson in the Skyrmion mass and size can also be verified by comparing the soliton wave functions shown in Fig. 3. This figure shows that the ρ meson shrinks the soliton wave functions, which can be seen by

TABLE II.   Skyrmion mass and size calculated in the HLS with the SS and BPS models with $a = 2$. The soliton mass M_{sol} and the Δ-N mass difference Δ_M are in units of MeV while $\sqrt{\langle r^2 \rangle_W}$ and $\sqrt{\langle r^2 \rangle_E}$ are in units of fm. The column of $O(p^2) + \omega_\mu B^\mu$ is "the minimal model" of Ref. [20] and that of $O(p^2)$ corresponds to the model of Ref. [19]. See the text for more details.

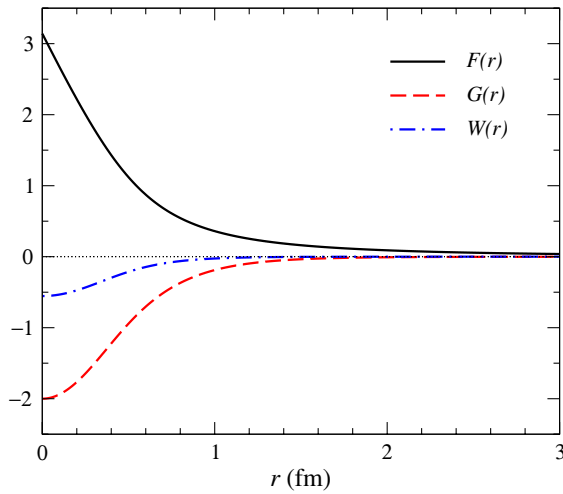|	$HLS_1(\pi, \rho, \omega)$	$HLS_1(\pi, \rho)$	$HLS_1(\pi)$	$BPS(\pi, \rho, \omega)$	$BPS(\pi, \rho)$	$BPS(\pi)$	$O(p^2) + \omega_\mu B^\mu$ [20]	$O(p^2)$ [19]
M_{sol}	1184	834	922	1162	577	672	1407	1026
Δ_M	448	1707	1014	456	4541	2613	259	1131
$\sqrt{\langle r^2 \rangle_W}$	0.433	0.247	0.309	0.415	0.164	0.225	0.540	0.278
$\sqrt{\langle r^2 \rangle_E}$	0.608	0.371	0.417	0.598	0.271	0.306	0.725	0.422

FIG. 1 (color online). The soliton wave functions obtained in the $HLS_1(\pi, \rho, \omega)$ model with $a = 2$. $F(r)$, $G(r)$, and $W(r)$ are given by the solid, dashed, and dot-dashed lines, respectively. Here, $F(r)$ and $G(r)$ are dimensionless, but $W(r)$ is in units of $1/fm$.
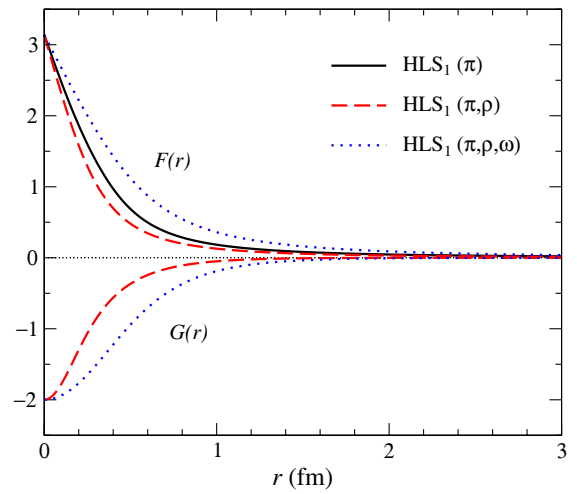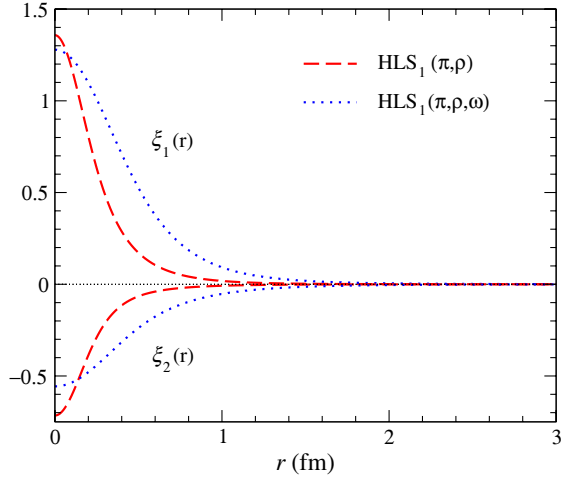

FIG. 3 (color online). Comparison of the soliton wave functions $F(r)$ and $G(r)$ in the three models, $HLS_1(\pi)$, $HLS_1(\pi, \rho)$, and $HLS_1(\pi, \rho, \omega)$, which are represented by the solid line, dashed lines, and dotted lines, respectively.

comparing the results from the $HLS_1(\pi)$ and the $HLS_1(\pi, \rho)$ models. However, as can be seen by the dotted lines, the inclusion of the ω meson expands the wave functions. All these behaviors can be found in the rms sizes $\sqrt{\langle r^2 \rangle_W}$ and $\sqrt{\langle r^2 \rangle_E}$ in Table II. Therefore, we conclude that the ρ meson decreases the soliton mass while the ω meson increases it.

(2) In the moment of inertia, or in the Δ-N mass difference Δ_M, through the collective quantization, the role of the ρ and ω mesons are the opposite to the case of the soliton mass. The mass difference Δ_M increases by the inclusion of the ρ meson, i.e., from 1014 MeV in the $HLS_1(\pi)$ to 1707 MeV in the

FIG. 2 (color online). The excitations of the soliton profile, $\xi_1(r)$ (solid line), $\xi_2(r)$ (dashed line), and $\varphi(r)$ (dot-dashed line) calculated in the $HLS_1(\pi, \rho, \omega)$ model with $a = 2$. Here, $\xi_1(r)$ and $\xi_2(r)$ are dimensionless, while $\varphi(r)$ is in units of fm.

