PHYSICAL REVIEW D **87,** 026006 (2013)

Correlation function of circular Wilson loop with two local operators and conformal invariance

E. I. Buchbinder[1,2,*] and A. A. Tseytlin[1,†]

[1]*The Blackett Laboratory, Imperial College, London SW7 2AZ, United Kingdom*

[2]*School of Physics (M013), The University of Western Australia, 35 Stirling Highway, Crawley, Western Australia 6009, Australia*

(Received 4 September 2012; published 24 January 2013)

We consider the correlation function of a circular Wilson loop with two local scalar operators at generic four positions a_1, a_2 in planar $\mathcal{N}=4$ supersymmetric gauge theory. We show that such a correlator is fixed by conformal invariance up to a function $F(\mathrm{u},\mathrm{v};\lambda)$ of two scalar combinations u, v of a_1, a_2 coordinates invariant under the conformal transformations preserving the circle as well as the 't Hooft coupling λ. We compute this function at leading orders at weak and strong coupling for some simple choices of local supersymmetric operators. We also check that correlators of an infinite line Wilson loop with local operators are the same as those for the circular loop.

DOI: 10.1103/PhysRevD.87.026006 PACS numbers: 11.25.Tq

I. INTRODUCTION

Supersymmetric Wilson loops [1] and their correlation functions with local operators in planar $\mathcal{N}=4$ SYM theory dual to $AdS_5 \times S^5$ string theory is presently an active subject of research. In this paper we will focus on correlators involving the simplest circular Wilson loop W_C [2–7]. The form of its correlator $\langle W_C \mathcal{O}(a)\rangle$ with one primary operator $\mathcal{O}$ [3,8] is completely fixed by conformal invariance up to a function of 't Hooft coupling λ that may be computed exactly [7,9] in the case when the operator is supersymmetric.

The correlator of W_C with *two* chiral primary operators can be again computed exactly [10] provided their locations and structure are special (so that at least 1/8 of supersymmetry is preserved [9]). Here we shall consider a "nonsupersymmetric" correlator $\langle W_C \mathcal{O}_1(a_1)\mathcal{O}_2(a_2)\rangle$ with *generic* positions a_1, a_2 in $\mathbb{R}^4$ for the simplest choices of supersymmetric operators $\mathcal{O}_i$.[1] As we shall find below, the conformal invariance restricts the dependence on locations of the circular loop and the two operators to just two scalar functions u, v of them, i.e., the above correlator is, in general, proportional to a function $F(u,\mathrm{v};\lambda)$. We shall compute this function at leading orders at weak and strong coupling λ.

The circular Wilson loop W_C is known to be closely related to the Wilson loop W_L defined by an infinite straight line [3,6]. Since the infinite line is related to the circle by a special conformal transformation, the expectation values of the two would be the same if not for an anomaly [5–7] (related to change of boundary conditions). Indeed, $\langle W_C\rangle = 1$ while $\langle W_C\rangle = \frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda})$ is a nontrivial function of λ [5–7,9]. However, if one considers the *normalized* correlators of W_C with *local* operators $\frac{\langle W\mathcal{O}_1(a_1)\ldots\mathcal{O}_n(a_n)\rangle}{\langle W\rangle}$ one may expect the anomaly to be absent, i.e., the result for W_C should be equivalent to the one for W_L.[2] This is clear, in particular, at strong coupling where the expression for such correlator (given by a product of the corresponding vertex operators evaluated on the minimal surface) is finite and thus should not be affected by the anomaly. At weak coupling, one can arrange the operators to stay away from the Wilson loop location before and after the conformal transformation. Below we will explicitly check the matching $\frac{\langle W_C\mathcal{O}_1(a_1)\mathcal{O}_2(a_2)\rangle}{\langle W_C\rangle} = \frac{\langle W_L\mathcal{O}_1(a_1)\mathcal{O}_2(a_2)\rangle}{\langle W_L\rangle}$ at leading order in λ for simplest 1/2 supersymmetric operators $\mathcal{O}_i$.

The dependence of the correlator of the circle or line Wilson loop with two local operators on just two invariants (u, v) is reminiscent of the familiar structure of the correlator of four scalar conformal primary operators. Heuristically, the fact that an infinite line may be specified by two points in $\mathbb{R}^4$ may be suggesting (by analogy with what was found in the null polygon Wilson loop cases [12–14]) a possible relation between $\langle W_C \mathcal{O}_1(a_1)\mathcal{O}_2(a_2)\rangle$ and some special four-point correlator. Another motivation for a study of such correlators is that they are special cases of correlators involving more general cusped Wilson loops (see, e.g., Refs. [4,15,16]).

The structure of this paper is as follows. In Sec. II we shall consider the conformal symmetry constraints on the correlator of a circular Wilson loop with two scalar conformal operators and explain why it is determined by the function of two invariants of the subset of six conformal transformations preserving the circular loop. In Sec. III we shall compute this function $F(\mathrm{u},\mathrm{v};\lambda)$ in the leading-order approximation at weak coupling for the case when the two local operators are a chiral primary of dimension two. In Sec. IV we shall discuss the strong coupling limit of the

*evgeny.buchbinder@uwa.edu.au

†Also at Lebedev Istitute, Moscow.
tseytlin@imperial.ac.uk

[1]To compare to Ref. [10] one would need to consider the special operators $\mathrm{Tr}(a_k\Phi_k + i\Phi_4)^J$ with coefficients depending on locations a_k that are restricted to the same $S^2 \subset \mathbb{R}^4$ to which the circle belongs.

[2]In the case of one-point correlator $\frac{\langle \mathcal{W}\,\mathcal{O}_1(\mathcal{A}_1)\rangle}{\langle \mathcal{W}\rangle}$ this equivalence was suggested by N. Drukker as mentioned in [11].

correlator $\langle W_C \mathcal{O}_1(a_1)\mathcal{O}_2(a_2)\rangle$ using the semiclassical string picture. We shall find that for two "light" operators (whose dimension does not scale with $\sqrt{\lambda}$) the correlator factorizes at strong coupling with the function F being constant. In the case when one of the two operators carries large "semiclassical" charge $J = \sqrt{\lambda}\mathcal{J}$, the expression for F will be given by a nontrivial integral that we shall evaluate for small and large $\mathcal{J}$.

In Sec. V we shall discuss the case of the Wilson loop W_L defined by an infinite line and check the agreement of its correlator with local operators with the corresponding correlators for the circular Wilson loop. Some technical remarks will be made in Appendices A, B, and C.

II. CONFORMAL INVARIANCE CONSTRAINTS ON CORRELATOR OF CIRCULAR WILSON LOOP WITH TWO SCALAR OPERATORS

In this section we shall first review the constraints on some of the simplest correlation functions in $\mathcal{N}=4$ gauge theory that follow from the conformal invariance and then consider the case of $\langle W_C \mathcal{O}_1(a_1)\mathcal{O}_2(a_2)\rangle$.

A. Conformal invariance constraints on some simple correlation functions

Let us start with correlation functions of scalar local operators $\mathcal{O}_i(a_i)$. As is well known, in conformal field theory their two- and three-point functions are fixed by conformal invariance up to a constant (function of coupling) while a four-point function is in general proportional to a function of two cross ratios (and coupling). This can be seen, for example, as follows. Given a set of n points in $\mathbb{R}^4$ we can act on them with 15 generators of the conformal group. However, there can be a subset of generators that leaves this set of points invariant. Let Γ_0 be the number of such generators. Then the number of conformally invariant combinations that one can construct out of n four coordinates is

$$d_n = 4n - (15 - \Gamma_0). \qquad (2.1)$$

If $n=2$ we can place one point at the origin and the other at infinity. This configuration preserves dilatations and all the Lorentz transformations that gives $\Gamma_0 = 7$. Then from (2.1) we get $d_2 = 0$. This means that one cannot construct any conformally invariant combinations and thus the two-point correlator is fixed up to a constant. As usual, the latter can be fixed to 1 by a choice of normalization, i.e.,

$$\langle \mathcal{O}(a_1)\mathcal{O}^\dagger(a_2)\rangle = \frac{1}{|a_1 - a_2|^{2\Delta}}, \qquad (2.2)$$

where $\Delta = \Delta(\lambda)$ is the dimension of the operator $\mathcal{O}$.

The $n=3$ case corresponds to adding an extra point at some finite distance from 0; that breaks dilatations and breaks Lorentz group to $SO(3)$. Hence for $n=3$ we get $\Gamma_0 = 3$ and $d_3 = 0$, meaning that the three-point function is also fixed by conformal symmetry up to a constant, i.e., is given by the well-known expression

$$\langle \mathcal{O}_1(a_1)\mathcal{O}_2(a_2)\mathcal{O}_3(a_3)\rangle = \frac{C_{123}(\lambda)}{|a_1-a_2|^{\Delta_1+\Delta_2-\Delta_3}|a_1-a_3|^{\Delta_1+\Delta_3-\Delta_2}|a_2-a_3|^{\Delta_2+\Delta_3-\Delta_1}}, \qquad (2.3)$$

where Δ_i are dimensions of $\mathcal{O}_i$.

Considering the $n=4$ case, i.e., adding one more point at a finite distance from the origin, one finds that the remaining symmetry is $SO(2)$, i.e., $\Gamma_0 = 1$ and thus $d_4 = 2$. This implies that the four-point correlator is fixed up to a function G of two conformally invariant variables

$$u = \frac{|a_1-a_2|^2|a_3-a_4|^2}{|a_1-a_3|^2|a_2-a_4|^2}, \qquad v = \frac{|a_1-a_4|^2|a_2-a_3|^2}{|a_1-a_3|^2|a_2-a_4|^2}. \qquad (2.4)$$

The general expression for a four-point function may then be written as[3]

$$\langle \mathcal{O}_1(a_1)\mathcal{O}_2(a_2)\mathcal{O}_3(a_3)\mathcal{O}_4(a_4)\rangle = \frac{G(u,v;\lambda)}{|a_1-a_2|^{q_1}|a_1-a_4|^{q_2}|a_2-a_4|^{q_3}|a_3-a_4|^{q_4}}, \qquad (2.5)$$

where q_i are fixed by demanding that the correlator has dimension $\Delta_1+\Delta_2+\Delta_3+\Delta_4$ and that it gets rescaled by $|a_1|^{2\Delta_1}|a_2|^{2\Delta_2}|a_3|^{2\Delta_3}|a_4|^{2\Delta_4}$ under the inversions (when $|a_i - a_j| \to \frac{|a_i-a_j|}{|a_i||a_j|}$)

$$\begin{aligned} q_1 &= \Delta_1+\Delta_2+\Delta_3-\Delta_4,\\ q_2 &= \Delta_1-\Delta_2-\Delta_3+\Delta_4,\\ q_3 &= -\Delta_1+\Delta_2-\Delta_3+\Delta_4,\\ q_4 &= 2\Delta_3. \end{aligned} \qquad (2.6)$$

Let us now consider examples of correlators of local operators with locally supersymmetric Wilson loop [1]

$$W = \frac{1}{N}\mathrm{Tr}\mathcal{P}\exp\left[\int d\tau (iA_\mu \dot{x}^\mu + \Phi_I\theta_I|\dot{x}|)\right]. \qquad (2.7)$$

Here (A_μ, Φ_I) are bosonic fields of $\mathcal{N}=4$ SYM theory $(I = 1,\ldots,6)$, $\theta_I\theta_I = 1$, and $x^\mu = x^\mu(\tau)$ defines a loop in $\mathbb{R}^4$. For example, in the case of W corresponding to the four-cusp null polygon, it was shown in Ref. [13] that the correlator $\frac{\langle W_4\mathcal{O}(a)\rangle}{\langle W_4\rangle}$ is fixed by conformal invariance up to a function depending on a single invariant variable ζ.[4] Indeed, let $x^{\mu(i)}$ $(i = 1, 2, 3, 4)$ be positions of the four cusps with $|x^{(i+1)} - x^{(i)}| = 0$. The total number of

[3]There is, obviously, more than one way to choose the scaling prefactor, but the ratio of any two such prefactors is conformally invariant and hence can be absorbed into the function $G(u,v)$.

