Unified covariant treatment of hyperfine splitting for heavy and light mesons

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This paper aims at proving the fundamental role of a relativistic formulation for quarkonia models. We present a completely covariant description of a two-quark system interacting by the Cornell potential with a Breit term describing the hyperfine splitting. Using an appropriate procedure to calculate the Breit correction, we find heavy meson masses in excellent agreement with experimental data. We finally use our approach to describe the light quarks: even by taking average values of the running coupling constant, we prove that covariance properties and hyperfine splitting are sufficient to explain the light meson spectrum and to give a very good agreement with the data.

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I. INTRODUCTION

Potential models of interacting quark systems have a long history and are still a very lively subject of investigation: this is witnessed by the large number of research papers and reviews that keep being published [1], which we refer to for bibliography and exhaustive details on the subject. Since the first papers that gave a rather complete overall picture of the subject [2], the starting point is often a Schrödinger equation with a potential having a Coulomb behavior at the origin and confining at infinity; the relativistic corrections, together with the spin-orbit and the spin-spin contributions, are taken into account by adding terms which are treated perturbatively. As one can easily imagine, already at the first perturbation order a non-negligible amount of work is required when calculating the spectrum, and even more is required when the eigenfunctions are needed for the evaluation of transition amplitudes. Attempts have also been made to overcome the limitations of a potential model due to asymptotic freedom at short distances and to light quark creation: a description of these effects has been tried by means of screened potentials softening the Coulomb interaction at the origin and by letting the confining term saturate at infinity. The spin dependent interactions are then modeled by the Breit-Fermi potential with a δ function centered at the origin, which in many cases yields difficulties in explaining the hyperfine splittings of the spectra. Although this approximation may be good for heavy mesons, a smearing of the δ function has been proposed to get a better description of the small distance behavior: recent results [3], however, show that this point has not been settled.

A major point of discussion has always been the relevance of relativistic properties of the systems, not only in the obvious case of light mesons, but also for heavy mesons. A truly covariant formulation going beyond the "relativized" treatment has often been invoked and approaches in such direction have actually been worked out [4–7]. Besides giving internal consistency to the models, a covariant description provides a substantial clarification of the dynamical role of the different terms of the potential, due to the complete inclusion of all the relativistic effects. It therefore allows a better extension to the investigation of light and heavy-light mesons, where a nonrelativistic treatment encounters serious difficulties; moreover, as we will show in the following, the actual calculations turn out to be simplified. Many of the existing models are connected with field theory along the lines of the Bethe-Salpeter equation, and the spectra of the resulting equations are not of straightforward computation. Few models deal with a consistent relativistic description. In Ref. [5] a full spinor treatment is presented. The confinement is essentially obtained by a cutoff of the wave function at a fixed interparticle separation, the Breit interaction is differently treated for light and heavy mesons, and an *ad hoc* contact interaction is introduced: the approach is interesting but not fully covariant. A covariant formulation is given in Ref. [6]; however, since the main subject of investigation is the Regge trajectories, the assumed potential is just linear in the radial variable. The papers in Ref. [7] study a well-formulated relativistic model with a two-body Dirac equation derived from constraint dynamics. The interaction is first introduced by a relativistic extension of the Adler-Piran potential and then improved by the addition of a timelike confining vector potential, yielding good results.

We present here a canonical description of quarkonium, focusing on the complete covariance of the model and on the fermionic nature of the elementary constituents. The formulation originates from a wave equation for two relativistic fermions with arbitrary masses obtained from two Dirac operators coupled by the interaction [8]. We refer to those papers for the proofs of the full covariance, of the Schrödinger and the one-particle Dirac limits, as well as of