$HLS_1(\pi, \rho)$, which worsens the situation phenomenologically. Furthermore, in the nucleon and Δ masses, the rotational energy at $O(1/N_c)$ is even larger than the soliton mass, which is of $O(N_c)$. This raises a serious problem as to the validity of the collective quantization method in these models. However, the inclusion of the ω meson reduces Δ_M and the rotational energy appreciably. The comparison of the soliton wave functions obtained from the $O(1/N_c)$ rotational energy can be found in Fig. 4. This shows that $\xi_{1,2}(r)$ are expanded further in the $HLS_1(\pi, \rho, \omega)$ model than in the $HLS_1(\pi, \rho)$ model. This leads to a larger value for the moment of inertia in the $HLS_1(\pi, \rho, \omega)$ model, which leads to a smaller Δ_M through the relation given in Eq. (25).

(3) The contributions from each term of the Lagrangian (7) to the soliton mass and the moment of inertia are also analyzed as functions of the HLS parameter a. The results are summarized in Fig. 5. The contributions from the $O(p^2)$ terms, $O(p^4)$ terms, and the hWZ terms are represented by the dotted, dashed, and dot-dashed lines, respectively, while the solid lines are their sums. We first verify that the contribution from the $O(p^2)$ terms to the soliton mass increases with a. On the contrary, the contribution from the $O(p^4)$ terms has a negative slope with a and its magnitude is smaller than the $O(p^2)$ terms,[6] which shows that this order counting is reasonable

---

[6]The magnitude of the $O(p^4)$ contribution is about 15% of the $O(p^2)$ contribution at $a = 2$. The results shown in Fig. 5 show that the contribution from the $O(p^4)$ terms is smaller than that of the $O(p^2)$ terms for $a > 1$, which shows that the power counting works for the soliton mass and the moment of inertia at $O(N_c)$.

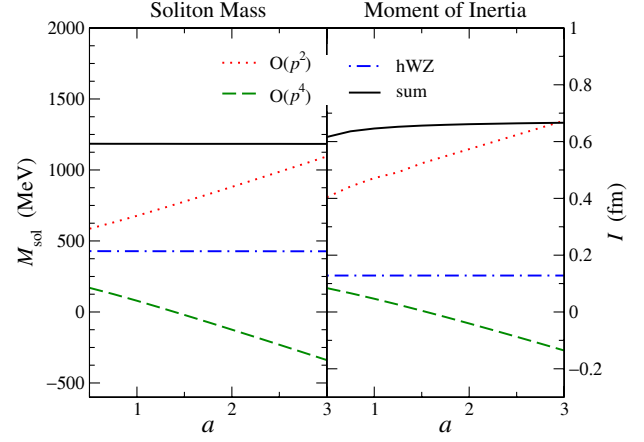FIG. 4 (color online). Same as Fig. 3 but for $\xi_1(r)$ and $\xi_2(r)$.

FIG. 5 (color online). Dependence of the soliton mass and the moment of inertia on the HLS parameter a in the $HLS_1(\pi, \rho, \omega)$ model. Dotted, dashed, and dot-dashed lines are the contributions from the $O(p^2)$, $O(p^4)$, and hWZ terms, respectively. Solid lines are their sums.

in the Skyrmion mass and size. However, the contribution from the hWZ terms that are connected to the ω meson is highly nontrivial. In particular, its contribution is stable as a becomes smaller while the $O(p^2)$ contribution decreases. As a result, when $a \rightarrow 1$, which corresponds to the value in the nuclear medium[7] [10], the contribution of the hWZ terms is close to that of the $O(p^2)$ terms. This evidently shows that the role of the ω meson may be even more addressed in nuclear matter. Therefore, it is highly desirable to investigate the role of the ω meson in more detail in Skyrmion matter. Our analysis shows that the three components of the Skyrmion mass represented by the dotted, dashed, and dot-dashed lines in Fig. 5 have very different behavior with a, but their sum is independent of the parameter a. Similar conclusions can be drawn from the decomposition of the moment of inertia as well. Here, we found a slight dependence on a, which might be related to the approximate method adopted by the collective quantization method. More detailed studies on the quantization method are, therefore, desirable.

All these observations show the importance of the ω meson in Skyrmions. The ω meson increases the soliton mass and decreases the moment of inertia, which is exactly the opposite to the role of the ρ meson. Furthermore, only when the ω meson is included is the rotational energy smaller than the soliton mass, and thus the standard collective quantization can be justified.

B. Skyrmion in the HLS induced from the BPS model

It was claimed in Refs. [27,28] that the BPS Skyrmion, i.e., the soliton in the flat-space 5D YM action, is a

---

[7]$1/N_c$ corrections are expected to be highly important in the medium, so this feature should be taken with caution.

potentially important feature in the Skyrmion structure. In this subsection, we determine the LECs of the HLS with the flat-space 5D YM action, which we call the BPS model. To investigate the warping-factor effect we consider a gauge theory in flat 5D Minkowski space-time. In the sense of large-'t Hooft parameter λ expansion, the flat space-time means that we are going to consider the $O(\lambda)$ terms since the warping effect is at $O(\lambda^0)$. In flat space-time, if the infinite tower of the KK mode is included and the CS term is turned off, the Skyrmion solution becomes the so-called BPS Skyrmion. In Refs. [27,28] the infinite tower is truncated to include low-lying isovector vector mesons. Here, we include the CS term to investigate the ω-meson effect. The flat space-time is defined by
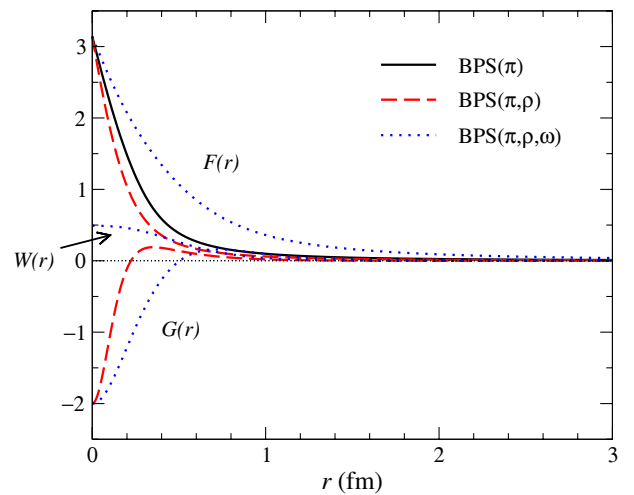
FIG. 6 (color online). Soliton wave functions $F(r)$ and $G(r)$ in BPS(π), BPS(π, ρ), and BPS(π, ρ, ω) models, which are represented by the solid line, dashed lines, and dotted lines, respectively. $W(r)$ is in units of $1/fm$.