[4]This correlator can thus be viewed as an "intermediate" case between the three-point and four-point functions of local operators.

coordinates of $4+1$ points is 20 but four null-line conditions reduce this number to 16. Acting with 15 conformal generators leaves only one conformally invariant combination[5]

$$\zeta = \frac{|a-x^{(2)}||a-x^{(4)}||x^{(1)}-x^{(3)}|}{|a-x^{(1)}||a-x^{(3)}||x^{(2)}-x^{(4)}|}, \tag{2.8}$$

and the correlator has the following general form (see Ref. [13] for details):

$$\frac{\langle W_4 \mathcal{O}(a)\rangle}{\langle W_4\rangle} = \frac{(|x^{(1)}-x^{(3)}||x^{(2)}-x^{(4)}|)^{\Delta/2}}{\prod_{i=1}^4 |a-x^{(i)}|^{\Delta/2}} \mathrm{F}(\zeta;\lambda). \tag{2.9}$$

In Ref. [13], the function $\mathrm{F}(\zeta;\lambda)$ was found to leading orders at weak and at strong coupling for $\mathcal{O}$ being the dilaton and the chiral primary. Recently it was computed [17] to the next-to-leading order at weak coupling for the case of the dilaton operator.

In determining the structure of (2.9) we assumed that the conformal transformations act on the operator as well as on the positions of the null cusps [in particular, ζ in (2.8) is invariant under all such conformal transformations]. Alternatively, we can view the loop as a fixed object and consider the correlation function as a function of the position of the operator only. Then the positions of the cusps are fixed constants and we can consider simply $\zeta' = \frac{|a-x^{(2)}||a-x^{(4)}|}{|a-x^{(1)}||a-x^{(3)}|}$, which is invariant only under the conformal transformations that preserve the null polygon.[6] Both approaches are of course equivalent.

B. Correlator of circular Wilson loop and one operator

Another special choice of W is a circular Wilson loop W_C. The correlator $\langle W_C \mathcal{O}(a)\rangle$ with one local operator also belongs to the class of simplest correlation functions: it is fixed by conformal invariance up to a constant (function of λ) [3,18,19]. This can be seen again by counting the free parameters. It is convenient to view the circle as a fixed object. For concreteness, we will assume that the circle is in the (x_1, x_2) plane in $\mathbb{R}^4$ with the center at the origin

$$x_1^2 + x_2^2 = R^2, \qquad x_3 = x_4 = 0. \tag{2.10}$$

As was shown in Ref. [20] (see also Appendix A), a circle in $\mathbb{R}^4$ is invariant under six conformal transformations. The configuration of a circle and an operator preserves $6-4=2$ of them. For example, if one places the operator at $a=\infty$ these twp conformal transformations are a rotation in the (x_1, x_2) plane and a rotation in the (x_3, x_4) plane. Then the number of combinations invariant under the conformal transformations preserving the circle is given by

$$d_{C,1} = 4 - (6-2) = 0. \tag{2.11}$$

This formula is analogous to (2.1) with the dimension of the full conformal group replaced with the dimension of the subgroup preserving the circle. The fact that $d_{C,1}=0$ means that we cannot construct any invariants and thus the correlation function of the circular Wilson loop and one local operator is fixed by the conformal invariance up to a constant (function of λ).

The explicit form of the correlator $\langle W_C \mathcal{O}(a)\rangle$ can be found, e.g., by using the fact that $\mathbb{R}^4$ is conformal to $\mathrm{AdS}_2 \times S^2$ [18,21]. Let us write the metric of $\mathbb{R}^4$ as

$$\begin{aligned} ds^2 &= dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 \\ &= dr^2 + r^2 d\psi^2 + dh^2 + h^2 d\varphi^2, \end{aligned} \tag{2.12}$$

where (r, ψ) and (h, φ) are the polar coordinates in the (x_1, x_2) and (x_3, x_4) planes. The circle (2.10) is at $r=R$, $h=0$. Let us transform to $\mathrm{AdS}_2 \times S^2$, i.e., change from (r, ψ, h, φ) to $(\rho, \psi, \theta, \varphi)$ as follows:

$$\begin{aligned} &r = \ell \sinh\rho, \quad h = \ell \sin\theta, \\ &\ell \equiv \frac{R}{\cosh\rho - \cos\theta} = \frac{\sqrt{(r^2+h^2-R^2)^2 + 4R^2h^2}}{2R}. \end{aligned} \tag{2.13}$$

In the new coordinates the metric becomes

$$\begin{aligned} ds^2 &= \ell^2(d\rho^2 + \sinh^2\rho\, d\psi^2 + d\theta^2 + \sin^2\theta\, d\varphi^2) \\ &= \ell^2 ds^2_{\mathrm{AdS}_2\times S^2}. \end{aligned} \tag{2.14}$$

Under this transformation the circular loop becomes the boundary of AdS_2 and, hence, is invariant under the isometries of $\mathrm{AdS}_2 \times S^2$. Then if we compute the correlator $\langle W_C \mathcal{O}(a)\rangle$ in gauge theory defined on $\mathrm{AdS}_2 \times S^2$ it can be invariant under the isometries only if it is a constant, i.e.,

$$\frac{\langle W_C \mathcal{O}(a)\rangle}{\langle W_C\rangle}\Big|_{\mathrm{AdS}_2\times S^2} = \mathrm{C}(\lambda). \tag{2.15}$$

To transform this back to $\mathbb{R}^4$ we note that under (2.13) we have $\mathcal{O}(a) \to \ell^{-\Delta}\mathcal{O}(a)$, so that

$$\begin{aligned} \frac{\langle W_C \mathcal{O}(a)\rangle}{\langle W_C\rangle} &= \frac{\mathrm{C}(\lambda)}{[\ell(a)]^\Delta} \\ &= \mathrm{C}(\lambda)\left[\frac{4R^2}{(r^2+h^2-R^2)^2+4R^2h^2}\right]^{\Delta/2}, \end{aligned} \tag{2.16}$$

where $r^2 = a_1^2 + a_2^2$ and $h^2 = a_3^2 + a_4^2$ (here a_μ are the coordinates of the point a). Note that in the limit when the position of the operator approaches a point on the circle this correlator diverges as $d^{-\Delta}$ where $d = \sqrt{(r-R)^2 + h^2}$ is the distance between the point a and a point on the circle. Also, (2.16) scales as $(r^2+h^2)^{-\Delta} = |a|^{-2\Delta}$ in the limit when the size of the circle goes to zero, in agreement with the operator product expansion (OPE) prediction [3] [cf. (2.2)].

For large λ the coefficient $\mathrm{C}(\lambda)$ is, in general, of order $\sqrt{\lambda}$ for large λ. For example, for $\mathcal{O}$ being the dilaton operator or chiral primary of fixed dimension j one gets [3]

---

[5] In this case $\Gamma_0 = 0$. One can show that the four-cusped null polygon is invariant under three conformal transformations. Addition of the operator(s) breaks all three of them.

[6] In (2.9) we can also absorb the a-independent numerator factor into the definition of $\mathrm{F}(\zeta')$.

$$C_{\rm dil}(\lambda) = \frac{\sqrt{6}\sqrt{\lambda}}{96N}, \qquad C_j(\lambda) = \frac{\sqrt{j}\sqrt{\lambda}}{2^{j+1}N}. \qquad (2.17)$$

For completeness, we present a derivation of these values in Appendix C.

C. Correlator of circular Wilson loop and two operators

Next, let us consider the case of our interest: the correlator of the circular Wilson loop (2.10) with two local operators

$$\frac{\langle W_C \mathcal{O}_1(a_1)\mathcal{O}_2(a_2)\rangle}{\langle W_C\rangle}. \qquad (2.18)$$

Let us again perform the counting of parameters. The two operators give $4+4=8$. In general, a configuration of a circle and two points is not invariant under any conformal transformations, i.e., here $\Gamma_0 = 0$. Then the number of remaining invariant parameters is

$$d_{C,2} = 8 - 6 = 2, \qquad (2.19)$$

and, hence, the correlator (2.18) is fixed by conformal symmetry up to a function of two variables (functions of a_1^μ, a_2^μ and location of the circle) and the coupling λ. These two variables, which we will denote as u and v, are invariant under six conformal transformations preserving the circle. As we shall now explain, u and v have a transparent geometric meaning.

Let us perform the change of coordinates (2.13), i.e., consider the correlator (2.18) in a theory defined on $AdS_2 \times S^2$. Since the circle is mapped to the boundary of AdS_2, it is invariant under the six isometries of $AdS_2 \times S^2$, and the same should apply to the correlator, i.e., the isometries of $AdS_2 \times S^2$ are precisely the six conformal transformations that preserve the circle (2.10) in $\mathbb{R}^4$. The natural two functions of the coordinates (a_1^μ, a_2^μ) invariant under the isometries of $AdS_2 \times S^2$ are the two geodesic distances between the two points: s in AdS_2 and ς in S^2. Thus

$$\frac{\langle W_C \mathcal{O}_1(a_1)\mathcal{O}_2(a_2)\rangle}{\langle W_C\rangle}\bigg|_{AdS_2\times S^2} = F(\mathrm{s}, \varsigma; \lambda). \qquad (2.20)$$

The two invariants (u, v) of the conformal transformations from $SO(1,2)\times SO(3) \subset SO(1,5)$ preserving the circle are then some functions of s and ς, e.g., $\mathrm{u} = \cosh \mathrm{s}$ and $\mathrm{v} = \cos\varsigma$. Given the two points $(\rho_1, \psi_1, \theta_1, \varphi_1)$ and $(\rho_2, \psi_2, \theta_2, \varphi_2)$ in $AdS_2 \times S^2$ corresponding to a_1 and a_2 in $\mathbb{R}^4$ via (2.12) and (2.13), i.e.,

$$(a_i^\mu) \to (r_i, \psi_i, h_i, \varphi_i) \to (\rho_i, \psi_i, \theta_i, \varphi_i), \qquad (2.21)$$

it is straightforward to construct the corresponding geodesics distances (see Appendix B).[7] Explicitly, one finds

$$\begin{aligned}\mathrm{u} &= \cosh \mathrm{s} \\ &= \cosh\rho_1 \cosh\rho_2 - \sinh\rho_1 \sinh\rho_2 \cos(\psi_2 - \psi_1),\end{aligned} \qquad (2.22)$$

$$\mathrm{v} = \cos\varsigma = \cos\theta_1\cos\theta_2 + \sin\theta_1 \sin\theta_2 \cos(\varphi_2 - \varphi_1), \qquad (2.23)$$

where from (2.13) we have $(i = 1, 2)$

$$\begin{aligned}\sinh\rho_i &= \frac{r_i}{\ell_i} = \frac{2Rr_i}{\sqrt{(r_i^2 + h_i^2 - R^2)^2 + 4R^2h_i^2}}, \\ \sin\theta_i &= \frac{h_i}{\ell_i} = \frac{2Rh_i}{\sqrt{(r_i^2 + h_i^2 - R^2)^2 + 4R^2h_i^2}}.\end{aligned} \qquad (2.24)$$

Transforming back to $\mathbb{R}^4$ we get [cf. (2.15) and (2.16)]

$$\begin{aligned}\mathcal{C}(W_C, a_1, a_2; \lambda) &= \frac{\langle W_C \mathcal{O}_1(a_1)\mathcal{O}_2(a_2)\rangle}{\langle W_C\rangle} \\ &= \frac{1}{[\ell(a_1)]^{\Delta_1}[\ell(a_2)]^{\Delta_2}} F(\mathrm{u}, \mathrm{v}; \lambda),\end{aligned} \qquad (2.25)$$

where Δ_i are the dimensions of $\mathcal{O}_i$ and we used that $\ell_i = \ell(a_i)$ where ℓ was defined in (2.13) and u, v depend on a_1, a_2 according to (2.12), (2.13), (2.21), and (2.23).