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the cyclicity of the relative time that avoids the difficulties of relative energy excitations. We observe that our construction has different assumptions from Ref. [7], so that the final equations and the results also are somewhat different. In Ref. [8] the hyperfine splitting of positronium was calculated, finding an agreement better than up to the fourth power of the fine structure constant with the results obtained by QED semiclassical expansions. In the present context we will use the simplest Cornell potential with a Breit term for the spin-spin interaction. Our purpose is to show that the full relativistic description and a proper perturbation treatment of the Breit term, avoiding the evaluation of a δ function at the origin, are already sufficient to give results in excellent agreement with the experimental data both for heavy and light mesons, contrary to some diffused ideas. Further improvements of the potential are an important issue which should be developed at a more phenomenological level of the investigation. For instance in our calculations we have used average values of the running coupling constant (RCC) for the different families of mesons, verifying *ex post* that the ratios of the assumed values are in agreement with those obtained from the well-known α_S curve [9]: this is possible because in each family the spread of the masses we have analyzed is sufficiently small, with the only irreducible exception of the pion. A fine-tuning of the RCC, modeled according to the α_S curve, should produce much better results.

II. THE TWO-FERMION WAVE EQUATION WITH CORNELL POTENTIAL AND BREIT TERM

The Dirac operators entering the wave equation prescribe the correct form for the interactions according to their tensorial nature: the Coulomb-like term of the Cornell potential is vectorial and thus minimally coupled to the energy; the linear term is scalar and therefore coupled to the mass. Indeed, only a scalar growing potential is actually confining, while an unbounded vector interaction is not [10]. We refer to Ref. [8] for the derivation of the radial system of the model. We call r_a, q_a the Wigner vectors of spin 1 given by the spatial parts of relative coordinates and momenta boosted to the frame with vanishing total spatial momentum, and we put $r = (r_a r_a)^{1/2}$ (sum over repeated indexes). We denote by $\gamma_{(i)}$ the gamma matrices acting in the spinor space of the ith fermion of mass $m_{(i)}$, $M = m_{(1)} + m_{(2)}$, and $\rho = |m_{(1)} - m_{(2)}|/M$. The vector and scalar couplings produce the terms $E + b/r$, $\frac{1}{2}(M + \sigma r)$, and the final wave equation reads

$$\Big[(\gamma^0_{(1)}\gamma_{(1)a} - \gamma^0_{(2)}\gamma_{(2)a})q_a + \frac{1}{2}(\gamma^0_{(1)} + \gamma^0_{(2)})(M + \sigma r) + \frac{1}{2}(\gamma^0_{(1)} - \gamma^0_{(2)})M\rho - \Big(E + \frac{b}{r}\Big) + V_B(r)\Big]\Psi(\vec{r}) = 0, \tag{1}$$

where

$$V_B(r) = \frac{b}{2r}\gamma^0_{(1)}\gamma_{(1)a}\gamma^0_{(2)}\gamma_{(2)b}\Big(\delta_{ab} + \frac{r_a r_b}{r^2}\Big) \tag{2}$$

is the Breit term generating the hyperfine splitting. As in Ref. [8], the first perturbation order of this term is evaluated by substituting $V_B(r)$ with $\varepsilon V_B(r)$ in (2) and taking the first derivative of the eigenvalues with respect to ε in $\varepsilon = 0$ from the numerical solutions of the differential equations. This could also be seen as an application of the spectral correspondence to the Feynman-Hellman theorem.

The radial system is obtained by diagonalizing angular momentum and parity. As in Ref. [8], it is formed by four algebraic plus four first order differential equations for each parity. Using the algebraic relations and defining the dimensionless variables Ω, w, s by

$$\sigma = \frac{M^2}{4}\Omega^{\frac{3}{2}}, \quad E = \frac{M}{2}(2 + \Omega w), \quad r = \frac{2}{M}\Omega^{-\frac{1}{2}}s, \tag{3}$$

the radial system for (1), replacing $V_B(r)$ by $\varepsilon V_B(r)$, is

$$\begin{pmatrix} u_1'(s) \\ u_2'(s) \\ u_3'(s) \\ u_4'(s) \end{pmatrix} + \begin{pmatrix} 0 & A_0(s) & -B_0(s) & 0 \\ A_\varepsilon(s) & 1/s & 0 & B_\varepsilon(s) \\ C_\varepsilon(s) & 0 & 2/s & A_\varepsilon(s) \\ 0 & D_\varepsilon(s) & A_0(s) & 1/s \end{pmatrix}\begin{pmatrix} u_1(s) \\ u_2(s) \\ u_3(s) \\ u_4(s) \end{pmatrix} = 0.$$

