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Topological duality between vortices and planar Skyrmions in BPS theories with area-preserving diffeomorphism symmetries

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The Bogomol'nyi-Prasad-Sommerfield (BPS) baby Skyrme models are submodels of baby Skyrme models, where the nonlinear sigma model term is suppressed. They have Skyrmion solutions saturating a BPS bound, and the corresponding static energy functional is invariant under area-preserving diffeomorphisms (APDs). Here we show that the solitons in the BPS baby Skyrme model, which carry a nontrivial topological charge $Q_b \in \pi_2(S^2)$ (a winding number), are dual to vortices in a BPS vortex model with a topological charge $Q_v \in \pi_1(S^1)$ (a vortex number), in the sense that there is a map between the BPS solutions of the two models. The corresponding energy densities of the BPS solutions of the two models are identical. A further consequence of the duality is that the dual BPS vortex models inherit the BPS property and the infinitely many symmetries (APDs) of the BPS baby Skyrme models. Finally, we demonstrate that the same topological duality continues to hold for the $U(1)$ gauged versions of the models.

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I. INTRODUCTION

There is an intimate relation between $SU(2)$ t' Hooft-Polyakov monopoles and Skyrmions. Indeed, their energy density distributions (for equal topological charges) almost coincide and possess exactly the same symmetries [1–5]. This is explained by rational maps [6], because there is a one-to-one correspondence between rational maps of degree N and N monopoles [7]. In particular, any rational map can be derived from a monopole field configuration and, conversely, each monopole defines a rational map. Rational maps are also behind the famous *rational map ansatz*, which is a powerful tool for the construction of approximate solutions in the *massless* Skyrme model [6].

This close correspondence no longer holds when a nonzero pion mass (i.e., a potential) is included. As this case is physically more relevant, any progress in the understanding of its Skyrmions is important (see, for example, Ref. [8], where Skyrmions with massless pions in hyperbolic space are used to approximate Skyrmions with massive pions in flat space).

Recently, inspired by the phenomenological deficiencies of the original Skyrme model, two Bogomol'nyi-Prasad-Sommerfield (BPS) Skyrme theories have been proposed. The first model is a *conformal* BPS model [9] with massless pions and an infinite tower of KK mesonic fields derived by a dimensional reduction from a higher-dimensional Yang-Mills theory. The second model is a *volume preserving diffeomorphism* (VPD) BPS model [10,11], with formally infinitely heavy perturbative pions (see also Refs. [12,13]). Of course, QCD is neither conformal nor VPD invariant, but a BPS model may be a proper starting point for the construction of a low energy effective action. In addition, the large symmetries of the BPS models make it easier to find classical solutions. The latter model contains a potential term, so it may also provide insight into the effects of non-trivial potentials, where, as said, no reliable approximation (except for full numerical simulations) has yet been found.

The present paper aims at further investigating properties of VPD BPS models. In particular, it would be interesting to study whether there is any correspondence between Skyrmions in the VPD BPS model (topological solitons with charges $Q \in \pi_3(S^3)$) and monopoles (solitons with charges $Q \in \pi_2(S^2)$).

As usual, we begin our investigation with a simpler, dimensionally reduced theory, i.e., the baby Skyrme model (a planar version of the original Skyrme model) [14]. There exists a $(2+1)$ counterpart of the VPD BPS Skyrme model called *area preserving diffeomorphism* (APD) BPS baby model [15–18] which consists of two parts: the topological current squared and a potential

$$\mathcal{S}_{\text{BPS baby}} = \int d^3x \left(-\frac{\lambda^2}{4} (\partial_\mu \vec{\phi} \times \partial_\nu \vec{\phi})^2 - \mu^2 V(\vec{n} \cdot \vec{\phi}) \right). \tag{1}$$

Here, instead of the chiral Skyrme field $U \in SU(2)$, one deals with a three component unit vector $\vec{\phi} \in S^2$. Hence, static baby Skyrmions are maps from compactified two dimensional base space $\mathbb{R}^2 \cup \{\infty\} \cong S^2$ into the target space S^2, classified by the corresponding winding number $Q_b \in \pi_2(S^2)$. Obviously, a lower-dimensional counterpart of monopoles are vortices with topological charge

$Q_v \in \pi_1(S^1)$. The precise aim of the present work is to show that BPS baby Skyrmions are related to some BPS vortices. Concretely, in Sec. II we construct a (2 + 1) dim vortex model with APD symmetry and prove that it is dual to the BPS baby Skyrme model, in the sense that there exists a field transformation mapping it into the latter. Further, we show that the vortex model has a BPS bound and that its static vortices saturate the bound. In Sec. III we repeat the same construction for the gauged models. Section IV contains our conclusions.