Note that as follows from (2.13)

$$\begin{aligned}|a_1 - a_2|^2 &= \frac{2[\cosh(\rho_1 - \rho_2) - \cos(\theta_1 - \theta_2)]}{(\cosh\rho_1 - \cos\theta_1)(\cosh\rho_1 - \cos\theta_2)} \\ &= 2\ell(a_1)\ell(a_2)(\mathrm{u} - \mathrm{v}).\end{aligned} \qquad (2.26)$$

According to the definitions in (2.23) we have $\mathrm{u} \geq 1$ and $|\mathrm{v}| \leq 1$. The values $\mathrm{u} = 1$, $\mathrm{v} = 1$ are achieved only when $\rho_1 = \rho_2$, $\psi_1 = \psi_2$, $\theta_1 = \theta_2$, $\varphi_1 = \varphi_2$, i.e., when the operators are at the coincident points $a_1 = a_2$. Hence the OPE limit $a_1 \to a_2$ is equivalent to $\mathrm{u} \to 1$, $\mathrm{v} \to 1$.

Another limiting case is when $\mathrm{u} = 1$ and $\mathrm{v} = -1$, corresponding, e.g., to $\rho_1 = \rho_2 = 0$, $\psi_1 = \psi_2$ and $\theta_1 = \theta_2 = \frac{\pi}{2}$, $\varphi_2 = \pi$, $\varphi_1 = 0$.[8] In this case $r_1 = r_2 = 0$, $h_1 = h_2 = R$ (with $\ell_1 = \ell_2 = R$), i.e., the two points are at the poles of the two sphere for which the circle is the equator, i.e., in Cartesian coordinates we have

$$\begin{aligned}a_1 &= (0, 0, R, 0), \\ a_1 &= (0, 0, -R, 0), \\ \mathrm{u} &= -\mathrm{v} = 1.\end{aligned} \qquad (2.27)$$

This case corresponds to a supersymmetric configuration considered in Ref. [10].

Let us note also that the limit when the radius R of the circle goes to 0 (or, equivalently, the locations a_i go to infinity) corresponds to $\rho_i \to 0$, $\theta_i \to 0$, so that again $\mathrm{u} \to 1$, $\mathrm{v} \to 1$. In this limit the Wilson loop can be

[7] For S^2 the geodesic distance is given by the "law of cosines"— a theorem in spherical trigonometry relating the sides and angles of spherical triangles.

[8] We thank S. Giombi for drawing our attention to this case.

represented as a sum of local operators [3], i.e., one has $W_C = \langle W_C\rangle[1 + \sum_k c_k R^{\Delta_k}\mathcal{O}_k(0) + \cdots]$ so that the first nontrivial term in the $R \to 0$ limit of the correlator (3.1) will be proportional to the corresponding three-point function.

Below we will explicitly compute the leading terms in $F(\mathrm{u}, \mathrm{v}; \lambda)$ for some simple cases of $\mathcal{O}_i$ at weak and at strong coupling.

III. THE CORRELATOR $\langle W_C \mathcal{O}_1(a_1)\mathcal{O}_2(a_2)\rangle$ AT WEAK COUPLING

Let us now consider the correlator

$$\mathcal{C}(W_C, a_1, a_2; \lambda) = \frac{\langle W_C \mathcal{O}_1(a_1)\mathcal{O}_2(a_2)\rangle}{\langle W_C\rangle} \tag{3.1}$$

at weak coupling $\lambda \ll 1$. We will choose the operators to be the simplest chiral primaries

$$\mathcal{O}_1(a_1) = c_2\mathrm{Tr}[Z^2(a_1)], \qquad \mathcal{O}_2(a_2) = c_2\mathrm{Tr}[\bar{Z}^2(a_2)],$$
$$Z = \Phi_1 + i\Phi_2, \qquad c_2 = \frac{4\pi^2}{\sqrt{2}N}. \tag{3.2}$$

For the unit-radius circle ($R = 1$)

$$x^\mu(\tau) = (\cos\tau, \sin\tau, 0, 0), \qquad |\dot{x}| = 1, \tag{3.3}$$

the Wilson loop (2.7) is given by

$$W = \frac{1}{N}\mathrm{Tr}\mathcal{P}\exp\left[g\int d\tau(iA_\mu\dot{x}^\mu + \Phi_1)\right]. \tag{3.4}$$

In (3.4) we assume that the fields in the Euclidean $\mathcal{N} = 4$ SYM Lagrangian $L = \frac{1}{2g^2}(\mathrm{Tr}F_{\mu\nu}^2 + \cdots)$ are rescaled by the gauge coupling constant g so that g appears only in the vertices. The 't Hooft coupling is defined as $\lambda = g^2N$. We will use the following conventions for the $SU(N)$ generators

$$A_\mu = A_\mu^a T^a, \qquad \Phi_I = \Phi_I^a T^a,$$
$$\mathrm{Tr}(T^aT^b) = \frac{1}{2}\delta^{ab}, \qquad a, b = 1, \ldots, N^2 - 1. \tag{3.5}$$

Then the propagators have the form

$$\langle A_\mu^a(a_1)A_\nu^b(a_2)\rangle = \frac{\delta_{\mu\nu}\delta^{ab}}{4\pi^2|a_1 - a_2|^2},$$
$$\langle Z^a(a_1)\bar{Z}^b(a_2)\rangle = \frac{\delta^{ab}}{2\pi^2|a_1 - a_2|^2}. \tag{3.6}$$

With the choice of c_2 in (3.2) the two-point function is canonically normalized[9]

$$\langle \mathcal{O}_1(a_1)\mathcal{O}_2(a_2)\rangle = \frac{1}{|a_1 - a_2|^4}. \tag{3.7}$$

We will choose the locations of the operators as [cf. (2.12)]

$$(a_1^\mu) = (r_1, 0, h_1, 0), \qquad (a_2^\mu) = (r_2, 0, h_2, 0), \tag{3.8}$$

i.e., the angles in (2.12) are $\psi_i = 0$, $\varphi_i = 0$. In this case, the variables u and v in (2.23) (invariant under the conformal transformations preserving the circle) take the simple form

$$\mathrm{u} = \cosh(\rho_1 - \rho_2), \qquad \mathrm{v} = \cos(\theta_1 - \theta_2). \tag{3.9}$$

The numerator of (3.1) contains a trivial disconnected contribution $\langle W_C\mathcal{O}_1(a_1)\mathcal{O}_2(a_2)\rangle \sim \langle W_C\rangle\langle\mathcal{O}_1(a_1)\mathcal{O}_2(a_2)\rangle$. Since the two-point function of chiral primary operators is not renormalized, this disconnected part coincides with the two-point function (3.7) to all orders in g

$$\mathcal{C}_{\mathrm{disc}} = \mathcal{C}_0 = \frac{1}{|a_1 - a_2|^4}. \tag{3.10}$$

Using (2.26) we see that this expression can indeed be written in the form (2.25)

$$\mathcal{C}_0 = \frac{F_0(\mathrm{u}, \mathrm{v})}{[\ell(a_1)]^2[\ell(a_2)]^2}, \qquad F_0(\mathrm{u}, \mathrm{v}) = \frac{1}{4(\mathrm{u} - \mathrm{v})^2}. \tag{3.11}$$

The first nontrivial (connected) contribution to $\mathcal{C}$ in (3.1) starts at order $g^2 \sim \lambda$

$$\mathcal{C}_1 = \frac{g^2c_2^2}{4N}\int_0^{2\pi}d\tau_1\int_0^{\tau_1}d\tau_2\langle\mathrm{Tr}[Z(\tau_1)\bar{Z}(\tau_2) + Z(\tau_2)\bar{Z}(\tau_1)]\mathrm{Tr}[Z^2(a_1)]\mathrm{Tr}[\bar{Z}^2(a_2)]\rangle_c, \tag{3.12}$$

where $\langle\ldots\rangle_c$ stands for the connected part of the correlator (here computed in free-theory approximation). As a result,

$$\mathcal{C}_1 = \frac{g^2Nc_2^2}{64\pi^6|a_1 - a_2|^2}\int_0^{2\pi}d\tau_1\int_0^{\tau_1}d\tau_2 \times\left[\frac{1}{|x(\tau_1) - a_1|^2|x(\tau_2) - a_2|^2} + (a_1 \leftrightarrow a_2)\right]. \tag{3.13}$$

Using that here $\frac{r_i^2 + h_i^2 + 1}{2r_i} = \coth\rho_i$ we get

$$\int_0^{2\pi}\int_0^{\tau_1}\frac{d\tau_1 d\tau_2}{|x(\tau_1) - a_1|^2|x(\tau_2) - a_2|^2} = \frac{1}{4r_1r_2}\int_0^{2\pi}\frac{d\tau_1}{\coth\rho_1 - \cos\tau_1}\int_0^{\tau_1}\frac{d\tau_2}{\coth\rho_2 - \cos\tau_2}. \tag{3.14}$$

The resulting expression for this integral is found to be

$$\frac{1}{4r_1r_2}2\pi^2\sinh\rho_1\sinh\rho_2. \tag{3.15}$$

The second integral in (3.13) produces the same contribution. Using that according to (2.13)

$$\frac{4r_1r_2}{\sinh\rho_1\sinh\rho_2} = 4\ell(a_1)\ell(a_2), \tag{3.16}$$

and taking into account the value of c_2 in (3.2) we get for (3.13)

[9] Below we will always consider only the planar approximation, i.e., the leading order in large N expansion.

$$\mathcal{C}_1 = \frac{\lambda}{8N^2}\frac{1}{\ell(a_1)\ell(a_2)|a_1-a_2|^2} = \frac{\lambda}{16N^2}\frac{1}{[\ell(a_1)]^2[\ell(a_2)]^2}\frac{1}{\mathrm{u}-\mathrm{v}}, \quad (3.17)$$

where we also used the relation (2.26). Thus the order $\lambda = g^2N$ term in the function $F(\mathrm{u},\mathrm{v};\lambda)$ in (2.25) is given by

$$F_1(\mathrm{u},\mathrm{v}) = \frac{\lambda}{16N^2}\frac{1}{\mathrm{u}-\mathrm{v}}. \quad (3.18)$$