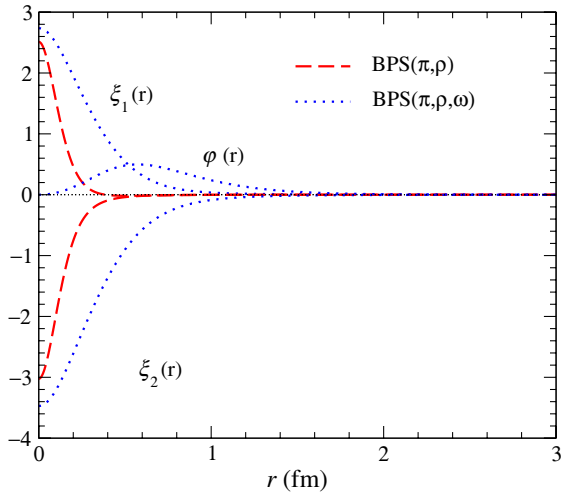FIG. 7 (color online). Comparison of the soliton wave functions $F(r)$ and $G(r)$ in the three models, BPS(π), BPS(π, ρ), and BPS(π, ρ, ω), which are represented by the solid line, dashed lines, and dotted lines, respectively. $W(r)$ is in units of $1/\mathrm{fm}$.

$$K_1(z) = K_2(z) = 1. \qquad (63)$$

As in the previous subsection, we consider the three models, namely, BPS(π), BPS(π, ρ), and BPS(π, ρ, ω). The obtained LECs and the soliton properties are also presented in Tables I and II, respectively. The soliton wave functions in the BPS models are shown in Figs. 6 and 7 as well.

We find that the role of the vector mesons is similar to the case of the HLS$_1$ model. Namely, the ρ meson shrinks the soliton while the ω meson expands it. Also, without the ω meson, the rotational energy, which is $O(1/N_c)$ in the baryon mass, is much larger than the soliton mass, which is of $O(N_c)$. This again raises serious questions on the validity of the collective rotation in the absence of the ω meson.

It is also interesting to note that the soliton mass and the moment of inertia obtained in the BPS(π, ρ, ω) model are
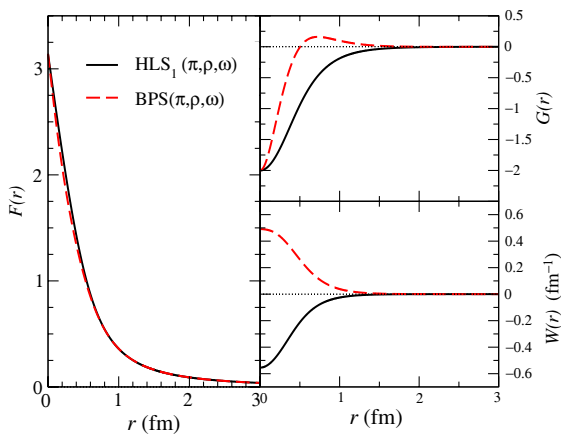
FIG. 8 (color online). Comparison of the soliton wave functions $F(r)$, $G(r)$, and $W(r)$ calculated in the HLS$_1(\pi, \rho, \omega)$ and BPS(π, ρ, ω) models.
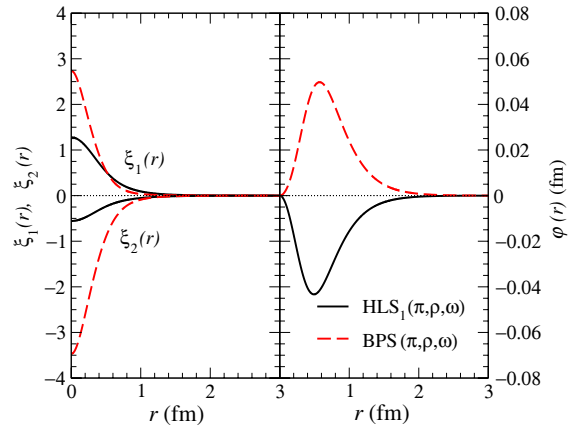

FIG. 9 (color online). Comparison of the soliton wave functions $\xi_1(r)$, $\xi_2(r)$, and $\varphi(r)$ calculated in the HLS$_1(\pi, \rho, \omega)$ and BPS(π, ρ, ω) models.

similar to those of the HLS$_1(\pi, \rho, \omega)$ model, while the difference of the corresponding results in the models without the ω meson is quite noticeable. Although the HLS$_1(\pi, \rho, \omega)$ and the BPS(π, ρ, ω) models give similar results for the soliton mass and the moment of inertia, the obtained soliton wave functions are very different. To understand this coincidence, we compare the soliton wave functions in these two models in Figs. 8 and 9, which clearly show the difference between the two models, especially in the wave functions of the ρ meson. The sign difference of the wave functions of ω_μ is due to the different sign in the source terms, i.e., the signs of c_is in Table II, and it can be seen that their magnitudes are similar in Figs. 8 and 9.

The difference between the HLS$_1(\pi, \rho, \omega)$ and the BPS(π, ρ, ω) models can be easily seen in the breakdown of the soliton mass and the moment of inertia. Shown in Fig. 10 are the contributions from $O(p^2)$, $O(p^4)$, and the hWZ terms that affect these physical quantities. We first found that the dependence of these quantities on a is very similar to that of the HLS$_1$ model. Contrary to the
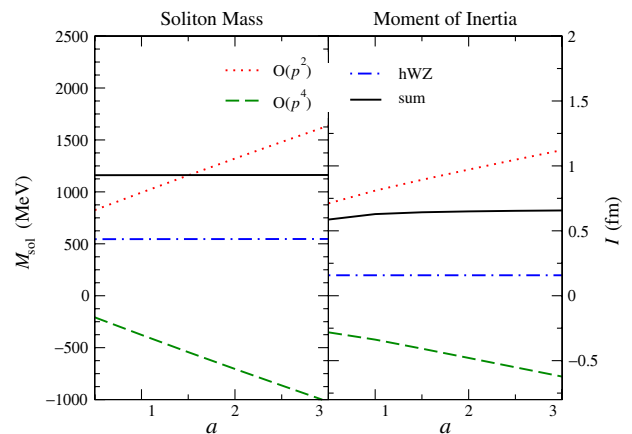

FIG. 10 (color online). Same as in Fig. 5 but in the BPS(π, ρ, ω) model.

