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Nonequilibrium thermodynamical inequivalence of quantum stress-energy and spin tensors

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It is shown that different pairs of stress-energy and spin tensors of quantum relativistic fields related by a pseudo-gauge transformation, i.e., differing by a divergence, imply different mean values of physical quantities in thermodynamical nonequilibrium situations. Most notably, transport coefficients and the total entropy production rate are affected by the choice of the spin tensor of the relativistic quantum field theory under consideration. Therefore, at least in principle, it should be possible to disprove a fundamental stress-energy tensor and/or to show that a fundamental spin tensor exists by means of a dissipative thermodynamical experiment.

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I. INTRODUCTION

In recent years, there has been considerable interest in theoretical relativistic hydrodynamics and its most general form, including dissipative terms [1]. This renewed interest has been mainly triggered by its successful application to the description of the quark gluon plasma dynamical evolution in ultrarelativistic heavy ion collisions [2]. Relativistic hydrodynamics can be seen as the theory describing the dynamical behavior of the mean value of the quantum stress-energy tensor $\hat{T}^{\mu\nu}$, that is, $\mathrm{tr}(\hat{\rho}\hat{T}^{\mu\nu})$. This tensor is generally assumed to be symmetric, although in *special* relativity it does not need to be such if it is accompanied by a nonvanishing rank three tensor, the so-called spin tensor $\hat{\mathcal{S}}^{\lambda,\mu\nu}$. In fact, in special relativistic quantum field theory, starting from particular stress-energy and spin tensors, different pairs can be generated (and are generally related) by means of a pseudo-gauge transformation [3,4], preserving the total energy, momentum and angular momentum:

$$
\begin{aligned}
\hat{T}'^{\mu\nu} &= \hat{T}^{\mu\nu} + \frac{1}{2}\partial_\alpha(\hat{\Phi}^{\alpha,\mu\nu} - \hat{\Phi}^{\mu,\alpha\nu} - \hat{\Phi}^{\nu,\alpha\mu}) \\
\hat{\mathcal{S}}'^{\lambda,\mu\nu} &= \hat{\mathcal{S}}^{\lambda,\mu\nu} - \hat{\Phi}^{\lambda,\mu\nu} + \partial_\alpha \hat{Z}^{\alpha\lambda,\mu\nu},
\end{aligned}
\tag{1}
$$

where $\hat{\Phi}$ is a rank three tensor field that is antisymmetric in the last two indices (often called and henceforth referred to as a *superpotential*), and $\hat{Z}$ is a rank four tensor that is antisymmetric in the pairs $\alpha\lambda$ and $\mu\nu$.

In a previous paper [5] we have shown that indeed different pairs $(\hat{T}, \hat{\mathcal{S}})$ and $(\hat{T}', \hat{\mathcal{S}}')$ are, in general, thermodynamically inequivalent as they imply different mean values of physical quantities for a rotating system at equilibrium. Particularly, for the free Dirac field, we showed that the canonical and Belinfante [obtained from the canonical one by setting $\hat{\Phi} = \hat{\mathcal{S}}$ and $\hat{Z} = 0$ in (1), hence with a vanishing new spin tensor $\hat{\mathcal{S}}'$] quantum stress-energy tensors result in different mean values for the momentum density and the total angular momentum density.

The thermodynamical inequivalence is (at least in our view) surprising because it was commonly believed that the only physical phenomenon which can discriminate between stress-energy tensors of a fundamental quantum field theory related by a transformation like (1) is gravity, or, in other words, the coupling to a metric tensor. In this paper we reinforce our previous finding by showing that the inequivalence extends to nonequilibrium thermodynamical quantities, specifically entropy production and transport coefficients. In summary, we will show that the use of different stress-energy tensors, related by (1), to calculate transport coefficients with the relativistic Kubo formula leads, in general, to different results. Therefore, at least in principle, an extremely accurate measurement of transport coefficients or total entropy in an experiment where dissipation is involved would allow one to *disprove* a candidate stress-energy or spin tensor, with obvious important consequences in relativistic gravitational theories. This finding means, in other words, that the existence of a fundamental spin tensor affects the microscopic number of degrees of freedom, or at least how quickly macroscopic information gets converted into microscopic, namely, entropy generation.

The paper is organized as follows: in Sec. II we will extend the framework of the nonequilibrium density operator introduced by Zubarev [6] to the case of a nonvanishing spin tensor. In Sec. III we will show that the nonequilibrium density operator is not invariant under a pseudo-gauge transformation (1); that is, it does depend on the chosen couple of stress-energy and spin tensors. In Sec. IV we will provide a general formula for the change of mean values of observables, and we will determine how entropy is affected by a pseudo-gauge transformation. In Sec. V we will show that transport coefficients are also modified and, particularly, we will focus on the modification of the Kubo formula for shear viscosity. Finally, in Sec. VI we will discuss the implications of this finding and draw our conclusions.

A. Notation

In this paper we adopt the natural units, with $\hbar = c = K = 1$. The Minkowskian metric tensor is diag(1, −1, −1, −1); for the Levi-Civita symbol we use the convention $\varepsilon^{0123} = 1$. We will use the relativistic notation with repeated indices assumed to be saturated. Operators in Hilbert space will be denoted by an upper hat, e.g., $\hat{\mathsf{R}}$, with the exception of the Dirac field operator which is denoted with a capital Ψ.