Let us now study some special limits of this expression. One is the OPE limit $a_2 \to a_1$. In general, in this limit we have the following leading singularity:

$$\mathcal{O}_1(a_1)\mathcal{O}_2(a_2) \sim \frac{1}{|a_1-a_2|^\delta}k_3\mathcal{O}_3(a_1) + \cdots, \qquad \delta = \Delta_1+\Delta_2-\Delta_3, \quad (3.19)$$

where $\mathcal{O}_3$ stands for an operator (or a linear combination of operators) of lowest dimension such that $k_3 \sim \langle\mathcal{O}_1(a_1)\mathcal{O}_2(a_2)\mathcal{O}_3(0)\rangle$ is nonzero. Substituting (3.19) into (3.1) gives

$$\mathcal{C}_1|_{a_2\to a_1} \to \frac{k_3}{|a_1-a_2|^\delta}\frac{\langle W_C\mathcal{O}_3(a_1)\rangle}{\langle W_C\rangle} = \frac{1}{[\ell(a_1)]^{\Delta_3}}\frac{1}{|a_1-a_2|^\delta}k_3\mathrm{C}_3(\lambda), \quad (3.20)$$

where we used that the correlator of the circular Wilson loop with one local operator is fixed by conformal invariance as in (2.16). In the limit $a_2 \to a_1$ (3.17) becomes

$$\mathcal{C}_1|_{a_2\to a_1} \to \frac{1}{[\ell(a_1)]^2}\frac{1}{|a_1-a_2|^2}\frac{\lambda}{8N^2}. \quad (3.21)$$

Comparing (3.21) with (3.20) we conclude that here $\delta = 2$ and $\Delta_3 = 2$. Thus the leading contribution in this limit should come from operators of dimension two that have a nonzero three-point function with $\mathrm{Tr}[\bar{Z}^2]$ and $\mathrm{Tr}[\bar{Z}^2]$. One obvious choice is a non-supersymmetric operator $O_3 = \mathrm{Tr}[Z\bar{Z}] + \cdots$. Another option is to consider $\mathcal{O}_3$ as a particular case of generic dimension two chiral primary operator

$$\mathcal{O} \sim \mathrm{Tr}[(n_I\Phi_I)^2], \qquad n\cdot n = 0, \qquad n\cdot\bar{n} = 2, \quad (3.22)$$

with $\mathcal{O}_1 \sim \mathrm{Tr}[Z^2]$ and $\mathcal{O}_2 \sim \mathrm{Tr}[\bar{Z}^2]$ corresponding, respectively, to $n_1 = (1, i, 0, 0, 0, 0)$ and $n_2 = \bar{n}_1 = (1, -i, 0, 0, 0, 0)$. Since $\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\rangle$ is proportional to $(n_1\cdot n_2)(n_1\cdot n_3)(n_2\cdot n_3)$ the necessary conditions on n_3 are $(n_3\cdot n_1) \neq 0$, $(n_3\cdot n_2) \neq 0$. The contribution of the supersymmetric operators to the OPE will dominate at higher orders as their dimension will not grow with λ.

Another special limit is when one of the two points, e.g., a_1, approaches a point on the circle, i.e., for the choice of coordinates in (3.8) this corresponds to $r_1 \to R = 1$, $h_1 \to 0$. In this limit $\ell(a_1)$ in (2.13) reduces to the distance $d(a_1) = \sqrt{(r_1-1)^2+h_1^2}$ from a_1 to the point (1, 0, 0, 0) on the circle while u and v stay finite. As could be expected, the behavior of the correlator (3.1) and (3.17) in this limit $\mathcal{C}_1 \to [d(a_1)]^{-2}$ is the same as of the single-operator correlator in (2.16).

Yet another special case related to the supersymmetric configurations considered in Ref. [10] is when the two points belong to the two sphere around the center of the circle, e.g., $a_1 = (0, 0, 1, 0)$, $a_2 = (0, 0, -1, 0)$, when $\mathrm{u} = 1$, $\mathrm{v} = -1$ [see (2.27); here $R = 1$, $\ell_i = \frac{h_i^2+R^2}{2R} = 1$]. Then from (3.17) we get

$$\mathcal{C}_{1(S^2)} = \frac{\lambda}{32N^2}. \quad (3.23)$$

As one can check, this agrees with the expression found in Eqs. (4.42), (4.43) in Ref. [10].[10]

IV. THE CORRELATOR $\langle W_C\mathcal{O}_1(a_1)\mathcal{O}_2(a_2)\rangle$ AT STRONG COUPLING

Let us now consider the correlator (3.1) at strong coupling using the $\mathrm{AdS}_5\times S^5$ string theory representation

$$\mathcal{C}(W_C, a_1, a_2) = \frac{1}{\langle W_C\rangle}\int_C \mathcal{D}\{X\}e^{-I(\{X\})}\mathrm{V}_1(a_1)\mathrm{V}_2(a_2). \quad (4.1)$$

Here $I(\{X\})$ is the string action proportional to the tension $T = \frac{\sqrt{\lambda}}{2\pi}$ and in the planar approximation the path integral is performed over the Euclidean wordsheets with the topology of a disc and boundary conditions set by the loop C. Local gauge invariant operators $\mathcal{O}(a)$ are represented by vertex operators "inserted" at the boundary point a of AdS_5

$$\mathrm{V}(a) = \int d^2\xi V(\{X(\xi)\}, a). \quad (4.2)$$

In the limit of large λ the path integral (4.1) is dominated by a classical solution with boundary conditions prescribed by the loop C (and possibly also by the vertex operators if they carry large charges of the same order as string tension $\sim\sqrt{\lambda}$). Semiclassical correlators of circular loop with one vertex operator were discussed, e.g., in Refs. [8,10,22,23]. Correlators with two operators similar to (4.1) were studied recently in Refs. [10,19].

We shall start with the case when the two operators are "light," i.e., have charges much smaller than $\sqrt{\lambda}$ so that they do not change the form of semiclassical surface that ends on the circular loop at the boundary. The leading term in the correlator (4.1) then factorizes into a product of $\langle W_C\mathcal{O}_1(a_1)\rangle$ and $\langle W_C\mathcal{O}_2(a_2)\rangle$. We shall then consider a less trivial case when one of the two operators is "heavy,"

[10]In Eqs. (4.42), (4.43) of Ref. [10] one has to set $J_1 = J_2 = 2$, $A_1 = A_2 = \frac{1}{2}A = 2\pi$, $s_2 = 1$, take into account the normalization of the chiral primary operators, and note that the Wilson loop in Ref. [10] was defined without the $1/N$ prefactor in front [i.e., there to the leading order $\langle W\rangle = N(1+\cdots)$].

i.e., has dimension $J \sim \sqrt{\lambda}$. In both cases the aim will be to check the general structure of the correlator (2.25) and to compute the leading strong coupling contribution to the function $F(\mathrm{u}, \mathrm{v}; \lambda)$.

A. Case of two light operators

In this case dimensions Δ_1 and Δ_2 are fixed, i.e., much less than $\sqrt{\lambda} \gg 1$. Then the classical solution that dominates the path integral (4.1) is the surface in AdS_5 [3,4,8] ending on the circle (2.10)[11]

$$z = \tanh\tau, \qquad x_1 = \frac{\cos\sigma}{\cosh\tau}, \qquad x_2 = \frac{\sin\sigma}{\cosh\tau},$$
$$x_3 = x_4 = 0, \qquad \tau \in [0, \infty), \qquad \sigma \in [0, 2\pi], \tag{4.3}$$

where the AdS_5 metric is $ds^2 = z^{-2}(dx^\mu dx^\mu + dz^2)$. Then Eq. (4.1) becomes

$$\mathcal{C}_{\sqrt{\lambda}\gg 1} = \int d\tau_1 d\sigma_1 V(z(\tau_1, \sigma_1), x^\mu(\tau_1, \sigma_1) - a_1^\mu)$$
$$\times \int d\tau_2 d\sigma_2 V(z(\tau_2, \sigma_2), x^\mu(\tau_2, \sigma_2) - a_2^\mu), \tag{4.4}$$

where $z(\tau, \sigma)$, $x^\mu(\tau, \sigma)$ is the solution (4.3). Each integral in (4.4) is the strong coupling limit of the correlation function of the circular loop with the corresponding local operator

$$\int d\tau_i d\sigma_i V(z(\tau_i, \sigma_i), x^\mu(\tau_i, \sigma_i) - a_i^\mu) = \frac{\langle W_C \mathcal{O}(a_i)\rangle}{\langle W_C\rangle}, \tag{4.5}$$

i.e., if $\Delta_1, \Delta_2 \ll \sqrt{\lambda}$ the correlator (4.1) factorizes in the strong coupling approximation[12]

$$\mathcal{C}_{\sqrt{\lambda}\gg 1} = \frac{\langle W_C \mathcal{O}(a_1)\rangle}{\langle W_C\rangle} \frac{\langle W_C \mathcal{O}(a_2)\rangle}{\langle W_C\rangle}. \tag{4.6}$$

Since the correlation function of a circular Wilson loop with a local operator is fixed by conformal invariance to have the form (2.16) we conclude, comparing to (2.25), that the function $F(\mathrm{u}, \mathrm{v}; \lambda)$ is constant (u, v independent) in this limit

$$\sqrt{\lambda} \gg 1\colon\ F(\mathrm{u}, \mathrm{v}; \lambda) = \mathrm{C}_1(\lambda)\mathrm{C}_2(\lambda). \tag{4.7}$$

Here $\mathrm{C}_i(\lambda)$ is the corresponding coefficient in (2.16) [given explicitly in (4.24) in the case when the light operator is the dilaton or the chiral primary of dimension j].

[11]Here we change the notation compared to (3.3) and use σ instead of τ to parametrize the unit-radius circle ($R = 1$). τ is then the second worldsheet coordinate, i.e., $\xi = (\tau, \sigma)$.

[12]This strong coupling factorization was also observed in Ref. [10].

B. Case of one heavy and one light operator

Let us now consider the case when one of the two operators (say $\mathcal{O}_1$) is chosen to be a heavy chiral primary operator with dimension $\Delta_1 = J \sim \sqrt{\lambda}$ so that

$$\mathcal{J} = \frac{J}{\sqrt{\lambda}} \tag{4.8}$$

is fixed in the large λ limit. $\mathcal{O}_2$ will be chosen to be the dilaton operator whose dimension is $\Delta_2 = 4 \ll \sqrt{\lambda}$. In the presence of $\mathcal{O}_1$ (inserted at infinity) the solution (4.3) is modified to [8]

$$z = e^{\mathcal{J}\tau}[\sqrt{\mathcal{J}^2 + 1}\tanh(\sqrt{\mathcal{J}^2 + 1}\tau + q) - \mathcal{J}],$$
$$x_1 = \mathrm{R}(\tau)\cos\sigma, \qquad x_2 = \mathrm{R}(\tau)\sin\sigma, \qquad x_3 = x_4 = 0,$$
$$\mathrm{R}(\tau) \equiv \frac{\sqrt{\mathcal{J}^2 + 1}\, e^{\mathcal{J}\tau}}{\cosh(\sqrt{\mathcal{J}^2 + 1}\tau + q)},$$
$$q \equiv \log(\sqrt{\mathcal{J}^2 + 1} + \mathcal{J}), \qquad \phi = i\mathcal{J}\tau, \tag{4.9}$$

where ϕ is an angle of big circle in S^5 and as in (4.3) here $\tau \in [0, \infty)$, $\sigma \in [0, 2\pi]$. The solution starts at $\tau = 0$ as the unit circle (2.10) [with $\mathrm{R}(0) = 1$] and at $\tau \to \infty$ approaches

$$z \sim e^{\mathcal{J}\tau}, \qquad x^\mu \to 0, \qquad \mathrm{R}(\tau) \to 0, \qquad \phi \sim i\mathcal{J}\tau. \tag{4.10}$$

This asymptotics corresponds to the chiral primary operator inserted at $z = \infty$, $x^\mu = 0$, i.e., the solution (4.9) "interpolates" between the circle and the operator.