HLS$_1(\pi, \rho, \omega)$ model, however, the $O(p^4)$ contribution is as large as 50% of that of the $O(p^2)$ terms at $a = 2$. Therefore, although the obtained soliton mass and the moment of inertia have similar values in both models, their breakdown clearly shows their difference. Since the $O(p^4)$ contribution in the BPS model is not suppressed enough, it would be interesting to study the contributions from the higher-order terms to see the convergence of the BPS model.

V. SUMMARY AND DISCUSSION

Solitonic solutions in holographic models have been studied in the literature in terms of the infinite tower of vector mesons. In this paper, we have investigated the role of vector mesons in the Skyrmion properties based on the HLS Lagrangian up to $O(p^4)$, which is obtained by integrating out the vector mesons other than the lowest ρ and ω mesons in the holographic model. In particular, by including the hWZ terms in the HLS, we have studied the role of the ω meson explicitly in the soliton structure. All the LECs of the HLS Lagrangian could be determined self-consistently from a class of holographic QCD models by making use of the general master formulas (35). In the present work, we considered two hQCD models, namely, the SS model and the BPS model. Equipped with the LECs of the HLS Lagrangian determined in this way, we have computed the Skyrmion properties and compared the results with those of the models of the $O(p^2)$ HLS.

The results summarized in Table II clearly show the important role of vector mesons, in particular, that of the ω vector meson. As claimed in the literature [16,27,28,34], we confirmed that the inclusion of the ρ meson reduces the size of the soliton and decreases the soliton mass so that the obtained Skyrmion mass becomes closer to the Bogomol'nyi bound. It also decreases the moment of inertia, which leads to a larger value of the Δ-N mass splitting. However, in the model of pion only or of pion and ρ meson, the obtained moment of inertia is very small so that the rotational energy of $O(1/N_c)$ becomes even larger than the soliton mass of $O(N_c)$. This strongly raises the question on the validity of the collective rotation in these models.[8]

We then studied the role of the ω meson that couples to the π and ρ mesons through the hWZ terms in the HLS, which are induced from the CS term in hQCD models. Contrary to the role of the ρ meson, the ω meson inflates the soliton size and increases the soliton mass, while it reduces the rotational energy by increasing the moment of inertia. Only when the ω meson is included is the rotational energy smaller than the soliton mass, which validates the application of the collective quantization. This shows that the ω meson has an important role in phenomenology as well.

We further confirmed that the obtained Skyrmion properties are independent of the HLS parameter a. This is the consequence of the generic properties of the hQCD models related to the normalization of the 5D wave functions. Given that this relation holds in the large-N_c limit, the a independence will necessarily break down in a dense medium where $1/N_c$ corrections are expected to be important.[9]

It should be pointed out that lessons from nuclear physics indicate that the ω meson is responsible for the repulsive interaction which prevents the nuclei from collapsing and the sigma meson (not the fourth component of the chiral four-vector in sigma models, but a scalar meson relating to the Casimir effects) causes the attractive interaction, and the near cancellation of the these two interactions gives the small binding energy of nuclear matter of ~ 16 MeV. Although it is very difficult to treat this Casimir effect given that we have a nonrenormalizable theory, it was shown that the one-loop corrections have a nontrivial role in the properties of nucleons. For example, it gives the Casimir contribution of the order of -500 MeV, which goes in the right direction with the correct order of magnitude [36,37]. Furthermore, in Ref. [37], by adopting a chiral Lagrangian of the pion, its effect was shown to also be important in evaluating many properties of the nucleon. Therefore, it would be desirable to estimate the one-loop corrections in a model with explicit vector mesons by employing the HLS Lagrangian employed in the present work.

Since our study shows the importance of the ω meson in the Skyrmion structure, it is natural to investigate its effects in the Skyrmion matter. The approach adopted in the present paper based on the HLS with self-consistently determined LECs from hQCD can be extended to the study on dense matter. This can be done by constructing the Skyrmion's crystal lattice to determine the critical density at which a Skyrmion (or an instanton) transforms into two half-Skyrmions [38,39] (or half-instantons/dyons [40]). This will be important in understanding the equation of state for compact-star matter, as shown in Ref. [41]. As suggested in Ref. [5], a reliable treatment will require low-mass scalar degrees of freedom, which will figure in at subleading order in N_c. Such work is in progress and will be reported elsewhere.

ACKNOWLEDGMENTS

---

[8]See, for example, Ref. [35] for the validity of the collective quantization in the Skyrme model.

[9]It is shown in Ref. [10] that the a independence in the meson sector is broken by the loop corrections and $1 \leq a \leq 2$ is preferred by phenomenology. Therefore, the single soliton properties would depend on the value of a when the loop effects are considered. This may be related to the Casimir effect and deserves further investigation.

APPENDIX A: THE SOLITON MASS AND THE EQUATIONS OF MOTION FOR $F(r)$, $W(r)$, AND $G(r)$

Using the wave functions defined in Eqs. (16) and (17) and the Lagrangian in Eq. (7), the soliton mass in the HLS up to $O(p^4)$ is obtained as

$$M_{\text{sol}} = 4\pi \int dr[M_{(2)}(r) + M_{(4)}(r) + M_{\text{anom}}(r)], \tag{A1}$$

where $M_{(2)}$, $M_{(4)}$, and M_{anom} are from $\mathcal{L}_{(2)}$, $\mathcal{L}_{(4)y} + \mathcal{L}_{(4)z}$, and $\mathcal{L}_{\text{anom}}$, respectively. Their explicit forms are

$$M_{(2)}(r) = \frac{f_\pi^2}{2}(F'^2 r^2 + 2\sin^2 F) - \frac{ag^2 f_\pi^2}{2} W^2 r^2 + af_\pi^2\left(G + 2\sin^2\frac{F}{2}\right)^2 - \frac{W'^2 r^2}{2} + \frac{G'^2}{g^2} + \frac{G^2}{2g^2 r^2}(G+2)^2, \tag{A2}$$