Here $A_0 = A_\varepsilon|_{\varepsilon=0}$, $B_0 = B_\varepsilon|_{\varepsilon=0}$, and $u'(s) = du(s)/ds$. Letting $J^2 = j(j+1)$, the even parity coefficients are

$$\begin{aligned} A_\varepsilon(s) &= \frac{2\sqrt{J^2}\rho}{\sqrt{\Omega}(sh(s) - 2\varepsilon b)}, \\ B_\varepsilon(s) &= \frac{(h^2(s)/2 - 2\rho^2/\Omega)s^2 - 2\varepsilon^2 b^2}{s^2 h(s) - 2\varepsilon b s}, \\ C_\varepsilon(s) &= \frac{h(s)}{2} + \frac{2\varepsilon b}{s} + \frac{2J^2}{2\varepsilon b s - s^2 h(s)} + \frac{2sk^2(s)}{4\varepsilon b - sh(s)}, \\ D_\varepsilon(s) &= \frac{2J^2}{s^2 h(s)} - \frac{4b^2\varepsilon^2 - s^2 h^2(s) + 4s^2 k^2(s)}{4\varepsilon b s - 2s^2 h(s)}, \end{aligned} \tag{4}$$

with $h(s) = (2 + \Omega w)/\sqrt{\Omega} + b/s$, $k(s) = (2 + \Omega s)/(2\sqrt{\Omega})$. The coefficients for the odd parity system are

$$\begin{aligned} A_\varepsilon(s) &= \frac{2\sqrt{J^2}k(s)}{2\varepsilon b - sh(s)}, \\ B_\varepsilon(s) &= \frac{4\varepsilon^2 b^2 - s^2 h^2(s) + 4s^2 k^2(s)}{4\varepsilon b s - 2s^2 h(s)}, \\ C_\varepsilon(s) &= \frac{h(s)}{2} + \frac{2J^2}{2\varepsilon b s - s^2 h(s)} + \frac{2\varepsilon b}{s} + \frac{2s\rho^2}{\Omega(4\varepsilon b - sh(s))}, \\ D_\varepsilon(s) &= -\frac{h(s)}{2} + \frac{2J^2}{s^2 h(s)} - \frac{\varepsilon b}{s} + \frac{2\rho^2 s}{\Omega(sh(s) - 2bs)}. \end{aligned} \tag{5}$$

Some details concerning the numerical method we have used are in order. The origin and infinity are the only singular points of the boundary value problem and no further singularities arise from the matrix of the

TABLE I. The $b\bar{b}$ levels in MeV. First column: term symbol, $I^G(J^{PC})$ numbers, particle name. $\sigma = 1.111$ GeV/fm, $\alpha = 0.3272$, $m_b = 4725.5$ MeV. Experimental data from Ref. [9]. Dots indicate that data are not available.