The correlator (4.1) can be written as follows:

$$\mathcal{C}_{J\sim\sqrt{\lambda}\gg 1} = \frac{\langle W_C \mathcal{O}_J(a_1)\mathcal{O}_{\mathrm{dil}}(a_2)\rangle}{\langle W_C\rangle}$$
$$= \frac{\langle W_C \mathcal{O}_J(a_1)\rangle}{\langle W_C\rangle} \frac{\langle W_C \mathcal{O}_J(a_1)\mathcal{O}_{\mathrm{dil}}(a_2)\rangle}{\langle W_C \mathcal{O}_J(a_1)\rangle}. \tag{4.11}$$

The first factor is the correlation function of the circular loop with the heavy operator found in Ref. [8] to be[13]

$$\frac{\langle W_C \mathcal{O}_J(a_1)\rangle}{\langle W_C\rangle} = \frac{\tilde{\mathrm{C}}_J}{[\ell(a_1)]^J}, \tag{4.12}$$

$$\tilde{\mathrm{C}}_J = 2^{-J}\exp(\sqrt{\lambda}[1 - \sqrt{\mathcal{J}^2 + 1} - \mathcal{J}\log(\sqrt{\mathcal{J}^2 + 1} - \mathcal{J})]). \tag{4.13}$$

The second factor in (4.11) is given by the light vertex operator evaluated on the classical solution (4.9)

[13]Our expression for $\tilde{\mathrm{C}}_J(\lambda)$ differs from the one in Ref. [8] by the factor 2^{-J} because our normalization of ℓ in (2.13) involves an extra factor of $\frac{1}{2}$.

$$\frac{\langle W_C \mathcal{O}_J(a_1)\mathcal{O}_{\rm dil}(a_2)\rangle}{\langle W_C \mathcal{O}_J(a_1)\rangle} = \int d\tau d\sigma V_{\rm dil}(z(\tau,\sigma), x^\mu(\tau,\sigma) - a_2^\mu, \phi(\tau,\sigma)). \tag{4.14}$$

Here the dilaton vertex operator is given by

$$V_{\rm dil}(a) = \hat{c}_{\rm dil}\Big[\frac{z}{z^2+(x^\mu-a^\mu)^2}\Big]^4 \mathcal{L}, \qquad \mathcal{L} = \frac{(\partial_a z)^2 + (\partial_a x^\mu)^2}{z^2} + (\partial_a\phi)^2, \tag{4.15}$$

where $\mathcal{L}$ is the $AdS_5 \times S^5$ Lagrangian in which we ignored the extra bosonic and fermionic coordinates that vanish on the classical solution. The normalization factor $\hat{c}_{\rm dil}$ is given by

$$\hat{c}_{\rm dil} = \frac{\sqrt{6}\sqrt{\lambda}}{8\pi N}. \tag{4.16}$$

To compute (4.14) for general enough values of a_1, a_2 [sufficient to restore the strong coupling limit of the function $F(\mathrm{u},\mathrm{v};\lambda)$ in (2.25)], we need the classical solution corresponding to the chiral primary operator inserted at a finite point on the AdS_5 boundary at $z=0$. It can be found by a conformal transformation applied to (4.9).[14] Since the correlator under consideration is fixed to a large extent by the conformal symmetry, it is sufficient to place the operators at some special points a_1, a_2 as long as the variables u and v remain independent. We found the following choice to be convenient [see (2.12)]:

$$a_1^\mu = (r_1, \psi_1, h_1, \varphi_1) = (0,0,h,0), \qquad a_2^\mu = (r_2,\psi_2,h_2,\varphi_2) = (r,0,0,0). \tag{4.17}$$

The chiral primary operator is then located above the center of the circle while the dilaton is inserted in the plane of the circle (here $r=1$ corresponds to a point of the circle). In this case [see (2.13), (2.23), and (2.24)]

$$\ell(a_1) = \frac{1}{2}(h^2+1), \qquad \ell(a_2) = \frac{1}{2}(r^2-1), \qquad \mathrm{u} = \frac{r^2+1}{r^2-1}, \qquad \mathrm{v} = \frac{h^2-1}{h^2+1}. \tag{4.18}$$

Let us now perform a finite conformal transformation (an isometry of AdS_5) that preserves the circle and maps the point ($z=\infty$, $x^\mu=0$) to the point ($z=0$, $x^\mu=a_1^\mu$). The transformation consisting of a dilatation (with parameter γ), a special conformal transformation (β_μ), and a translation (α_μ) can be written as

$$z' = \frac{\gamma z}{1+2\gamma\beta\cdot x + \gamma^2\beta^2(z^2+x^2)}, \qquad x'_\mu = \frac{\gamma[x_\mu + \gamma\beta_\mu(z^2+x^2)]}{1+2\gamma\beta\cdot x+\gamma^2\beta^2(z^2+x^2)} + \alpha_\mu. \tag{4.19}$$

We will choose $\alpha^\mu = (0,0,\alpha,0)$, $\beta^\mu=(0,0,\beta,0)$. Then the circle $x_1^2+x_2^2=1$, $x_3=x_4=0$ at $z=0$ is transformed into

$$x'_1 = \frac{\gamma}{1+\gamma^2\beta^2}x_1, \qquad x'_2 = \frac{\gamma}{1+\gamma^2\beta^2}x_2, \qquad x'_3 = \frac{\gamma^2\beta}{1+\gamma^2\beta^2}+\alpha, \qquad x'_4 = 0, \tag{4.20}$$

so that to preserve it we have to require

$$\frac{\gamma}{1+\gamma^2\beta^2} = 1, \qquad \frac{\gamma^2\beta}{1+\gamma^2\beta^2}+\alpha = 0, \qquad \text{i.e.} \quad \alpha = -\sqrt{\gamma-1}, \qquad \beta = \frac{\sqrt{\gamma-1}}{\gamma}. \tag{4.21}$$

Note that this conformal transformation preserves the entire plane $x_3=x_4=0$. Acting with (4.19) on the solution (4.9) we obtain a new (conformally equivalent) solution[15]

$$z' = \gamma w(\tau)z(\tau), \qquad x'_1 = \gamma w(\tau)\mathrm{R}(\tau)\cos\sigma, \qquad x'_2 = \gamma w(\tau)\mathrm{R}(\tau)\sin\sigma, \qquad x'_4 = 0, \tag{4.22}$$

$$x'_3 = \sqrt{\gamma-1}\,w(\tau)[z^2(\tau)+\mathrm{R}^2(\tau)-1], \qquad w(\tau) \equiv \frac{1}{1+(\gamma-1)[z^2(\tau)+\mathrm{R}^2(\tau)]}, \tag{4.23}$$

where $z(\tau)$ and $\mathrm{R}(\tau)$ were given in (4.9). For $\tau\to 0$ this solution still approaches the circle (2.10) while for $\tau\to\infty$ we obtain

$$z'=0, \qquad x'_1 = x'_2 = x'_4 = 0, \qquad x'_3 = \frac{1}{\sqrt{\gamma-1}}. \tag{4.24}$$

To match the location a_1 of the chiral primary operator in (4.17), we then need to fix γ as

$$\gamma = \frac{h^2+1}{h^2}. \tag{4.25}$$

Let us now use this transformed solution to compute the contribution of the light vertex operator in (4.14) and (4.15). Taking into account that the position of the dilaton operator is chosen as in (4.17) and that the value of $\mathcal{L}$ in (4.15) is

$$\mathcal{L} = \frac{2(1+\mathcal{J}^2)}{\sinh^2(\sqrt{\mathcal{J}^2+1}\tau)}, \tag{4.26}$$

we can present (4.14) in the form

$$\frac{\langle W_C \mathcal{O}_J(a_1)\mathcal{O}_{\rm dil}(a_2)\rangle}{\langle W_C\mathcal{O}_J(a_1)\rangle} = \frac{\hat{c}_{\rm dil}}{8(\mathcal{J}^2+1)r^4}\int_0^\infty d\tau \sinh^2(\sqrt{\mathcal{J}^2+1}\tau) \times \int_0^{2\pi}\frac{d\sigma}{[y(\tau)-\cos\sigma]^4}. \tag{4.27}$$

[14] A similar conformal transformation was considered in Ref. [10] and also in Ref. [23].

[15] The solution for ϕ is of course unchanged and is still given by (4.9).

Here

$$y(\tau) \equiv \frac{\gamma^2(z^2+\mathrm{R}^2)+(\gamma-1)(z^2+\mathrm{R}^2-1)^2+r^2[1+(\gamma-1)(z^2+\mathrm{R}^2)]^2}{2\gamma r\mathrm{R}[1+(\gamma-1)(z^2+\mathrm{R}^2)]}, \tag{4.28}$$

with $z=z(\tau)$ and $\mathrm{R}=\mathrm{R}(\tau)$ given in (4.9). Recall that in view of (4.18) and (4.25) we have

$$r=\sqrt{\frac{\mathrm{u}+1}{\mathrm{u}-1}}, \qquad h=\sqrt{\frac{1+\mathrm{v}}{1-\mathrm{v}}}, \qquad \gamma=\frac{2}{\mathrm{v}+1}. \tag{4.29}$$

Doing the integral over σ we end up with

$$\frac{\langle W_C\mathcal{O}_J(a_1)\mathcal{O}_{\mathrm{dil}}(a_2)\rangle}{\langle W_C\mathcal{O}_J(a_1)\rangle}=\frac{\pi\hat{c}_{\mathrm{dil}}}{8(\mathcal{J}^2+1)r^4}I(\mathrm{u},\mathrm{v},\mathcal{J}), \tag{4.30}$$

$$I(\mathrm{u},\mathrm{v},\mathcal{J})=\int_0^\infty d\tau \sinh^2(\sqrt{\mathcal{J}^2+1}\tau)\frac{[2y^2(\tau)+3]y(\tau)}{[y^2(\tau)-1]^{7/2}}, \tag{4.31}$$

where we assume that r and γ in y are expressed in terms of u and v as in (4.29).

Combining (4.12) and (4.30) according to (4.11) and comparing to the general expression (2.25) for the correlator in question, we conclude that

$$\lambda \gg 1,$$
$$\mathcal{J}=\frac{J}{\sqrt{\lambda}}:\ F(\mathrm{u},\mathrm{v};\lambda)=\frac{\pi\tilde{\mathrm{C}}_J\hat{c}_{\mathrm{dil}}}{8(\mathcal{J}^2+1)(\mathrm{u}^2-1)^2}I(\mathrm{u},\mathrm{v},\mathcal{J}), \tag{4.32}$$

where we used (4.18) [i.e., $[\ell(a_2)]^{-4}=16(r^2-1)^{-4}$].

In the special case of $\mathrm{u}=1$, $\mathrm{v}=-1$ [see (2.27)] corresponding here to $r\to\infty$, $\gamma\to\infty$ we get a finite expression for the function $F(\mathrm{u},\mathrm{v};\lambda)$ in (4.32). Indeed, in this limit

$$y\to r\frac{z^2+\mathrm{R}^2}{2\mathrm{R}}, \tag{4.33}$$

and then the y-dependent factor in the integrand of (4.31) becomes

$$\frac{(2y^2+3)y}{(y^2-1)^{7/2}}\to\frac{1}{y^4}\to r^4\left(\frac{2\mathrm{R}}{z^2+\mathrm{R}^2}\right)^4. \tag{4.34}$$

The singular factor r^4 in (4.30) then cancels out, so that the correlator becomes a finite constant (a function of $\mathcal{J}$ only).