II. VORTICES WITH THE APD SYMMETRY

For models with a standard kinetic term, finite energy vortices can be obtained only if a $U(1)$ gauge field is added (leading, e.g., to the well-known Abelian-Higgs model [19]), because the standard quadratic kinetic term has infinite energy for vortex boundary conditions. In theories with generalized kinetic terms, where the standard kinetic term is absent, this argument no longer holds, allowing to construct planar models of a complex field which do support finite energy vortices. The omission of the "usual" quadratic kinetic term might seem a drastic modification. However, the kinetic terms we consider here are still quadratic in first time derivatives, such that a standard Hamiltonian exists. These (non-negative) kinetic terms are suppressed for some field configurations (e.g., for vacuum configurations), therefore the Cauchy problem is not well defined for these initial data. The model is, thus, an effective model with extra symmetries which allow to better understand its static field configurations. The complete model will possess the usual kinetic terms (either added explicitly or generated, e.g., by quantum corrections), but we hope most other properties will not be altered too much. From a more physical perspective, omitting the quadratic kinetic term may correspond to an approximation which is reliable in a nonperturbative regime (where the energy density is rather large), whereas it does not reproduce the near vacuum behavior (linear fluctuations about the vacuum are suppressed).

A. The model

The Lagrangian [on (2 + 1) dim Minkowski space, u is a complex field] is

$$\mathcal{L}_{\text{BPS vortex}} = \lambda^2 \mathcal{L}_4 - V(u\bar{u}) \tag{2}$$

and consists of a potential V and a fourth derivative term (a Skyrme-like term)

$$\begin{aligned} \mathcal{L}_4 &= -(u_\mu \bar{u}^\mu)^2 + u_\mu^2 \bar{u}_\nu^2 = -K_\mu u^\mu, \\ K_\mu &= (u_\nu \bar{u}^\nu)\bar{u}_\mu - \bar{u}_\nu^2 u_\mu. \end{aligned} \tag{3}$$

Its static energy functional is invariant under base space APDs [11,18] and has BPS solutions, so we call it *APD BPS vortex* model. The Euler-Lagrange equations are

$$\partial_\mu K^\mu - \frac{1}{2\lambda^2} V_u = 0, \tag{4}$$

and in the static case reduce to

$$\nabla \cdot \vec{K} - \frac{1}{2\lambda^2} V_u = 0, \qquad \vec{K} = (\nabla u \nabla \bar{u})\nabla \bar{u} - (\nabla \bar{u})^2 \nabla u. \tag{5}$$

Concretely, we choose the Abelian-Higgs type potentials

$$V = \frac{\mu^2}{4}(1 - u\bar{u})^\alpha, \tag{6}$$

where $\alpha \geq 1$. With the static axially symmetric ansatz

$$u(r, \phi) = f(r)e^{in\phi} \tag{7}$$

($n \in Z$ is the winding number and f is a profile function) we get

$$f\frac{1}{r}\partial_r\left(\frac{ff_r}{r}\right) + f\frac{\alpha\mu^2}{16n^2\lambda^2}(1 - f^2)^{\alpha-1} = 0, \tag{8}$$

where we assume the vortex boundary conditions $f(0) = 0$, $f(\infty) = 1$. Introducing the new variables $h = 1 - f^2$ and $x = r^2/2$, we find

$$h_{xx} - \frac{\alpha\mu^2}{8n^2\lambda^2}h^{\alpha-1} = 0 \tag{9}$$

and the boundary conditions become $h(x = 0) = 1$, $h(x = \infty) = 0$. For the standard Abelian-Higgs potential ($\alpha = 2$), the topologically nontrivial solution is