State	Exp.	Num.
$(1^1s_0)0^+(0^{-+})\eta_b$	9390.90 ± 2.8	9390.39
$(1^3s_1)0^-(1^{--})\Upsilon$	9460.30 ± .25	9466.10
$(1^3p_0)0^+(0^{++})\chi_{b0}$	9859.44 ± .73	9857.41
$(1^3p_1)0^+(1^{++})\chi_{b1}$	9892.78 ± .57	9886.70
$(1^1p_1)0^-(1^{+-})h_b$	9898.60 ± 1.4	9895.35
$(1^3p_2)0^+(2^{++})\chi_{b2}$	9912.21 ± .57	9908.14
$(2^1s_0)0^+(0^{-+})\eta_b$	⋯	9971.14
$(2^3s_1)0^-(1^{--})\Upsilon$	10023.26 ± .0003	10009.04
$(1^3d_1)0^-(1^{--})\Upsilon$	⋯	10143.84
$(1^3d_2)0^-(2^{--})\Upsilon_2$	10163.70 ± 1.4	10152.69
$(1^1d_2)0^+(2^{-+})\eta_{b2}$	⋯	10154.79
$(1^3d_3)0^-(3^{--})\Upsilon_3$	⋯	10160.91
$(2^3p_0)0^+(0^{++})\chi_{b0}$	10232.50 ± .0009	10232.36
$(2^3p_1)0^+(1^{++})\chi_{b1}$	10255.46 ± .0005	10256.58
$(2^1p_1)0^-(1^{+-})h_b$	⋯	10263.61
$(2^3p_2)0^+(2^{++})\chi_{b2}$	10268.65 ± .0007	10274.26
$(3^1s_0)0^+(0^{-+})\eta_b$	⋯	10334.98
$(3^3s_1)0^-(1^{--})\Upsilon$	10355.20 ± .0005	10364.52
$(3^3p_0)0^+(0^{++})\chi_{b0}$	⋯	10534.86
$(3^3p_1)0^+(1^{++})\chi_{b1}$	$\langle 10530 \pm .014\rangle_J$	10556.59
$(3^3p_2)0^+(2^{++})\chi_{b2}$	⋯	10572.44
$(4^3s_1)0^-(1^{--})\Upsilon$	10579.40 ± .0012	10655.34
$(5^3s_1)0^-(1^{--})\Upsilon$	10876 ± 11	10910.35

TABLE II. The $c\bar{c}$ levels in MeV. $\sigma = 1.111$ GeV/fm, $\alpha = 0.435$, $m_c = 1394.5$ MeV. Experimental data from Ref. [9]. Question marks indicate that the assignments of the corresponding quantum numbers are not available.

State	Exp.	Num.
$(1^1s_0)0^+(0^{-+})\eta_c$	2978.40 ± 1.2	2978.26
$(1^3s_1)0^-(1^{--})J/\psi$	3096.916 ± .011	3097.91
$(1^3p_0)0^+(0^{++})\chi_{c0}$	3414.75 ± .31	3423.88
$(1^3p_1)0^+(1^{++})\chi_{c1}$	3510.66 ± .07	3502.83
$(1^1p_1)0^-(1^{+-})h_c$	3525.41 ± .16	3523.67
$(1^3p_2)0^+(2^{++})\chi_{c2}$	3556.20 ± .09	3555.84
$(2^1s_0)0^+(0^{-+})\eta_c$	3637 ± 4	3619.64
$(2^3s_1)0^-(1^{--})\psi$	3686.09 ± .04	3692.91
$(1^3d_1)0^-(1^{--})\psi$	3772.92 ± .35	3808.48
$(1^3d_2)0^-(2^{--})$	⋯	3833.62
$(1^1d_2)0^+(2^{-+})$	⋯	3839.20
$(1^3d_3)0^-(3^{--})$	⋯	3855.18
$(2^3p_0)0^+(0^{++})\chi_{c0}$	⋯	3898.00
$0^+(?^{?+})$ X(3872)	3871.57 ± .25	⋯
$(2^3p_1)0^+(1^{++})\chi_{c1}$	⋯	3961.21
$(2^1p_1)0^-(1^{+-})h_c$	⋯	3977.71
$0^+(?^{?+})$ X(3915)	3917.4 ± 2.7	⋯
$(2^3p_2)0^+(2^{++})\chi_{c2}$	3927 ± 2.6	4003.93
$?^+(?^{??})$ X(3940)	3942 ± 13	⋯
$(3^1s_0)0^+(0^{-+})\eta_c$	⋯	4064.21
$(3^3s_1)0^-(1^{--})\psi$	4039 ± 1	4122.95
$(2^3d_1)0^-(1^{--})\psi$	4153 ± 3	4200.51
$(4^3s_1)0^-(1^{--})\psi$	4421 ± 4	4479.22

coefficients. The solution was obtained by a double shooting method, the spectral relation being the vanishing of the 4×4 determinant of the matching conditions at a crossing point [8]. Padé techniques have been used to improve the accuracy of the approximate solutions at zero and infinity. The integration precision has always been kept very high and tested against the stability of the spectral values. The results are displayed in Tables I, II, III, IV, and V.