In general, the integral $I(\mathrm{u},\mathrm{v},\mathcal{J})$ in (4.31) appears to be too complicated to be computable analytically for arbitrary $\mathcal{J}$ but it can be easily evaluated in the limiting cases of small and large $\mathcal{J}$.

1. Small $\mathcal{J}$ limit

For $\mathcal{J}=0$ the solution (4.9) and (4.23) becomes the original circle solution (4.3) and (4.31) reduces to the correlator of the circular Wilson loop with the dilaton operator

$$\mathcal{J}\to 0:\ \frac{\langle W_C\mathcal{O}_J(a_1)\mathcal{O}_{\mathrm{dil}}(a_2)\rangle}{\langle W_C\mathcal{O}_J(a_1)\rangle}\to\frac{\langle W_C\mathcal{O}_{\mathrm{dil}}(a_2)\rangle}{\langle W_C\rangle}=\frac{\mathrm{C}_{\mathrm{dil}}(\lambda)}{[\ell(a_2)]^4}, \tag{4.35}$$

where $\mathrm{C}_{\mathrm{dil}}$ was given in (2.17), i.e., in this limit the function $F(\mathrm{u},\mathrm{v};\lambda)$ is constant

$$\lambda\gg 1, \qquad \mathcal{J}\ll:\ F(\mathrm{u},\mathrm{v};\lambda)=\bar{\mathrm{C}}_J\mathrm{C}_{\mathrm{dil}}[1+\mathcal{O}(\mathcal{J})], \tag{4.36}$$

with

$$\bar{\mathrm{C}}_J=(\tilde{\mathrm{C}}_J)_{\mathcal{J}\ll 1}=2^{-J}\exp\left(\frac{1}{2}\sqrt{\lambda}[\mathcal{J}^2+O(\mathcal{J}^4)]\right). \tag{4.37}$$

To find the linear $\mathcal{J}$ term in F we expand the solution (4.9) and thus y in (4.28) in powers of $\mathcal{J}$

$$\begin{aligned} z(\tau)&=\tanh\tau[1+\mathcal{J}(\tau-\tanh\tau)+O(\mathcal{J}^2)],\\ \mathrm{R}(\tau)&=\frac{1}{\cosh\tau}[1+\mathcal{J}(\tau-\tanh\tau)+O(\mathcal{J}^2)],\\ y(\tau)&=\frac{1+r^2}{2r}\cosh\tau+\mathcal{J}\frac{(\gamma-2)(r^2-1)}{2\gamma r}\\ &\quad\times(\tau\cosh\tau-\sinh\tau)+O(\mathcal{J}^2). \end{aligned} \tag{4.38}$$

Then the order $\mathcal{J}$ term in (4.30) becomes

$$\left(\frac{\langle W\mathcal{O}_J(a_1)\mathcal{O}_{\mathrm{dil}}(a_2)\rangle}{\langle W\mathcal{O}_J(a_1)\rangle}\right)_{\mathcal{J}}=-16\mathcal{J}\pi\hat{c}_{\mathrm{dil}}\frac{\gamma-2}{\gamma}(r^2-1)\mathcal{I}(r), \tag{4.39}$$

$$\begin{aligned} \mathcal{I}(r)&=\int_0^\infty d\tau\frac{\sinh^2\tau(\tau\cosh\tau-\sinh\tau)}{[(1+r^2)^2\cosh^2\tau-4r^2]^{9/2}}\\ &\quad\times[6r^4+12r^2(1+r^2)^2\cosh^2\tau+(1+r^2)^4\cosh^4\tau]\\ &=\frac{7+4r^2+2r^4-12r^6-r^8+4(1+r^2)^2\log\frac{2r^2}{1+r^2}}{12(r^2-1)^6(1+r^2)^3}. \end{aligned} \tag{4.40}$$

Expressing γ and r in terms of v and u according to (4.29), extracting the factor $[\ell(a_2)]^{-4}$, and also using that $\mathrm{C}_{\mathrm{dil}}=\frac{\pi}{12}\hat{c}_{\mathrm{dil}}$ [see (2.17)] we finally get for the order $\mathcal{J}$ term in F in (4.36)

$$F(\mathrm{u},\mathrm{v};\lambda)=\bar{\mathrm{C}}_J\mathrm{C}_{\mathrm{dil}}\left[1+\mathcal{J}\frac{\mathrm{v}}{\mathrm{u}^3}\left(1+2\mathrm{u}^2-4\mathrm{u}^3+4\mathrm{u}^4\log\frac{\mathrm{u}+1}{\mathrm{u}}\right)+\mathcal{O}(\mathcal{J}^2)\right]. \tag{4.41}$$

2. Large $\mathcal{J}$ limit

In the limit of large $\mathcal{J}$ one finds, to leading order,

$$z(\tau) = \frac{1}{\mathcal{J}}\sinh\mathcal{J}\tau, \qquad \mathrm{R}(\tau) = 1, \qquad y = \frac{1+r^2}{2r} + \frac{1+(\gamma-1)r^2}{2\mathcal{J}^2\gamma r}\sinh^2(\mathcal{J}\tau). \tag{4.42}$$

Let us rescale $\mathcal{J}\tau \to \tau$ and use y as the new integration variable. Then up to terms subleading at large $\mathcal{J}$ the integral (4.31) can be written as

$$I(\mathrm{u}, \mathrm{v}, \mathcal{J} \gg 1) = \frac{\mathcal{J}\gamma r}{1+(\gamma-1)r^2}\int_{\frac{1+r^2}{2r}}^{\infty} dy \frac{(3+2y^2)y}{(y^2-1)^{7/2}}. \tag{4.43}$$

The integral over y gives

$$\int_{\frac{1+r^2}{2r}}^{\infty} dy \frac{(3+2y^2)y}{(y^2-1)^{7/2}} = \frac{16}{3}\frac{r^3(1+4r^2+r^4)}{(r^2-1)^5}. \tag{4.44}$$

As a result, from (4.32) we get

$$\lambda \gg 1, \mathcal{J} \gg 1: F(\mathrm{u}, \mathrm{v}; \lambda) = \frac{\hat{\mathrm{C}}_J \mathrm{C}_{\mathrm{dil}}}{2\mathcal{J}}\left[\frac{3\mathrm{u}^2 - \mathrm{v}}{\mathrm{u}-\mathrm{v}} + \mathcal{O}\left(\frac{1}{\mathcal{J}}\right)\right], \tag{4.45}$$

where

$$\hat{\mathrm{C}}_J = (\tilde{\mathrm{C}}_J)_{\mathcal{J}\gg 1} = 2^{-J}\exp(\sqrt{\lambda}[\mathcal{J}(\log(2\mathcal{J})-1)+1+O(\mathcal{J}^{-1})]). \tag{4.46}$$

Note that the leading singularity in the OPE limit $a_1 \to a_2$ is still $(\mathrm{u}-\mathrm{v})^{-1} \sim |a_1 - a_2|^{-2}$ just like at weak coupling [see (3.18)]. Explicitly, in this limit

$$(\mathcal{C}_{\mathcal{J}\gg 1})_{a_1\to a_2} \to \frac{1}{[\ell(a_1)]^{J+2}}\frac{1}{|a_1-a_2|^2}\frac{2\tilde{\mathrm{C}}_J \mathrm{C}_{\mathrm{dil}}}{\mathcal{J}}, \tag{4.47}$$

where we used Eq. (2.26) and that in this limit $\mathrm{u} \to 1$, $\mathrm{v} \to 1$. Comparing with (3.20) we see that here $\delta = 2$ and that the leading contribution should come from an operator of dimension $\Delta_3 = J + 2$. This is consistent with (3.19) and (3.20) as we have $\Delta_1 = J$, $\Delta_2 = 4$.

V. CORRELATOR OF INFINITE LINE WILSON LOOP WITH LOCAL OPERATORS

The locally supersymmetric Wilson loop [1,3,4] defined by an infinite straight line (which we will denote as W_L) is a 1/2 supersymmetric object with trivial expectation value, $\langle W_L\rangle = 1$. If we choose the line along the x_1 direction, i.e.,

$$x_1 = \tau, \qquad x_2 = x_3 = x_4 = 0, \tag{5.1}$$

the field combination in (2.7) becomes "chiral" $(iA_1 + \Phi_1)$.[16] The infinite line (5.1) is related [3,6] to the circle (2.10) of radius R with the center at 0 by a particular conformal transformation [cf. (4.19)]

$$x_1' = \frac{x_1}{1+\beta^2 x_1^2}, \qquad x_2' = \frac{\beta x_1^2}{1+\beta^2 x_1^2} - R, \qquad \beta \equiv \beta_2 = \frac{1}{2R}, \qquad x_3' = x_4' = 0, \tag{5.2}$$

where $x_1'^2 + x_2'^2 = R^2$.[17] The need to regularize the correlator (and the fact that the inversion changes boundary conditions at infinity or changes topology of the world surface on the string side) leads to an anomaly [5–7], explaining why the expectation value of the circular Wilson loop is no longer equal to 1: its expression is given in terms of the modified Bessel function of $\sqrt{\lambda}$, $\langle W_C\rangle = \frac{2}{\sqrt{\lambda}}I_1(\sqrt{\lambda}) = 1 + \frac{\lambda}{8} + \frac{\lambda^2}{192} + \cdots$.

As was mentioned in the Introduction, one may expect that despite $\langle W_L\rangle \neq \langle W_C\rangle$ the transformation (5.2) may still be relating the normalized correlators of W_L and W_C with local operators, i.e., the anomaly should be absent in the local correlators.

Let us first discuss the conformal symmetries preserved by the configuration involving a straight line (5.1). As in the circle case we may perform a conformal map from $\mathbb{R}^4$ to $\mathrm{AdS}_2 \times S^2$ with the line becoming the boundary of AdS_2. Here it is natural to use the Poincare coordinates for AdS_2. Explicitly, going first to spherical coordinates in the (x_2, x_3, x_4) subspace we get

$$ds^2 = dx^2 + dz^2 + z^2(d\theta^2 + \sin^2\theta d\varphi^2) = z^2\left[\frac{dx^2+dz^2}{z^2} + ds^2_{S^2}\right], \tag{5.3}$$

$$x \equiv x_1, \qquad z = \sqrt{x_2^2 + x_3^2 + x_4^2}. \tag{5.4}$$

An analysis similar to the one in Appendix A shows that the line (5.1) is preserved by six conformal transformations: dilatations, translations along the line, special conformal transformations along the line, and three rotations in the orthogonal space. These may be interpreted as the isometries of $\mathrm{AdS}_2 \times S^2$ preserving the boundary (line x) of AdS_2.

As in the case of the circle, the correlation function of a line with one local operator is fixed by conformal symmetry: since the line is invariant under the six isometries, it is impossible to construct an invariant depending on the four position of the operator, i.e., by the same argument as in Sec. II B we get (here a^μ are the Cartesian coordinates of the point a with the direction of the line being $x = a^1$)

[16] Note that the expectation value of any function of $iA_1 + \Phi_1$ over the Gaussian measure defined by $L = (\partial_\mu A_1)^2 + (\partial_\mu \Phi_1)^2 + \cdots$ vanishes.