$$h = e^{-\frac{\mu}{2n\lambda}x} \Rightarrow f = \sqrt{1 - e^{-\frac{\mu}{4n\lambda}r^2}}, \tag{10}$$

and the energy of this solution becomes

$$\begin{aligned} E &= 2\pi n^2\lambda^2 \int_0^\infty dx\left(h_x^2 + \frac{\mu^2}{4n^2\lambda^2}h^2\right) \\ &= \pi\mu^2 \int_0^\infty dx h^2 = \pi\mu\lambda n. \end{aligned} \tag{11}$$

Hence, E is proportional to the topological charge of the soliton, typical for BPS solutions. For $\alpha > 2$, we find the (powerlike localized) solutions

$$h(x) = \left(\frac{x_0}{x + x_0}\right)^{\frac{2}{\alpha-2}}, \qquad x_0 = \frac{4n\lambda}{\mu(\alpha - 2)}. \tag{12}$$

Localization becomes weaker with growing α. For $\alpha \in [1, 2)$ we find compact vortices, i.e., solitons for which the field takes its vacuum value at a finite distance x_0,

$$h(x) = \begin{cases} \left(1 - \frac{x}{x_0}\right)^{\frac{2}{2-\alpha}} & x \in [0, x_0] \\ 0 & x \geq x_0 \end{cases} \qquad x_0 = \frac{4n\lambda}{\mu(2 - \alpha)}. \tag{13}$$

Specifically, for $\alpha = 1$ we get the standard signum-Gordon compacton with a parabolic approach to the vacuum [20].

B. Bogomol'nyi equation and BPS bound

The static energy of the model has a topological bound which is saturated by solutions of a first order BPS equation. Completing the square, we get

$$E_{\mathrm{BPS\ vortex}} = \int d^2x (i\lambda\epsilon_{ij}\nabla_i u \nabla_j \bar{u} \pm \sqrt{V})^2 \mp 2i\lambda \int d^2x \epsilon_{ij}\nabla_i u \nabla_j \bar{u}\sqrt{V}, \tag{14}$$

which implies

$$E_{\mathrm{BPS\ vortex}} \geq B_{\mathrm{BPS}} \equiv \left| 2i\lambda \int d^2x \epsilon_{ij}\nabla_i u \nabla_j \bar{u}\sqrt{V}\right|, \tag{15}$$

with equality if and only if the BPS equation $\lambda\epsilon_{ij}\nabla_i u\nabla_j\bar{u} = \pm i\sqrt{V}$ holds. This bound is topological, because the base space two-form $d^2x\epsilon_{ij}\nabla_i u\nabla_i\bar{u}\sqrt{V}$ is the pullback of a target space two form, so the base space integral may be replaced by n times a target space integral. The target space is $\mathbb{R}^2$, so the two form on target space is exact (a total derivative), but it gives a nontrivial result, nevertheless, because of the nontrivial boundary conditions imposed by the Higgs potential. Introducing the fields X, Y via $u = X + iY$, we find

$$B_{\mathrm{BPS}} = 4\lambda \int d^2x \epsilon_{ij}\nabla_i X \nabla_j Y \sqrt{V} = 4\lambda n \int d^2X\sqrt{V(X^2+Y^2)} = 4\pi\lambda n \int_0^1 dt\sqrt{V(t)}, \tag{16}$$

where $t = X^2 + Y^2$. Here, the Higgs type potentials take their vacuum values at $t = 1$, $V(t = 1) = 0$. The bound only depends on the model (i.e., the potential and the coupling constants) and on the vortex number, as required for a topological BPS bound.