$$\begin{aligned}
M_{(4)}(r) = {} & -y_1\frac{r^2}{8}\left(F'^2 + \frac{2}{r^2}\sin^2 F\right)^2 - y_2\frac{r^2}{8}F'^2\left(F'^2 - \frac{4}{r^2}\sin^2 F\right) - y_3\frac{r^2}{2}\left[\frac{g^2 W^2}{2} - \frac{1}{r^2}\left(G + 2\sin^2\frac{F}{2}\right)^2\right]^2 \\
& - y_4\frac{g^2 W^2 r^2}{2}\left\{\frac{g^2 W^2}{4} - \frac{1}{r^2}\left(G + 2\sin^2\frac{F}{2}\right)^2\right\} + \frac{y_5}{4}(r^2 F'^2 + 2\sin^2 F)\left[\frac{g^2 W^2}{2} - \frac{1}{r^2}\left(G + 2\sin^2\frac{F}{2}\right)^2\right] \\
& + \left(y_8 - \frac{y_7}{2}\right)\frac{\sin^2 F}{r^2}\left(G + 2\sin^2\frac{F}{2}\right)^2 + y_9\left\{\frac{g^2 W^2 r^2}{8}\left(F'^2 + \frac{2}{r^2}\sin^2 F\right) + \frac{F'^2}{4}\left(G + 2\sin^2\frac{F}{2}\right)^2\right\} \\
& + z_4\left\{G'F'\sin F + \frac{\sin^2 F}{2r^2}G(G+2)\right\} + \frac{z_5}{2r^2}G(G+2)\left(G + 2\sin^2\frac{F}{2}\right)^2, \tag{A3}
\end{aligned}$$

$$M_{\text{anom}}(r) = \alpha_1 F'W\sin^2 F + \alpha_2 WF'\left(G + 2\sin^2\frac{F}{2}\right)^2 - \alpha_3\left\{G(G+2)WF' + 2\sin F\left[WG' - W'\left(G + 2\sin^2\frac{F}{2}\right)\right]\right\}, \tag{A4}$$

where

$$\alpha_1 = \frac{3gN_c}{16\pi^2}(c_1 - c_2), \qquad \alpha_2 = \frac{gN_c}{16\pi^2}(c_1 + c_2), \qquad \alpha_3 = \frac{gN_c}{16\pi^2}c_3. \tag{A5}$$

The Euler-Lagrange equations for $F(r)$ and $G(r)$ are obtained as

$$\mathcal{A}_1 F'' + \mathcal{A}_2 G'' = \mathcal{B}, \qquad \mathcal{A}_3 G'' + \mathcal{A}_4 F'' = \mathcal{D}, \tag{A6}$$

where

$$\mathcal{A}_1 = f_\pi^2 r^2 - \frac{3}{2}(y_1 + y_2)r^2 F'^2 - (y_1 - y_2)\sin^2 F + (y_5 + y_9)\frac{g^2 W^2 r^2}{4} - (y_5 - y_9)\frac{1}{2}\left(G + 2\sin^2\frac{F}{2}\right)^2, \tag{A7}$$

$$\mathcal{A}_2 = z_4\sin F, \tag{A8}$$

$$\mathcal{A}_3 = 1, \tag{A9}$$

$$\mathcal{A}_4 = \frac{g^2}{2}z_4\sin F, \tag{A10}$$

and

$$\mathcal{B} = -2f_\pi^2 rF' + f_\pi^2 \sin 2F + 2af_\pi^2 \sin F\left(G + 2\sin^2\frac{F}{2}\right) + f_\pi^2 m_\pi^2 r^2 \sin F + (y_1 + y_2)rF'^3 + (y_1 - y_2)\frac{\sin 2F}{2}F'^2$$

$$- y_1\frac{\sin 2F}{r^2}\sin^2 F + (y_3 + y_4)g^2 W^2\left(G + 2\sin^2\frac{F}{2}\right)\sin F - y_3\frac{2}{r^2}\left(G + 2\sin^2\frac{F}{2}\right)^3 \sin F$$

$$- (y_5 + y_9)\frac{g^2 rW}{2}(W + rW')F' + (y_5 + y_9)\frac{g^2 W^2}{4}\sin 2F + (y_5 - y_9)\left(G + 2\sin^2\frac{F}{2}\right)\left(G' + \frac{1}{2}\sin FF'\right)F'$$

$$- y_5\frac{\sin 2F}{2r^2}\left(G + 2\sin^2\frac{F}{2}\right)^2 - y_5\frac{\sin^3 F}{r^2}\left(G + 2\sin^2\frac{F}{2}\right) + (2y_8 - y_7)\frac{\sin 2F}{2r^2}\left(G + 2\sin^2\frac{F}{2}\right)^2$$

$$+ (2y_8 - y_7)\frac{\sin^3 F}{r^2}\left(G + 2\sin^2\frac{F}{2}\right) + z_4\frac{\sin 2F}{2r^2}G(G + 2) + z_5\frac{G(G + 2)}{r^2}\left(G + 2\sin^2\frac{F}{2}\right)\sin F$$

$$- \alpha_1\sin^2 FW' - \alpha_2 W'\left(G + 2\sin^2\frac{F}{2}\right)^2 - 2\alpha_2 WG'\left(G + 2\sin^2\frac{F}{2}\right)$$

$$+ \alpha_3\left[G(G + 2)W' + 2\left(G + 2\sin^2\frac{F}{2}\right)G'W + 2\cos F\left(G + 2\sin^2\frac{F}{2}\right)W' + 2\sin^2 FW'\right], \tag{A11}$$

$$\mathcal{D} = ag^2 f_\pi^2\left(G + 2\sin^2\frac{F}{2}\right) + \frac{1}{r^2}G(G + 1)(G + 2) + y_3 g^2\left[\frac{g^2 W^2}{2} - \frac{1}{r^2}\left(G + 2\sin^2\frac{F}{2}\right)^2\right]\left(G + 2\sin^2\frac{F}{2}\right)$$

$$+ y_4\frac{g^4 W^2}{2}\left(G + 2\sin^2\frac{F}{2}\right) - y_5\frac{g^2}{4}\left(F'^2 + \frac{2}{r^2}\sin^2 F\right)\left(G + 2\sin^2\frac{F}{2}\right) + (2y_8 - y_7)\frac{g^2}{2r^2}\sin^2 F\left(G + 2\sin^2\frac{F}{2}\right)$$

$$+ y_9\frac{g^2}{4}F'^2\left(G + 2\sin^2\frac{F}{2}\right) - z_4\frac{g^2}{2}\cos FF'^2 + z_4\frac{g^2}{2r^2}\sin^2 F(G + 1) + z_5\frac{g^2}{2r^2}\left[(G + 1)\left(G + 2\sin^2\frac{F}{2}\right)\right.$$

$$\left. + G(G + 2)\right]\left(G + 2\sin^2\frac{F}{2}\right) + \alpha_2 g^2 WF'\left(G + 2\sin^2\frac{F}{2}\right) + \alpha_3 g^2\left[2W'\sin F - WF'\left(G + 2\sin^2\frac{F}{2}\right)\right]. \tag{A12}$$