[17] To get the standard parametrization of the circle in (3.3) we need also to change $\tau \to \tau'$, $\cos\tau' = \frac{\tau}{1+\frac{\tau^2}{4R^2}}$.

$$\frac{\langle W_L \mathcal{O}(a)\rangle}{\langle W_L\rangle} = \frac{C_L(\lambda)}{[\ell_L(a)]^{\Delta}},\qquad \ell_L(a) \equiv z = \sqrt{(a^2)^2 + (a^3)^2 + (a^4)^2}. \tag{5.5}$$

Note that $\ell_L(a)$ is just the distance from the position of the operator to the line (5.1).

Let us compute $\frac{\langle W_L \mathcal{O}(a)\rangle}{\langle W_L\rangle}$ to leading order in λ for $\mathcal{O}$ being a chiral primary operator and compare it with the corresponding expression for the circular Wilson loop. Using the definitions of the Wilson loop in (3.4) and the $\Delta = 2$ operator in (3.2), we get for the order λ term:

$$\frac{\langle W_L \mathcal{O}(a)\rangle}{\langle W_L\rangle} = \frac{\lambda c_2}{32\pi^4}\int_{-\infty}^{\infty} d\tau_1 \int_{-\infty}^{\tau_1} d\tau_2 \times \frac{1}{|x(\tau_1)-a|^2 |x(\tau_2)-a|^2}, \tag{5.6}$$

where the line is parametrized as $x(\tau) = (\tau, 0, 0, 0)$. Performing the integrals gives

$$\frac{\langle W_L \mathcal{O}(a)\rangle}{\langle W_L\rangle} = \frac{\lambda}{16\sqrt{2}N}\frac{1}{[\ell_L(a)]^2}. \tag{5.7}$$

This is the same result as found in the case of the circle [7].[18] In general, one should have [cf. (2.16) and (5.5)]

$$[\ell_L(a)]^{-\Delta}\frac{\langle W_L \mathcal{O}(a)\rangle}{\langle W_L\rangle} = [\ell_C(a)]^{-\Delta}\frac{\langle W_C \mathcal{O}(a)\rangle}{\langle W_C\rangle}, \tag{5.8}$$

for all conformal operators and for all values of λ.

The exact expression for the correlator (5.6) is found by replacing λ in (5.7) by $4\sqrt{\lambda}\frac{I_2(\sqrt{\lambda})}{I_1(\sqrt{\lambda})} = \lambda - \frac{1}{24}\lambda^2 + \cdots$ [7,9].[19] Since the dimension four dilaton operator is in the same supermultiplet with the $\Delta = 2$ chiral primary operator, one may expect that its normalized correlator with the circular Wilson loop should also be proportional to $\sqrt{\lambda}\frac{I_2(\sqrt{\lambda})}{I_1(\sqrt{\lambda})}$. This is indeed what one finds if one observes that[20]

$$\frac{d}{d\sqrt{\lambda}}\log\langle W_C\rangle = \frac{d}{d\sqrt{\lambda}}\log\left[\frac{2}{\sqrt{\lambda}}I_1(\sqrt{\lambda})\right] = \sqrt{\lambda}\frac{I_2(\sqrt{\lambda})}{I_1(\sqrt{\lambda})}, \tag{5.9}$$

and that differentiating $\langle W_C\rangle$ over the coupling produces the insertion of the *integrated* (over four space) dilaton operator. The latter is the gauge theory action if the correlator is understood in terms of the gauge theory path integral or the string theory action if it is defined in terms of the string path integral.

There is, however, a subtlety if one tries to use this argument to deduce the value of the coefficient $C_{\rm dil}(\lambda)$ in the local correlator (2.16): integrating (2.16) over the position a one gets (for $\Delta_{\rm dil} = 4$) the integral $\int d^4a[\ell(a)]^{-4} \sim R^4 \int_0^\infty \int_0^\infty \frac{r dr h dh}{[(r^2+h^2-R^2)^2+4h^2R^2]^2}$, which is linearly UV divergent ($\sim R\int_0^\infty \frac{dh}{h^2}$) at $h \to 0$. As usual, this UV divergence is to be regularized away to make the comparison to (5.9) possible.[21]

In the case of the line where $\langle W_L\rangle = 1$, the analog of (5.9) vanishes but this does not imply that $C_{\rm dil}(\lambda)$ should vanish [which would be in contradiction with (5.8)]. Indeed, the corresponding integral $\int d^4a[\ell_L(a)]^{-4} = \int_{-\infty}^{\infty} da_1 \int_{-\infty}^{\infty} \frac{d^3\vec{a}}{|\vec{a}|^4}$ is now not only UV but also IR divergent (along the infinite direction of the line). Its subtracted value should be zero, thus reconciling the fact that $\frac{d}{d\sqrt{\lambda}}\langle W_L\rangle = 0$ with the expected relation (5.8).

Let us now turn to the case of the correlator of W_L with two operators. Like for the case of the circle, the correlator of the line with two operators

$$\mathcal{C}(W_L, a_1, a_2) = \frac{\langle W_L \mathcal{O}_1(a_1)\mathcal{O}_2(a_2)\rangle}{\langle W_L\rangle} \tag{5.10}$$

is also fixed up to a function of $8 - 6 = 2$ variables u, v related to the geodesic distances in AdS_2 and S^2 [see (2.23)] that are invariant under the conformal transformations preserving the line. Here the variable u should be written in terms of the Poincare coordinates. Using the relation between the global and the Poincare coordinates in AdS_2 [cf. (2.14)]

$$\cosh\rho = \frac{1 + x^2 + z^2}{2z},\qquad \cos\psi = \frac{2x}{\sqrt{(x^2 + z^2 - 1)^2 + 4x^2}}, \tag{5.11}$$

we find that for the two points in AdS_2 with coordinates (x_1, z_1) and (x_2, z_2) corresponding to (ρ_1, ψ_1) and (ρ_2, ψ_2)[22]

$$\mathrm{u} = 1 + \frac{(x_1 - x_2)^2 + (z_1 - z_2)^2}{2z_1 z_2}. \tag{5.12}$$

Hence

$$\mathcal{C}(W_L, a_1, a_2) = \frac{1}{[\ell_L(a_1)]^{\Delta_1}[\ell_L(a_2)]^{\Delta_2}} F_L(\mathrm{u}, \mathrm{v}; \lambda). \tag{5.13}$$

[18]In Ref. [7] the leading contribution at weak coupling is given in Eq. (1.17). The operator $\mathcal{O}$ that we used is $\frac{1}{\sqrt{2}}(\mathcal{O}_2^1 + i\mathcal{O}_2^2)$ in the notation of Ref. [7].

[19]For dimension k chiral primary one is to replace I_2 by I_k [7].

[20]In general, $x\frac{d}{dx}I_k(x) = k + x\frac{I_{k+1}(x)}{I_k(x)}$.

[21]See Sec. 2.1 in Ref. [24] for a related discussion of integrated dilaton insertion into correlation functions where one also needs to introduce a UV cutoff (see also Ref. [25]). Note that similar divergence is found at strong coupling if one simply evaluates the string action on the corresponding minimal surface [4] (see also Sec. 4 in Ref. [24]).

[22]Note that since this is just a coordinate transformation in Euclidean AdS_2 that should not change geodesic distances, the variables u and v are actually the same in both cases.

Here $x_{1,2}$ are first components of $a_{1,2}$, i.e., $x_i = a_i^1$, while $\ell_L(a_i) = z_i = \sqrt{(a_i^2)^2 + (a_i^3)^2 + (a_i^4)^2}$ and the distance between the points a_1 and a_2 is again given by (2.26)

$$\begin{aligned} |a_1 - a_2|^2 &= (x_1 - x_2)^2 + z_1^2 + z_2^2 - 2z_1 z_2 \mathrm{v} \\ &= 2\ell_L(a_1)\ell_L(a_2)(\mathrm{u} - \mathrm{v}). \end{aligned} \tag{5.14}$$

As an example, let us compute the correlator (5.10) to leading order at weak coupling for the case when the operators are the chiral primaries in (3.2). The leading connected contribution is still given by (3.13) but with different integration limits

$$\begin{aligned} \mathcal{C}_1 = & \frac{g^2 c_2^2}{64\pi^6 |a_1 - a_2|^2} \int_{-\infty}^{\infty} d\tau_1 \int_{-\infty}^{\tau_1} d\tau_2 \\ & \times \Big[\frac{1}{|x(\tau_1) - a_1|^2 |x(\tau_2) - a_2|^2} + (a_1 \leftrightarrow a_2) \Big], \end{aligned} \tag{5.15}$$

where in the present case of the line $x(\tau) = (\tau, 0, 0, 0)$. To find F_L in (5.13) it is sufficient to make a special choice of coordinates of the points a_1 and a_2 [here we list the values of the $AdS_2 \times S^2$ coordinates, i.e., $a_i = (x_i, z_i, \theta_i, \varphi_i)$]

$$a_1 = (0, z_1, \theta_1, 0), \qquad a_2 = (0, z_2, \theta_2, 0). \tag{5.16}$$

In this case it is straightforward to evaluate the integrals in (5.15) to obtain [see (3.2) and (5.14)]

$$\begin{aligned} \mathcal{C}_1 &= \frac{\lambda c_2^2}{64\pi^6 |a_1 - a_2|^2} \frac{\pi^2}{\ell_L(a_1)\ell_L(a_2)} \\ &= \frac{\lambda}{16N^2} \frac{1}{[\ell_L(a_1)]^2 [\ell_L(a_2)]^2} \frac{1}{\mathrm{u} - \mathrm{v}}. \end{aligned} \tag{5.17}$$

Thus the leading-order term in F_L in (5.13) is

$$F_{1L}(\mathrm{u}, \mathrm{v}) = \frac{\lambda}{16N^2} \frac{1}{\mathrm{u} - \mathrm{v}}, \tag{5.18}$$

which is the same as F_1 in (3.18) found for the circular Wilson loop.[23] Similar agreement should be present also at higher orders in λ and for more general correlators.

ACKNOWLEDGMENTS

We are grateful to N. Drukker, S. Giombi, R. Roiban, and K. Zarembo for very useful suggestions and discussions. This work was supported by STFC Grant No. ST/J000353/1. A. A. T. was also supported by the ERC Advanced Grant No. 290456.

APPENDIX A: INFINITESIMAL CONFORMAL TRANSFORMATION PRESERVING THE CIRCLE

Here we shall review the count of conformal symmetries preserved by the circle before and after adding local operators (see Refs. [19,20]). A general infinitesimal conformal transformation acts as follows:

$$\delta x_\mu = \alpha_\mu + \omega_{\mu\nu} x_\nu + \sigma x_\mu + x^2 \beta_\mu - 2(\beta \cdot x) x_\mu, \tag{A1}$$

where the parameters α^μ, $\omega_{\mu\nu}$, σ, and β^μ correspond to translations, Lorentz transformations, dilatations, and special conformal transformations, respectively. Let us split $x^\mu = (x_1, x_2, x_3, x_4)$ into the components in the plane of the circle (2.10) and in the orthogonal plane, $x_l = (x_1, x_2)$ and $x_t = (x_3, x_4)$. Below we will fix the radius to be $R = 1$. Taking into account that $x_l^2 = 1$, $x_t = 0$ we get $(l, m = 1, 2)$

$$\begin{aligned} \delta x_l &= \alpha_l + \omega_{lm} x_m + \sigma x_l + \beta_l - (\beta_m x_m) x_l, \\ \delta x_t &= \alpha_t + \omega_{tm} x_m + \beta_t. \end{aligned} \tag{A2}$$

Now using $\delta x_t = 0$, $x_l \delta x_l = 0$ we obtain

$$\omega_{tm} = 0, \qquad \sigma = 0, \qquad \alpha_t = -\beta_t, \qquad \alpha_l = \beta_l. \tag{A3}$$

This means that the surviving 6 transformations are generated by

$$\begin{aligned} &\omega_{12}, \qquad \omega_{34}, \qquad \alpha_l, \qquad \alpha_t, \\ &\beta_l = \alpha_l, \qquad \beta_t = -\alpha_t. \end{aligned} \tag{A4}$$

An addition of an operator at a generic point of four space will break four out of six conformal transformations (A4). Introduction of the second operator will break all the conformal transformations.