III. DISCUSSION OF THE NUMERICAL RESULTS

As stated in the Introduction, in order to have the best possible test of the relevance of the relativistic dynamics in quarkonium models, we have aimed at choosing the least number of fit parameters. Flavor independence could be expected for heavy quarks. In fact, by doing separate fits for $b\bar{b}$, $b\bar{s}$, and $c\bar{c}$, we find that the string tensions turn out to be the same within the computation precision. The same values of σ and of the masses are taken for the unique measured Bc state. We introduce $\alpha = (3/4)b$, where b is the parameter of the Cornell potential appearing in Eqs. (4) and (5). We have previously said that, due to the small spread of respective masses, we will assume for each family of mesons a constant α determined by a separate fit. We then verify that the ratios of the α parameters numerically obtained for the different families of mesons are very close to the corresponding ratios of the values of α_S, evaluated at an approximately average mass of the mesons of the family; equivalently, the ratios of α_S with respect to the corresponding α have an average equal to 0.55 with a standard deviation equal to 0.013. This is shown in Table VI. This indicates that, up to a proportionality factor, our values of α reproduce the behavior of the RCC α_S within the allowed uncertainty bounds [9]. In Table VII, we finally give some explicit values of the Breit corrections $\Delta_{q\bar{q}}$ for different states. As expected, the corrections decrease for increasing j and become more and more important for decreasing quark masses.

The spectra show common features, generally shared by all potential models: the states group into doublets of s states and quadruplets of $p, d, \ldots$ states. It clearly appears that the results are in very good agreement with experimental data below the thresholds of B and D mesons [9] for $b\bar{b}$ and $c\bar{c}$, respectively. Above the thresholds, the calculated energies of the levels are larger than the experimental ones and a softened potential could make a sensible difference in reproducing the data of higher levels. The regularity of the pattern is, however, maintained. From Table II, for instance, as the resonance X(3782) has the two possible assignments $J^{PC} = 1^{++}$ and 2^{-+} [9], the

TABLE III. The $s\bar{s}$ levels in MeV. $\sigma = 1.34$ GeV/fm, $\alpha = 0.6075$, $m_s = 134.27$ MeV. Experimental data from Ref. [9].

State	Exp.	Num.
$(1^1s_0)0^+(0^{-+})$	...	818.12
$(1^3s_1)0^-(1^{--})\phi$	$1019.455 \pm .020$	1019.44
$(1^3p_0)0^+(0^{++})$	...	1206.44
$(1^3p_1)0^+(1^{++})$ f$_1$(1420)	$1426.4 \pm .9$	1412.84
$(1^1p_1)0^-(1^{+-})$	...	1458.59
$(1^3p_2)0^+(2^{++})$ f$'_2$(1525)	1525 ± 5	1525.60
$(2^1s_0)0^+(0^{-+})$	...	1554.68
$(2^3s_1)0^-(1^{--})\phi$	1680 ± 20	1698.41
$?^?(1^{--})$ X(1750)	1753.5 ± 3.8	...
$(1^3d_1)0^-(1^{--})$	...	1776.53
$(1^3d_2)0^-(2^{--})$	...	1838.72
$(2^3p_0)0^+(0^{++})$	...	1841.12
$(1^1d_2)0^+(2^{-+})$	...	1851.44
$(1^3d_3)0^-(3^{--})$ ϕ_3(1850)	1854 ± 7	1880.85
$(2^3p_1)0^+(1^{++})$	...	1988.38
$(2^1p_1)0^-(1^{+-})$	...	2021.97
$(2^3p_2)0^+(2^{++})$ f$_2$(2010)	2011 ± 70	2073.15
$(3^1s_0)0^+(0^{-+})$	...	2099.15
$(3^3s_1)0^-(1^{--})\phi$	2175 ± 15	2217.57

model could indicate a χ_{c1} classification. On the other hand, nothing can be suggested for X(3915) and X (3940), having no accepted quantum numbers. The situation is simpler in Table I, where there are no unclassified physical states. We point out the good estimate of the recently discovered $\chi_b(3P)$ resonance [9], staying just below the B production threshold. On the contrary, the calculated values for $\Upsilon(4^3s_1)$ and $\Upsilon(5^3s_1)$ exceed the experimental data.