The equation of motion of W reads

$$W'' = -\frac{2}{r}W' + ag^2 f_\pi^2 W + (y_3 + y_4)g^2 W\left[\frac{g^2 W^2}{2} - \frac{1}{r^2}\left(G + 2\sin^2\frac{F}{2}\right)^2\right] - (y_5 + y_9)\frac{g^2 W}{4}\left(F'^2 + \frac{2}{r^2}\sin^2 F\right)$$

$$- \alpha_1\frac{\sin^2 F}{r^2}F' - \alpha_2\frac{F'}{r^2}\left(G + 2\sin^2\frac{F}{2}\right)^2 + \frac{\alpha_3}{r^2}\left[(G^2 + 2G + 2\sin^2 F)F' + 2\cos F\left(G + 2\sin^2\frac{F}{2}\right)F' + 4\sin FG'\right]. \tag{A13}$$

This evidently shows that the hWZ terms, i.e., the c_i terms, are the source terms of $W(r)$.

APPENDIX B: MOMENT OF INERTIA AND EQUATIONS OF MOTION OF THE EXCITED FIELDS

When the collective rotation is introduced, the Lagrangian can be written as

$$L = -M_{\text{sol}} + \mathcal{I}\,\text{Tr}(\dot{A}\dot{A}^\dagger), \tag{B1}$$

where the moment of inertia $\mathcal{I}$ is summarized as

$$\mathcal{I} = 4\pi\int dr[\mathcal{I}_{(2)}(r) + \mathcal{I}_{(4)}(r) + \mathcal{I}_{\text{anom}}(r)]. \tag{B2}$$

The contributions from $\mathcal{L}_{(2)}$, $\mathcal{L}_{(4)y} + \mathcal{L}_{(4)z}$, and $\mathcal{L}_{\text{anom}}$ are represented by $\mathcal{I}_{(2)}(r)$, $\mathcal{I}_{(4)}(r)$, and $\mathcal{I}_{\text{anom}}(r)$, respectively. We further write

$$\mathcal{I}_{(4)} = \sum_i y_i\mathcal{I}_{y_i} + \sum_i z_i\mathcal{I}_{z_i}. \tag{B3}$$

The moment of inertia from $\mathcal{L}_{(2)}$ is obtained as

$$I_2(r) = \frac{2}{3} f_\pi^2 r^2 \sin^2 F + \frac{1}{3} a f_\pi^2 r^2 \left[(\xi_1 + \xi_2)^2 + 2\left(\xi_1 - 2\sin^2 \frac{F}{2}\right)^2 \right] - \frac{1}{6} a g^2 f_\pi^2 \varphi^2 - \frac{1}{6}\left(\varphi'^2 + \frac{2\varphi^2}{r^2}\right)$$

$$+ \frac{r^2}{3g^2}(3\xi_1'^2 + 2\xi_1'\xi_2' + \xi_2'^2) + \frac{4}{3g^2} G^2(\xi_1 - 1)(\xi_1 + \xi_2 - 1) + \frac{2}{3g^2}(G^2 + 2G + 2)\xi_2^2. \tag{B4}$$

Each term of $I_{(4)}$ is calculated as

$$I_{y_1}(r) = -\frac{1}{3} r^2 \sin^2 F\left(F'^2 + \frac{2}{r^2}\sin^2 F\right), \tag{B5}$$

$$I_{y_2}(r) = \frac{1}{3} r^2 \sin^2 F F'^2, \tag{B6}$$

$$I_{y_3}(r) = -\frac{1}{12} g^2 \varphi^2 \left[g^2 W^2 - \frac{4}{r^2}\left(G + 2\sin^2 \frac{F}{2}\right)^2 \right] + \frac{2}{3} g^2 W \varphi\left(G + 2\sin^2 \frac{F}{2}\right)\left(\xi_1 - 2\sin^2 \frac{F}{2}\right)$$

$$+ \left[\frac{1}{2} r^2 g^2 W^2 - \frac{1}{3}\left(G + 2\sin^2 \frac{F}{2}\right)^2 \right]\left[(\xi_1 + \xi_2)^2 + 2\left(\xi_1 - 2\sin^2 \frac{F}{2}\right)^2 \right], \tag{B7}$$

$$I_{y_4}(r) = \frac{r^2}{2} g^2 W^2 \left[(\xi_1 + \xi_2)^2 + 2\left(\xi_1 - 2\sin^2 \frac{F}{2}\right)^2 \right] - \frac{1}{12} g^2 W \varphi\left[g^2 W \varphi - 8\left(G + 2\sin^2 \frac{F}{2}\right)\left(\xi_1 - 2\sin^2 \frac{F}{2}\right) \right]$$

$$+ \frac{1}{3}\left(G + 2\sin^2 \frac{F}{2}\right)^2 \left[\frac{g^2 \varphi^2}{r^2} + (\xi_1 + \xi_2)^2 \right], \tag{B8}$$

$$I_{y_5}(r) = \frac{1}{6}\sin^2 F\left[r^2 g^2 W^2 - 2\left(G + 2\sin^2 \frac{F}{2}\right)^2 \right] - \frac{r^2}{12}\left(F'^2 + \frac{2}{r^2}\sin^2 F\right)\left[2\left(\xi_1 - 2\sin^2 \frac{F}{2}\right)^2 + (\xi_1 + \xi_2)^2 - \frac{g^2 \varphi^2}{2r^2} \right], \tag{B9}$$

$$I_{y_6}(r) = \frac{1}{6}\sin^2 F\left(rgW - \frac{g\varphi}{2r} \right)^2, \tag{B10}$$

$$I_{y_7}(r) = \frac{1}{6}\sin^2 F\left[\left(rgW - \frac{g\varphi}{2r} \right)^2 + 4\left(G + 2\sin^2 \frac{F}{2}\right)\left(\xi_1 - 2\sin^2 \frac{F}{2}\right) \right], \tag{B11}$$

$$I_{y_8}(r) = \frac{1}{3}\sin^2 F\left[\left(rgW - \frac{g\varphi}{2r} \right)^2 - 4\left(G + 2\sin^2 \frac{F}{2}\right)\left(\xi_1 - 2\sin^2 \frac{F}{2}\right) \right], \tag{B12}$$

$$I_{y_9}(r) = \frac{r^2}{6} g^2 W^2 \sin^2 F + \frac{r^2}{6} F'^2\left(\xi_1 - 2\sin^2 \frac{F}{2}\right)^2 - \frac{r^2}{12}\left(F'^2 - \frac{2}{r^2}\sin^2 F\right)(\xi_1 + \xi_2)^2 + \frac{1}{24} g^2 \varphi^2\left(F'^2 + \frac{2}{r^2}\sin^2 F\right), \tag{B13}$$

and

$$I_{z_4}(r) = \frac{2}{3}\sin^2 F[G(1 - \xi_1) + \xi_2] - \frac{2}{3} r^2 \sin F F' \xi_1', \tag{B14}$$

$$I_{z_5}(r) = -\frac{2}{3}\left(G + 2\sin^2 \frac{F}{2}\right)\{[G(1 - \xi_1 - \xi_2) - \xi_2](\xi_1 + \xi_2) + [G(1 - \xi_1) + \xi_2]\left(\xi_1 - 2\sin^2 \frac{F}{2}\right)\}. \tag{B15}$$