C. Duality between BPS vortices and BPS baby Skyrmions

The BPS baby Skyrme model in its CP^1 formulation,

$$\mathcal{L}_{\mathrm{BPS\ baby}} = -\lambda^2 \frac{K_\mu u^\mu}{(1+|u|^2)^4} - \frac{\mu^2}{4}\left(\frac{|u|^2}{1+|u|^2}\right)^\alpha, \tag{17}$$

differs from the quartic vortex model by a target space factor multiplying the first term and by a different family of potentials (the old baby Skyrme potentials). These models are different also at a deeper level, because they support solitons with different topology. Baby Skyrmions are classified by a winding number $Q \in \pi_2(S^2)$, which for the ansatz (7) implies that the profile function f_b must cover the whole semiline, i.e., $f_b(0) = \infty$ and $f_b(\infty) = 0$. Now, let us demonstrate that the solutions of the two models are related by the transformation

$$f_v^2(x_\mu) = \frac{1}{1+f_b^2(x_\mu)}, \qquad \Phi_v(x_\mu) = \Phi_b(x_\mu), \tag{18}$$

where the real functions f_v, f_b and Φ_v, Φ_v are defined as

$$u_v(x_\mu) = f_v(x_\mu)e^{i\Phi_v(x_\mu)}, \quad u_b(x_\mu) = f_b(x_\mu)e^{i\Phi_b(x_\mu)}. \tag{19}$$

The transformation implies that $f_v \in [0, 1]$, so it only holds for finite energy solutions, i.e., solutions where f_v approaches its vacuum value, $\lim_{|\vec{x}|\to\infty} f_v(x_\mu) = 1$. For the proof, we show that the corresponding Lagrange densities are connected by this transformation. Indeed,

$$-\mathcal{L}_{\mathrm{BPS\ vortex}} = 4\lambda^2 f_v^2[(\partial_\mu f_v)^2(\partial_\nu\Phi_v)^2 - (\partial_\mu f_v\partial^\mu\Phi_v)^2] + \frac{\mu^2}{4}(1-f_v^2)^\alpha = \tag{20}$$

$$4\lambda^2\frac{f_b^2}{(1+f_b^2)^4}[(\partial_\mu f_b)^2(\partial_\nu\Phi_b)^2 - (\partial_\mu f_b\partial^\mu\Phi_b)^2] + \frac{\mu^2}{4}\left(\frac{f_b^2}{1+f_b^2}\right)^\alpha = -\mathcal{L}_{\mathrm{BPS\ baby}}. \tag{21}$$

Obviously, general potentials are related via $V_b(f_b^2) = V_v((1+f_b^2)^{-1})$.

We remark that

(i) a static vortex is transformed into a baby Skyrmion with the same topological charge. This is obvious for the axially symmetric ansatz where $\Phi_v = \Phi_b = in\phi \Rightarrow Q_b = Q_v = n$. For general field configurations it follows, e.g., from the invariance of topological charges under continuous deformations.

(ii) The BPS equation of one model is mapped into the BPS equation of the other.

(iii) Both models possess the same *master* static dimensionally reduced energy integral [using the ansatz (7)]

$$E = 2\pi\int_0^\infty dx(n^2\lambda^2 h_x^2 + \mu^2 h^\alpha), \tag{22}$$

as may be shown easily by inserting the expression

$$h = 1 - \frac{1}{1+f_b^2} = 1 - f_v^2 \tag{23}$$

into the corresponding static energy functionals. Moreover, h obeys exactly the same boundary conditions in both cases, $h(0) = 1$ and $h(\infty) = 0$.

(iv) Time-dependent spinning configurations rotate with the same frequency. Spinning solutions are obtained from the ansatz $u = f(r)e^{in\phi+i\omega t}$. Hence, using $\Phi_v = \Phi_b$ we get $(n, \omega)_{\mathrm{vortex}} = (n, \omega)_{\mathrm{baby}}$.

These results show that both models are dual to each other. They describe exactly the same physics (identical energy densities and symmetries) but by means of two different topological objects, so we call this duality a *topological duality*. Such a duality is rather unusual, relating two distinct topological charges, whereas normally dualities transform, e.g., a topological charge into a Noether charge (e.g., T-duality [21], or Montonen-Olive duality [22]).

We found that the link between baby Skyrmions and vortices in the APD BPS models is more intimate than for the usual Skyrmions and monopoles. Here, they are not only qualitatively similar, they are essentially identical.