We next consider the $s\bar{s}$ system, for which there are few accepted experimental states. The much lighter mass of the s quark highly enhances the relativistic character of the $s\bar{s}$ composite system and the fundamental role of the Breit corrections, giving rise to large hyperfine splittings. Due to these reasons, the string tension σ has not been given the same value of the previous systems but has been considered a fitting parameter, finding a value larger than in $b\bar{b}$. We report our results in Table III, where we have also included the unassigned f1(1420), X(1750), ϕ_3(1850), and ϕ(2170). Although we cannot have a complete phenomenological confidence in the numerical results, a fair number of experimental data can still be accommodated with a good accuracy. For instance, the model could suggest a 1^3d_1 assignment for X(1750).

We then study mesons composed of quarks with different flavors: we use the mass of the s quark together with the b and c masses to determine the levels of the Bs and Ds mesons, reported in Table IV. The discussion of some of these states is more delicate. Such is, in particular, the classification of D$_{s0}$(2317) and D$_{s1}$(2460), due to the fact that they are very narrow and their observation occurred through isospin violating decays. Theoretical analyses have been produced to explain these mesons as tetraquark structures or DK molecules (see, e.g., Ref. [11]). Different proposals in favor of the $c\bar{s}$ nature of these states have also been given (see, e.g., Ref. [12]). More recently, however, the data from their radiative decays have been argued to be consistent with their interpretation

TABLE IV. The Bc, Bs, and Ds levels in MeV. $\sigma = 1.111$, 1.111, 1.227 GeV/fm and $\alpha = 0.3591$, 0.3975, 0.5344, respectively.

State	Exp.	Num.
$(1^1s_0)0(0^-)$ $B_c^\pm$	$6277 \pm .006$	6277
$(1^1s_0)0(0^-)$ B_s^0	$5366.77 \pm .24$	5387.41
$(1^3s_1)0(1^-)$ B_s^*	5415.4 ± 2.1	5434.34
$(1^3p_0)0(0^+)$	...	5711.71
$(1^3p_1)0(1^+)$	...	5753.89
$(1^1p_1)0(1^+)$ $B_{s1}(5830)^0$	$5829.4 \pm .7$	5817.80
$(1^3p_2)0(2^+)$ $B_{s2}(5840)^0$	$5839.7 \pm .6$	5829.33
$(1^1s_0)0(0^-)$ D_s	$1968.49 \pm .32$	1961.24
$(1^3s_1)0(1^-)$ D_s^*	$2112.3 \pm .50$	2101.78
$(1^3p_0)0(0^+)$ $D_{s0}(2317)$	$2317.8 \pm .6$	2339.94
$(1^3p_1)0(1^+)$ $D_{s1}(2460)$	$2459.6 \pm .6$	2466.15
$(1^1p_1)0(1^+)$ $D_{s1}(2536)$	$2535.12 \pm .13$	2535.82
$(1^3p_2)0(2^+)$ $D_{s2}^*(2573)$	$2571.9 \pm .8$	2574.92
$(2^1s_0)0(0^-)$ $D_s(2632)$	2632.6 ± 1.6	2613.98
$(2^3s_1)0(1^-)$ $D_{sJ}^*(2710)$	2709 ± 9	2716.67
$(1^3d_1)0(1^-)$	...	2821.30
$(1^3d_2)0(2^-)$	...	2857.08
$(1^1d_2)0(2^-)$	...	2881.48
$(2^3p_0)0(0^+)$	...	2885.44
$(1^3d_3)0(3^-)$ $D_{sJ}(2860)$	2862 ± 7	2900.14
$(2^3p_1)0(1^+)$	...	2983.53
$(2^1p_1)0(1^+)$ $D_{sJ}(3040)$	3044 ± 38	3029.01
$(2^3p_2)0(2^+)$	...	3062.61

TABLE V. The $u\bar{d}$ levels in MeV. $\sigma = 1.34$ GeV/fm, $\alpha = 0.656$, $m_d = 6.1$ MeV, $m_u = 2.94$ MeV.