The hWZ terms give

$$I_{\text{anom}}(r) = \frac{gN_c}{8\pi^2}(c_1 - c_2)\varphi F' \sin^2 F - \frac{gN_c}{24\pi^2}(c_1 + c_2)\varphi F'\left(G + 2\sin^2 \frac{F}{2}\right)\left(\xi_1 - 2\sin^2 \frac{F}{2}\right)$$

$$+ \frac{gN_c}{24\pi^2} c_3\{\varphi F'(G\xi_1 - G - \xi_1 - 2\xi_2) + \varphi \sin F(\xi_1' - G') + \varphi' \sin F\left(G - \xi_1 + 4\sin^2 \frac{F}{2}\right)\}. \tag{B16}$$

Then the equations of motion for ξ_1, ξ_2, and φ are obtained as

$$\xi_1'' = -\frac{2}{r}\xi_1' + ag^2f_\pi^2\left(\xi_1 - 2\sin^2\frac{F}{2}\right) + \frac{G^2}{r^2}(\xi_1 - 1) - \frac{2}{r^2}(G+1)\xi_2 + \frac{3g^2}{4r^2}(\mathcal{F}_1 - \mathcal{F}_2)$$
$$- \frac{g^3N_c}{32\pi^2r^2}(c_1 + c_2)\varphi F'\left(G + 2\sin^2\frac{F}{2}\right) - \frac{g^3N_c}{32\pi^2r^2}c_3\left[2\varphi'\sin F - \varphi F'\left(G + 2\sin^2\frac{F}{2}\right)\right], \tag{B17}$$

$$\xi_2'' = -\frac{2}{r}\xi_2' + ag^2f_\pi^2\left(\xi_2 + 2\sin^2\frac{F}{2}\right) + \frac{G^2}{r^2}(\xi_1 + 2\xi_2 - 1) + \frac{6}{r^2}(G+1)\xi_2 + \frac{3g^2}{4r^2}(3\mathcal{F}_2 - \mathcal{F}_1)$$
$$+ \frac{g^3N_c}{32\pi^2r^2}(c_1 + c_2)\varphi F'\left(G + 2\sin^2\frac{F}{2}\right) + \frac{g^3N_c}{32\pi^2r^2}c_3[2\varphi'\sin F - \varphi F'(G + 5 - \cos F)], \tag{B18}$$

$$\varphi'' = \frac{2}{r^2}\varphi + ag^2f_\pi^2\varphi - 3\mathcal{F}_3 - \frac{3gN_c}{8\pi^2}(c_1 - c_2)F'\sin^2 F + \frac{gN_c}{8\pi^2}(c_1 + c_2)F'\left(G + 2\sin^2\frac{F}{2}\right)\left(\xi_1 - 2\sin^2\frac{F}{2}\right)$$
$$+ \frac{gN_c}{8\pi^2}c_3\left\{2\sin F(G' - \xi_1') + F'\left[G(1 + \cos F) - \xi_1\left(G - 2\sin^2\frac{F}{2}\right) + 2\xi_2 + 3\sin^2 F - 4\sin^4\frac{F}{2}\right]\right\}, \tag{B19}$$

where

$$\mathcal{F}_1 = y_3\left\{\frac{2}{3}g^2W\varphi\left(G + 2\sin^2\frac{F}{2}\right) + \left[r^2g^2W^2 - \frac{2}{3}\left(G + 2\sin^2\frac{F}{2}\right)^2\right]\left(3\xi_1 + \xi_2 - 4\sin^2\frac{F}{2}\right)\right\}$$
$$+ y_4\left\{r^2g^2W^2\left(3\xi_1 + \xi_2 - 4\sin^2\frac{F}{2}\right) + \frac{2}{3}g^2W\varphi\left(G + 2\sin^2\frac{F}{2}\right) + \frac{2}{3}\left(G + 2\sin^2\frac{F}{2}\right)^2(\xi_1 + \xi_2)\right\}$$
$$- y_5\frac{r^2}{6}\left(F'^2 + \frac{2}{r^2}\sin^2 F\right)\left(3\xi_1 + \xi_2 - 4\sin^2\frac{F}{2}\right) + (y_7 - 2y_8)\frac{2}{3}\sin^2 F\left(G + 2\sin^2\frac{F}{2}\right)$$
$$+ y_9\frac{r^2}{3}\left\{F'^2\left(\xi_1 - 2\sin^2\frac{F}{2}\right) - \frac{1}{2}\left(F'^2 - \frac{2}{r^2}\sin^2 F\right)(\xi_1 + \xi_2)\right\} + z_4\frac{2}{3}\left\{-G\sin^2 F + r^2\cos F F'^2\right.$$
$$+ r^2\sin F\left(F'' + \frac{2}{r}F'\right)\right\} - z_5\frac{4}{3}G\left(G + 2\sin^2\frac{F}{2}\right)\left(1 - 2\xi_1 - \xi_2 + \sin^2\frac{F}{2}\right), \tag{B20}$$

$$\mathcal{F}_2 = y_3\left[r^2g^2W^2 - \frac{2}{3}\left(G + 2\sin^2\frac{F}{2}\right)^2\right](\xi_1 + \xi_2) + y_4\left[r^2g^2W^2 + \frac{2}{3}\left(G + 2\sin^2\frac{F}{2}\right)^2\right](\xi_1 + \xi_2)$$
$$- y_5\frac{r^2}{6}\left(F'^2 + \frac{2}{r^2}\sin^2 F\right)(\xi_1 + \xi_2) - y_9\frac{r^2}{6}\left(F'^2 - \frac{2}{r^2}\sin^2 F\right)(\xi_1 + \xi_2) + z_4\frac{2}{3}\sin^2 F$$
$$- z_5\frac{2}{3}\left(G + 2\sin^2\frac{F}{2}\right)\left[G(1 - 2\xi_1) - 2(G+1)\xi_2 - 2\sin^2\frac{F}{2}\right], \tag{B21}$$