APPENDIX B: GEODESIC DISTANCES IN S^2 AND AdS_2

Let us present an analytic derivation of the well-known expressions for the geodesic distances in S^2 and AdS_2 used in (2.23).

The geodesic $[\theta = \theta(t), \varphi = \varphi(t)]$ on S^2 connecting the points (θ_1, φ_1) and (θ_2, φ_2) can be obtained by minimizing the functional

$$\varsigma = \int dt [(\partial_t \theta)^2 + \sin^2\theta (\partial_t \varphi)^2] \tag{B1}$$

and evaluating it on the solution. Integrating the equations for the geodesic gives

$$\begin{aligned} \partial_t(\sin^2\theta \partial_t \varphi) &= 0, \\ \partial_t^2 \theta - \sin\theta \cos\theta (\partial_t \varphi)^2 &= 0, \end{aligned} \tag{B2}$$

[23] This agreement is not too surprising. As was argued in Ref. [6], the anomaly (leading to $\langle W_L \rangle \neq \langle W_C \rangle$) comes from a nontrivial transformation of the gauge vector propagator under the inversions [and, hence, under the special conformal transformation in (5.2)]. Since in the above example the vector propagators did not contribute, we should get the same answer for both the line and the circle.

$$\cot\theta(t) = \frac{C_2}{[C_1^2 + (C_1^2 + C_2^2)\cot^2(\sqrt{C_1^2 + C_2^2}(t - t_0))]^{1/2}},$$
$$\cot(\varphi(t) - \varphi_0) = \frac{\sqrt{C_1^2 + C_2^2}}{C_1}\cot(\sqrt{C_1^2 + C_2^2}(t - t_0)), \tag{B3}$$

where C_1, C_2, t_0, φ_0 are integration constants. Eliminating t we can write the geodesic passing through the points (θ_1, φ_1) and (θ_2, φ_2) in the form

$$\cot\theta(\varphi) = A\sin\varphi + B\cos\varphi, \tag{B4}$$

$$A = \frac{\tan\theta_2\cos\varphi_2 - \tan\theta_1\cos\varphi_1}{\tan\theta_2\tan\theta_1\sin(\varphi_1 - \varphi_2)},\quad B = \frac{\tan\theta_1\sin\varphi_1 - \tan\theta_2\sin\varphi_2}{\tan\theta_2\tan\theta_1\sin(\varphi_1 - \varphi_2)}. \tag{B5}$$

Then the geodesic length may be written as

$$\varsigma = \int_{\varphi_1}^{\varphi_2}\frac{d\varphi\sqrt{1 + A^2 + B^2}}{1 + (A\sin\varphi + B\cos\varphi)^2} = \arctan\frac{AB + (1 + A^2)\tan\varphi_2}{\sqrt{1 + A^2 + B^2}} - (\varphi_2 \to \varphi_1). \tag{B6}$$

As a result, one finds

$$\cos\varsigma = \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos(\varphi_2 - \varphi_1). \tag{B7}$$

A similar analysis in AdS_2 gives for the corresponding geodesic distance s

$$\cosh s = \cosh\rho_1\cosh\rho_2 - \sinh\rho_1\sinh\rho_2\cos(\psi_2 - \psi_1). \tag{B8}$$

The two expressions are, of course, related by the analytic continuation $\theta_k \to i\rho_k$, $\varphi_k \to \psi_k$.

APPENDIX C: CORRELATOR OF CIRCULAR WILSON LOOP WITH ONE LIGHT SUPERSYMMETRIC OPERATOR AT STRONG COUPLING

Here we will review the derivation of the correlation function of a circular Wilson loop with one local operator that will be chosen to be the dilaton or the chiral primary (with dimension j fixed, i.e., not scaling with λ).

The correlator in question appeared in (2.16) and (4.5); i.e., in the leading large λ approximation it is given by

$$\mathcal{C}(W_C, a) = \frac{\langle W_C \mathcal{O}(a)\rangle}{\langle W_C\rangle} = \int d\tau d\sigma V(z(\tau,\sigma), x^\mu(\tau,\sigma) - a^\mu)), \tag{C1}$$

where $[z(\tau,\sigma), x^\mu(\tau,\sigma)]$ represents the circular loop solution (4.3). Since this correlator is fixed by conformal invariance up to a constant (2.16), we can choose the position of the operator to be at $a^\mu = (0, 0, h, 0)$.

Evaluating the dilaton vertex operator (4.15) on the solution (4.9) gives [see (2.13)]

$$\mathcal{C}(W_C, a) = \frac{4\pi\hat{c}_{\rm dil}}{(L^2 + 1)^4}\int_0^\infty d\tau\frac{\sinh^2\tau}{\cosh^4\tau} = \frac{4\pi\hat{c}_{\rm dil}}{3(h^2 + 1)^4} = \frac{\pi\hat{c}_{\rm dil}}{12}\frac{1}{[\ell(a)]^4}. \tag{C2}$$

Using the value of the normalization coefficient (4.16) we obtain $C_{\rm dil}(\lambda)$ in (2.17).

The bosonic part of the chiral primary vertex operator of dimension $\Delta = j$ is given by [3,26]

$$V(a) = \hat{c}_j\int d\tau d\sigma\left[\frac{z}{z^2 + (x^\mu - a^\mu)^2}\right]^j e^{ij\phi}U,\quad \hat{c}_j = \frac{\sqrt{\lambda}}{8\pi N}\sqrt{j}(j+1), \tag{C3}$$

where ϕ is the relevant angle in $S^1 \subset S^5$ and the two-derivative part U is given by [27]

$$U = U_1 + U_2 + U_2,$$
$$U_1 = \frac{1}{z^2}[(\partial_a x^\mu)^2 - (\partial_a z)^2] - \mathcal{L}_{S^5},$$
$$U_2 = \frac{8}{[z^2 + (x^\mu - a^\mu)^2]^2}[(x^\mu - a^\mu)^2(\partial_a z)^2 - [(x^\mu - a^\mu)\partial_a x^\mu]^2],$$
$$U_3 = \frac{8[(x^\mu - a^\mu)^2 - z^2]}{z[z^2 + (x^\mu - a^\mu)^2]^2}[(x_\nu - a_\nu)\partial_a x_\nu]\partial_a z, \tag{C4}$$

where $\mathcal{L}_{S^5}$ is the S^5 part of the bosonic Lagrangian. Evaluating U on the semiclassical Wilson loop background (4.3) (note that here $\phi = 0$) gives

$$U_1 = \frac{2}{\cosh^2\tau},\quad U_2 = -U_3 = \frac{8}{h^2 + 1}\frac{1}{\cosh^6\tau}(h^2\cosh^2\tau + 1 - \sinh^2\tau). \tag{C5}$$

Thus U_2 and U_3 cancel each other and we end up with

$$\mathcal{C}(W_C, a) = \frac{4\pi\hat{c}_j}{(h^2 + 1)^j}\int_0^\infty d\tau\frac{\tanh^j\tau}{\cosh^2\tau} = \frac{1}{[\ell(a)]^j}\frac{\pi\hat{c}_j}{2^{j-2}(j+1)}. \tag{C6}$$

Using the normalization $\hat{c}_j$ in (C3) we find that the coefficient $C_j(\lambda)$ in (2.16) is given by (2.17).

[1] J. M. Maldacena, Phys. Rev. Lett. **80**, 4859 (1998).
[2] D. J. Gross and H. Ooguri, Phys. Rev. D **58**, 106002 (1998).
[3] D. E. Berenstein, R. Corrado, W. Fischler, and J. M. Maldacena, Phys. Rev. D **59**, 105023 (1999).
[4] N. Drukker, D. J. Gross, and H. Ooguri, Phys. Rev. D **60**, 125006 (1999).
[5] J. K. Erickson, G. W. Semenoff, and K. Zarembo, Nucl. Phys. **B582**, 155 (2000).
[6] N. Drukker and D. J. Gross, J. Math. Phys. (N.Y.) **42**, 2896 (2001).
[7] G. W. Semenoff and K. Zarembo, Nucl. Phys. **B616**, 34 (2001); Nucl. Phys. B, Proc. Suppl. **108**, 106 (2002).
[8] K. Zarembo, Phys. Rev. D **66**, 105021 (2002).
[9] S. Giombi and V. Pestun, J. High Energy Phys. 10 (2010) 033; V. Pestun, J. High Energy Phys. 12 (2012) 067; Commun. Math. Phys. **313**, 71 (2012).
[10] S. Giombi and V. Pestun, arXiv:1207.7083.
[11] B. Fiol, B. Garolera, and A. Lewkowycz, J. High Energy Phys. 05 (2012) 093.
[12] L. F. Alday, B. Eden, G. P. Korchemsky, J. Maldacena, and E. Sokatchev, J. High Energy Phys. 09 (2011) 123.
[13] L. F. Alday, E. I. Buchbinder, and A. A. Tseytlin, J. High Energy Phys. 09 (2011) 034.
[14] T. Adamo, J. High Energy Phys. 12 (2011) 006.
[15] N. Drukker and V. Forini, J. High Energy Phys. 06 (2011) 131.
[16] D. Correa, J. Henn, J. Maldacena, and A. Sever, J. High Energy Phys. 06 (2012) 048.
[17] L. F. Alday, P. Heslop, and J. Sikorowski, arXiv:1207.4316.
[18] J. Gomis, S. Matsuura, T. Okuda, and D. Trancanelli, J. High Energy Phys. 08 (2008) 068.
[19] L. F. Alday and A. A. Tseytlin, J. Phys. A **44**, 395401 (2011).
[20] M. Bianchi, M. B. Green, and S. Kovacs, J. High Energy Phys. 04 (2002) 040.
[21] N. Drukker and S. Kawamoto, Phys. Rev. D **74**, 046002 (2006).
[22] A. Miwa and T. Yoneya, J. High Energy Phys. 12 (2006) 060; M. Sakaguchi and K. Yoshida, Nucl. Phys. **B798**, 72 (2008); N. Drukker and B. Fiol, J. High Energy Phys. 01 (2006) 056; R. Hernandez, Nucl. Phys. **B862**, 751 (2012).
[23] T. Enari and A. Miwa, Phys. Rev. D **86**, 106004 (2012).
[24] R. Roiban and A. A. Tseytlin, Phys. Rev. D **82**, 106011 (2010).
[25] M. S. Costa, R. Monteiro, J. E. Santos, and D. Zoakos, J. High Energy Phys. 11 (2010) 141.
[26] K. Zarembo, J. High Energy Phys. 09 (2010) 030.
[27] E. I. Buchbinder and A. A. Tseytlin, J. High Energy Phys. 02 (2011) 072.