III. ABELIAN-HIGGS MODEL WITH THE APD SYMMETRY

A. The model

The APD BPS vortex model minimally coupled to the Maxwell field is

$$\mathcal{L}_{\text{gauged BPS vortex}} = -\lambda^2[(D_\mu u D^\mu \bar{u})^2 - (D_\mu u)^2(D_\nu \bar{u})^2] - V(u\bar{u}) - \frac{1}{4g^2}F_{\mu\nu}^2, \tag{24}$$

where the covariant derivative is $D_\mu u = u_\mu + iA_\mu u$. This is just the APD version of the Abelian-Higgs model. The equations of motion are

$$\bar{D}_\mu \mathcal{K}^\mu - \frac{1}{2}V_u = 0, \tag{25}$$

where

$$\mathcal{K}^\mu = (D_\nu u D^\nu \bar{u})D^\mu \bar{u} - (D_\nu \bar{u})^2 D^\mu u, \qquad \bar{D}_\mu \mathcal{K}^\mu = (\partial_\mu - iA_\mu)\mathcal{K}^\mu, \tag{26}$$

and

$$\frac{1}{g^2}\partial_\mu F^{\mu\nu} - 2ie[(D_\mu u D^\mu \bar{u})(\bar{u}D^\nu u - uD^\nu \bar{u}) - (D_\mu u)^2 \bar{u} D^\nu \bar{u} - (D_\mu \bar{u})^2 u D^\nu u] = 0. \tag{27}$$

Again, we assume the static ansatz (7) and $A_0 = A_r = 0$, $A_\phi = na(r)$. This leads to

$$D_i u D_i \bar{u} = f_r^2 + \frac{n^2 f^2}{r^2}(1+a)^2, \qquad (D_i \bar{u})^2 = \left(f_r^2 - \frac{n^2 f^2}{r^2}(1+a)^2\right)e^{-2in\phi}, \tag{28}$$

and the static field equations become ($x = r^2/2$, $h = 1 - f^2$ and $V = \mu^2 h^\alpha$, as above)

$$\partial_x[h_x(1+a)^2] - \frac{\alpha\mu^2}{8n^2\lambda^2}h^{\alpha-1} = 0, \tag{29}$$

$$a_{xx} = 2g^2 h_x^2(1+a), \tag{30}$$

which are exactly equal to the static field equations for the gauged BPS baby Skyrme model with potential $V = 4h^\alpha$, see section V.C. of Ref. [23].

B. Duality between vortices and BPS baby Skyrmions

The Lagrangian of the gauged BPS baby Skyrme model with the "old" baby potentials is

$$\mathcal{L}_{\text{gauged BPS baby}} = -\lambda^2\frac{(D_\mu u D^\mu \bar{u})^2 - (D_\mu u)^2(D_\nu \bar{u})^2]}{(1+|u|^2)^4} - \mu^2\left(\frac{|u|^2}{1+|u|^2}\right)^\alpha - \frac{1}{4g^2}F_{\mu\nu}^2. \tag{31}$$

Assuming a gauged extended version of the duality map, i.e., (18) and $A_\mu^v(x_\mu) = A_\mu^b(x_\mu) \equiv A_\mu$ we find that, again, one Lagrangian is transformed in to the other. The static energy of (31) becomes [using the ansatz (7) and (23)]

$$E = 2\pi\int_0^\infty dx\left[\lambda^2 n^2 h_x^2(1+a)^2 + \mu^2 h^\alpha + \frac{n^2}{4g^2}a_x^2\right], \tag{32}$$

which exactly agrees with the static energy of the gauged BPS vortex model.