$$\mathcal{F}_3 = -y_3\frac{1}{6}g^2\varphi\left[g^2W^2 - \frac{4}{r^2}\left(G + 2\sin^2\frac{F}{2}\right)^2\right] + y_3\frac{2}{3}g^2W\left(G + 2\sin^2\frac{F}{2}\right)\left(\xi_1 - 2\sin^2\frac{F}{2}\right)$$
$$- y_4\frac{1}{6}g^2W\left[g^2W\varphi - 4\left(G + 2\sin^2\frac{F}{2}\right)\left(\xi_1 - 2\sin^2\frac{F}{2}\right)\right] + y_4\frac{2}{3}\frac{g^2\varphi}{r^2}\left(G + 2\sin^2\frac{F}{2}\right)^2$$
$$+ (y_5 + y_9)\frac{1}{12}g^2\varphi\left(F'^2 + \frac{2}{r^2}\sin^2 F\right) - (y_6 + y_7 + 2y_8)\frac{g}{6}\sin^2 F\left(gW - \frac{g\varphi}{2r^2}\right). \tag{B22}$$

[1] G. E. Brown and M. Rho, *The Multifaceted Skyrmions* (World Scientific, Singapore, 2010).

[2] H.-J. Lee, B.-Y. Park, D.-P. Min, M. Rho, and V. Vento, Nucl. Phys. **A723**, 427 (2003).

[3] M. Harada, S. Matsuzaki, and K. Yamawaki, Phys. Rev. D **74**, 076004 (2006).

[4] M. Harada, S. Matsuzaki, and K. Yamawaki, Phys. Rev. D **82**, 076010 (2010).

[5] Y.-L. Ma, Y. Oh, G.-S. Yang, M. Harada, H. K. Lee, B.-Y. Park, and M. Rho, Phys. Rev. D **86,** 074025 (2012).

[6] T. H. R. Skyrme, Proc. R. Soc. A **260,** 127 (1961).

[7] T. H. R. Skyrme, Nucl. Phys. **31,** 556 (1962).

[8] M. Bando, T. Kugo, S. Uehara, K. Yamawaki, and T. Yanagida, Phys. Rev. Lett. **54,** 1215 (1985).

[9] M. Bando, T. Kugo, and K. Yamawaki, Phys. Rep. **164,** 217 (1988).

[10] M. Harada and K. Yamawaki, Phys. Rep. **381,** 1 (2003).

[11] D. T. Son and M. A. Stephanov, Phys. Rev. D **69,** 065020 (2004).

[12] Y. Kim, I. J. Shin, and T. Tsukioka, arXiv:1205.4852.

[13] D. K. Hong, M. Rho, H.-U. Yee, and P. Yi, Phys. Rev. D **76,** 061901 (2007).

[14] D. K. Hong, M. Rho, H.-U. Yee, and P. Yi, J. High Energy Phys. 09 (2007) 063.

[15] D. K. Hong, M. Rho, H.-U. Yee, and P. Yi, Phys. Rev. D **77,** 014030 (2008).

[16] K. Nawa, H. Suganuma, and T. Kojo, Phys. Rev. D **75,** 086003 (2007).

[17] H. Hata, T. Sakai, S. Sugimoto, and S. Yamato, Prog. Theor. Phys. **117,** 1157 (2007).

[18] T. Fujiwara, Y. Igarashi, A. Kobayashi, H. Otsu, T. Sato, and S. Sawada, Prog. Theor. Phys. **74,** 128 (1985).

[19] Y. Igarashi, M. Johmura, A. Kobayashi, H. Otsu, T. Sato, and S. Sawada, Nucl. Phys. **B259,** 721 (1985).

[20] U.-G. Meissner, N. Kaiser, and W. Weise, Nucl. Phys. **A466,** 685 (1987).

[21] P. Jain, R. Johnson, U.-G. Meissner, N. W. Park, and J. Schechter, Phys. Rev. D **37,** 3252 (1988).

[22] E. Witten, Nucl. Phys. **B160,** 57 (1979).

[23] M. Tanabashi, Phys. Lett. B **316,** 534 (1993).

[24] G. S. Adkins, C. R. Nappi, and E. Witten, Nucl. Phys. **B228,** 552 (1983).

[25] J. Erlich, E. Katz, D. T. Son, and M. A. Stephanov, Phys. Rev. Lett. **95,** 261602 (2005).

[26] L. Da Rold and A. Pomarol, Nucl. Phys. **B721,** 79 (2005).

[27] P. Sutcliffe, J. High Energy Phys. 08 (2010) 019.

[28] P. Sutcliffe, J. High Energy Phys. 04 (2011) 045.

[29] U.-G. Meissner, Phys. Rep. **161,** 213 (1988).

[30] T. Sakai and S. Sugimoto, Prog. Theor. Phys. **113,** 843 (2005).

[31] T. Sakai and S. Sugimoto, Prog. Theor. Phys. **114,** 1083 (2005).

[32] B.-Y. Park, M. Rho, and V. Vento, Nucl. Phys. **A736,** 129 (2004).

[33] H.-J. Lee, B.-Y. Park, M. Rho, and V. Vento, Nucl. Phys. **A726,** 69 (2003).

[34] K. Nawa, A. Hosaka, and H. Suganuma, Phys. Rev. D **79,** 126005 (2009).

[35] R. A. Battye, S. Krusch, and P. M. Sutcliffe, Phys. Lett. B **626,** 120 (2005).

[36] M. A. Nowak, M. Rho, and I. Zahed, *Chiral Nuclear Dynamics* (World Scientific, Singapore, 1996).

[37] F. Meier and H. Walliser, Phys. Rep. **289,** 383 (1997).

[38] H. K. Lee, B.-Y. Park, and M. Rho, Phys. Rev. C **83,** 025206 (2011); **84,** 059902(E) (2011).

[39] H. K. Lee and M. Rho, arXiv:0905.0235.

[40] M. Rho, S.-J. Sin, and I. Zahed, Phys. Lett. B **689,** 23 (2010).

[41] H. Dong, T. T. S. Kuo, H. K. Lee, R. Machleidt, and M. Rho, arXiv:1207.0429.