State	Exp.	Num.
$(1^1s_0)0^+(0^{-+})$ $\pi^\pm$	$139.57018 \pm .00035$	616.45[a]
$(1^3s_1)1^+(1^{--})$ $\rho(770)$	$775.49 \pm .39$	826.14
$(1^3p_0)1^-(0^{++})$ $a_0(980)$	$980. \pm 20$	970.34
$(1^3p_1)1^-(1^{++})$ $a_1(1260)$	$1230. \pm 40$	1204.66
$(1^1p_1)1^+(1^{+-})$ $b_1(1235)$	1229.5 ± 3.2	1274.76
$(1^3p_2)1^-(2^{++})$ $a_2(1320)$	$1318.3 \pm .6$	1325.40
$(2^1s_0)1^-(0^{-+})$ $\pi(1300)$	1300 ± 100	1337.36
$(2^3s_1)1^+(1^{--})$ $\rho(1450)$	1465 ± 25	1497.63
$(1^3d_1)1^+(1^{--})$ $\rho(1570)$	1570[b]	1565.42
$(3^1s_0)1^-(0^{-+})$ $\pi(1800)$	1812 ± 12	1882.30
$(3^3s_1)1^+(1^{--})$ $\rho(1900)$	1900[b]	2016.35

[a]Since the curve of the RCC α_S has a steep increase for low masses, the value for $\pi^\pm$ in the table is obviously affected by a very large error. The experimental value is reproduced by $\alpha = 0.99$. To a smaller extent, the argument holds also for $\rho(770)$.
[b]Meson Summary Table [9].

TABLE VI. Behavior of α_{num} vs α_S for average values $\Lambda_S = 0.221, 0.296, 0.349$ GeV for $n_f = 5, 4, 3$.

Ratios of α_{num}	Ratios of α_S
$\alpha_{b\bar{b}}/\alpha_{c\bar{c}} = 0.752$	$\alpha_S(\chi_{b1,1p})/\alpha_S(\chi_{c0,1p}) = 0.754$
$\alpha_{b\bar{b}}/\alpha_{b\bar{c}} = 0.911$	$\alpha_S(\chi_{b1,1p})/\alpha_S(B_c^{\pm}) = 0.914$
$\alpha_{b\bar{c}}/\alpha_{b\bar{s}} = 0.903$	$\alpha_S(B_c^{\pm})/\alpha_S(B_s^*) = 0.955$
$\alpha_{b\bar{c}}/\alpha_{c\bar{s}} = 0.672$	$\alpha_S(B_c^{\pm})/\alpha_S(D_c^{*\pm}) = 0.686$
$\alpha_{c\bar{c}}/\alpha_{s\bar{s}} = 0.716$	$\alpha_S(\chi_{c0,1p})/\alpha_S(f_{1,1p}) = 0.714$
$\alpha_{s\bar{s}}/\alpha_{u\bar{d}} = 0.926$	$\alpha_S(f_{1,1p})/\alpha_S(a_{1,1p}) = 0.933$

as usual $c\bar{s}$ mesons, thus completing the p-wave multiplet [13]. Our model is in full agreement with this point of view. A similar agreement is also met with $D_{sJ}^*(2710)$ corresponding the first radial excitation of D_s^*. The model would then suggest a classification of $D_s(2632)$ as 2^1s_0. Moreover the state $D_{sJ}(3040)$ appears to be one of the two states with $J^P = 1^+$, $n = 2$, as proposed in Ref. [13]. More debated, again, is the interpretation of $D_{sJ}(2860)$ [14,15]. However, since this state decays to two pseudoscalars, its quantum numbers can be $J^P = 0^+, 1^-, 2^+, \ldots$. In Refs. [13,15] for $D_{sJ}(2860)$, a $J^P = 3^-$ assignment was suggested: since our model finds the d-wave masses for the $c\bar{s}$ mesons considerably lower than many quark models, we find a very good agreement with such a prediction. Finally we recall that it has been observed that systems with different mass components are more affected by the relativistic effects (see, e.g., Richard in Ref. [1]): this is probably the reason why our covariant framework gives a better agreement with the experimental data for the Ds mesons than most nonrelativistic models [2], where the masses are generally overestimated.