C. BPS bound and BPS equations

The static energy density of the gauged vortex model is

$$\mathcal{E}_v = \lambda^2 \mathcal{Q}_v^2 + V_v + \frac{1}{g^2}B^2, \tag{33}$$

where the covariant "topological density" $\mathcal{Q}_v$ takes the form

$$\mathcal{Q}_v \equiv i\epsilon_{ij}D_i u D_j \bar{u} = i\epsilon_{ij}u_i\bar{u}_j + \epsilon_{ij}A_i\partial_j|u|^2 \equiv \mathbf{q}_v + \epsilon_{ij}A_i\partial_j|u|^2 \tag{34}$$

$$= \epsilon_{ij}\partial_i(f_v)^2(\Phi_j - A_j) = \epsilon_{ij}\partial_i h(A_j - \Phi_j). \tag{35}$$

On the other hand, the static energy density of the gauged BPS baby Skyrme model is given by

$$\mathcal{E}_b = \lambda^2 \mathcal{Q}_b^2 + V_b + \frac{1}{g^2}B^2, \tag{36}$$

where its covariant topological density $\mathcal{Q}_b$ is

$$\mathcal{Q}_b \equiv -i(1+u\bar{u})^{-2}\epsilon_{ij}D_i u D_j \bar{u} = -(1+u\bar{u})^{-2}\epsilon_{ij}(iu_i\bar{u}_j + \epsilon_{ij}A_i\partial_j|u|^2) = \epsilon_{ij}\partial_i(1+f_v^2)^{-1}(\Phi_j - A_j) = \epsilon_{ij}\partial_i h(A_j - \Phi_j). \tag{37}$$

The two topological densities are identical in terms of the "master function" h. The energy densities are, therefore, identical if the potentials as functions of h are the same, i.e., $V_v(h) = V_b(h)$, which just means $V_b(f_b^2) = V_v((1+f_b^2)^{-1})$. It follows that the gauged vortex model and the gauged BPS baby Skyrme model expressed in terms of h have exactly the same BPS bound and equations. Briefly, the BPS equations are $\mathcal{Q} = W_h$, $B = g^2\lambda^2 W$, where the "superpotential" $W(h)$ must obey the superpotential equation

$$\lambda^2 W_h^2 + \lambda^4 g^2 W^2 = V(h) \tag{38}$$

and the boundary condition $W(0) = 0$. The BPS bound for the energy then is

$$E \geq 2\lambda^2 \left| \int d^2 x \mathbf{q} W_h \right| = 4\pi\lambda^2 |nW(1)|, \qquad (39)$$

where n is the topological charge. For details we refer to Ref. [23] (the same bound for the gauged BPS baby Skyrme model has been derived in Ref. [24]). Equation (38) is called the superpotential equation because it also arises in supergravity coupled to a scalar field; see, e.g., Ref. [25].

IV. SUMMARY

In this paper, we introduced planar models with APD symmetries of the static energy which support topological vortices. These solitons are BPS solutions (they solve a Bogomol'nyi equation) and their energy grows linearly with the topological charge. The action does not contain the standard kinetic sigma model term, allowing to find vortices both without and with gauge fields. Moreover, we were able to find exact solutions for arbitrary topological charge. We remark that other generalizations of the Abelian-Higgs model supporting BPS vortices have been introduced recently [26].

Our main result is the duality between APD BPS vortex models and APD BPS baby Skyrme models. There exists a map between the fields of the two theories such that solutions of one model are transformed into solutions of the other. Also their energy densities and topological charges coincide. In other words, the baby Skyrmions of the APD BPS baby Skyrme model can be equivalently described in terms of vortices of the APD BPS vortex model. This is an extreme version of the approximate correspondence between 3 dimensional Skyrmions and magnetic monopoles. Here, different topological objects are not only similar, but exactly equivalent.

As the dual transformation holds also for time dependent solutions, the theories remain dual even at the semiclassical quantization level. Hence, the relevant excitations should be the same.

An important question is whether this duality still exists in (3 + 1) dimensions, where now Skyrmions of the VPD BPS Skyrme model would be dual to monopoles of some (currently unknown) VPD BPS monopole model, because VPD BPS Skyrmions may provide a useful starting point for the description of physical nuclei. This duality would offer a dual description of baryons and atomic nuclei in terms of monopoles, allowing to identify monopole-like substructures within nuclei, which might provide valuable insight into their understanding. We remark that in Ref. [27] a Yang-Mills-Higgs model with a Skyrme type term quartic in covariant derivatives was investigated. The authors found that the resulting "Skyrmed monopoles" behave differently from Skyrmions. Specifically their symmetries are different.

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