We lastly look at the lightest $u\bar{d}$ mesons, for which the Breit correction, as commonly calculated, is considered to be insufficient to reproduce the data. We have again fitted the data with a constant RCC. The fit includes also the very light $\rho(770)$, but obviously excludes the $\pi^{\pm}$ for which the use of a higher α cannot be avoided, due to the steepness of the α_S curve for very low masses. The

TABLE VII. The Breit correction Δ_B in MeV for some levels of $b\bar{b}$, $c\bar{c}$, $s\bar{s}$.

State	$\Delta_B(b\bar{b})$	$\Delta_B(c\bar{c})$	$\Delta_B(s\bar{s})$	$\Delta_B(u\bar{d})$
$(1^1s_0)0^+(0^{-+})$	92.31	155.22	296.81	600.12[a]
$(1^3s_1)0^-(1^{--})$	18.09	38.80	94.37	106.21
$(1^3p_0)0^+(0^{++})$	44.30	117.41	297.14	334.57
$(1^3p_1)0^+(1^{++})$	19.98	52.14	127.83	142.63
$(1^1p_1)0^-(1^{+-})$	15.95	43.24	110.77	124.42
$(1^3p_2)0^+(2^{++})$	7.51	21.10	55.93	63.72
$(2^3s_1)0^-(1^{--})$	24.31	60.02	134.22	147.94
$(1^3d_1)0^-(1^{--})$	17.49	49.32	123.85	139.59

[a]The Breit correction for $\pi^{\pm}$ has been calculated using the value $\alpha = 0.99$ that reproduces the physical mass.

results are not very sensitive to the mass ratio ρ that we fix at the physical value 0.35; the string tension appears to be the same found for $s\bar{s}$. As for some of the Ds mesons previously mentioned, we recall that also the nature of the state $a_0(980)$ has been widely discussed [16]: we will only say that the results coming from our model agree with the arguments exposed in Badalian [16], explaining that this state is likely to be identified with the lowest $^3p_0$. Finally, as observed in the caption to Table V, the steep increase of the RCC at low energies does not allow us to reproduce a sensible value for the pion mass: we could instead use the experimental datum for giving an estimate of α_S at this energy, largely below the domain of applicability of the renormalization group analysis. Assuming that the scaling parameter remains in the range 0.55 ± 0.013 found for the other states, we immediately get $\alpha_S(m_{\pi^{\pm}}) = 0.545 \pm 0.013$. As a final remark, notice that the u and d masses turn out to be close to current algebra masses, as opposed to constituent masses (see also Ref. [5]) that are usually much higher in potential models.

IV. CONCLUSIONS

We summarize the results we have obtained in this paper. In the first place we stress again that a fully covariant formulation of the two-fermion problem is a great conceptual and effective simplification for dealing with meson spectra, as it makes the model consistent, it avoids the introduction of a very large number of correction terms related to the relativistic effects, it renders a more clear physical picture, it simplifies the calculations, and it allows a more sound extension to light mesons. We have proved that the proper setting of the Breit perturbation term gives an effective unified way of treating the hyperfine interaction without the use of eigenfunctions, eliminating the ambiguity connected with the spread at the origin. This is indeed sufficient to explain with good accuracy even the spectrum of light and heavy-light mesons; for the latter, in particular, our model brings some arguments into the present debate on the nature of some mesons. By analyzing the behavior of the parameter α of the electromagneticlike term of the Cornell potential, we have found a very good relationship to the curve α_S of the QCD running coupling constant. Finally we would like to observe that our covariant treatment should allow a simpler and more effective numerical approach to the calculation of the transition amplitudes, since the wave functions can be determined by a perturbation procedure similar to the one we used for the eigenvalues. Work in this direction is in progress.

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