II. NONEQUILIBRIUM DENSITY OPERATOR

A suitable formalism to calculate transport coefficients for relativistic quantum fields without going through kinetic theory was developed by Zubarev [6,7], extending to the relativistic domain a formalism already introduced by Kubo [8]. In this approach, a nonequilibrium density operator is introduced which reads [9][1]

$$\hat{\rho} = \frac{1}{Z}\exp[-\hat{\Upsilon}] = \frac{1}{Z}\exp\left[-\lim_{\varepsilon\to 0}\varepsilon\int_{-\infty}^{t'} dt e^{\varepsilon(t-t')} \times \int d^3x(\hat{T}^{0\nu}\beta_\nu(x) - \hat{j}^0\xi(x))\right], \qquad (2)$$

where $\hat{j}$ is a conserved current, the four-vector field β is a point-dependent inverse temperature four-vector ($\beta = u/T_0$, u being a four-velocity field and T_0 the comoving or invariant temperature) and $\xi = \mu_0/T_0$ is a scalar function whose physical meaning is that of a point-dependent ratio between comoving chemical potential μ_0 and comoving temperature T_0; the Z factor is analogous to a partition function, i.e., a normalization factor to have $\mathrm{tr}\hat{\rho} = 1$. The operators in the exponential of Eq. (2) are in the Heisenberg representation. It should be stressed that in the formula (2) covariance is broken from the very beginning by the choice of a specific inertial frame and its time. However, it can be shown that the operator $\hat{\rho}$ is, in fact, time independent [9], namely, independent of t', so that $\hat{\rho}$ is a good density operator in the Heisenberg representation.

In the formula (2) the possible contribution of a spin tensor is simply disregarded; therefore, the formula is correct only if the stress-energy tensor is the symmetrized Belinfante one (or improved ones; see the last section), whose associated spin tensor is vanishing. It is the aim of this section to find the appropriate extension of the formula (2) with a spin tensor.

Using the identity

$$e^{\varepsilon(t-t')}(\hat{T}^{0\nu}\beta_\nu(x) - \hat{j}^0\xi(x)) = \left(\frac{\partial}{\partial x^\mu}\frac{e^{\varepsilon(t-t')}}{\varepsilon}\right)(\hat{T}^{\mu\nu}\beta_\nu(x) - \hat{j}^\mu\xi(x)),$$

integrating by parts and taking into account the continuity equations $\partial_\mu\hat{T}^{\mu\nu} = \partial_\mu\hat{j}^\mu = 0$, the operator $\hat{\Upsilon}$ in Eq. (2) can be rewritten as

$$\hat{\Upsilon} = \int d^3x(\hat{T}^{0\nu}\beta_\nu(t',\mathbf{x}) - \hat{j}^0\xi(t',\mathbf{x})) + \lim_{\varepsilon\to 0}\int_{-\infty}^{t'} dt e^{\varepsilon(t-t')}\int dS n_i(\hat{T}^{i\nu}\beta_\nu(x) - \hat{j}^i\xi(x)) - \lim_{\varepsilon\to 0}\int_{-\infty}^{t'} dt e^{\varepsilon(t-t')}\int d^3x(\hat{T}^{\mu\nu}\partial_\mu\beta_\nu(x) - \hat{j}^\mu\partial_\mu\xi(x)). \qquad (3)$$

The first term is the so-called local thermodynamical equilibrium one, which is defined by the same formula of the global equilibrium [10,11] with x-dependent four-temperature and chemical potentials, whereas the term dependent on their derivatives is interpreted as a perturbation.

At equilibrium, the right-hand side should reduce to the known form, which, at least for the most familiar form of thermodynamical equilibrium with $\beta^{\rm eq} = (1/T, \mathbf{0}) = $ const and $\xi^{\rm eq} = \mu/T = $ const, is readily recognized in the first term, setting $\beta = \beta^{\rm eq}$ and $\xi = \xi^{\rm eq}$:

$$\begin{aligned}\hat{\Upsilon}^{\rm eq} &= \int d^3x(\hat{T}^{0\nu}\beta^{\rm eq}_\nu - \hat{j}^0\xi^{\rm eq}) + \lim_{\varepsilon\to 0}\int_{-\infty}^{t'} dt e^{\varepsilon(t-t')}\int dS n_i(\hat{T}^{i\nu}\beta^{\rm eq}_\nu - \hat{j}^i\xi^{\rm eq}) - \lim_{\varepsilon\to 0}\int_{-\infty}^{t'} dt e^{\varepsilon(t-t')}\int d^3x(\hat{T}^{\mu\nu}\partial_\mu\beta^{\rm eq}_\nu - \hat{j}^\mu\partial_\mu\xi^{\rm eq}) \\ &= \hat{H}/T - \mu\hat{Q}/T + \lim_{\varepsilon\to 0}\int_{-\infty}^{t'} dt e^{\varepsilon(t-t')}\int dS n_i(\hat{T}^{i\nu}\beta^{\rm eq}_\nu - \hat{j}^i\xi^{\rm eq}) - \lim_{\varepsilon\to 0}\int_{-\infty}^{t'} dt e^{\varepsilon(t-t')}\int d^3x(\hat{T}^{\mu\nu}\partial_\mu\beta^{\rm eq}_\nu - \hat{j}^\mu\partial_\mu\xi^{\rm eq}).\end{aligned} \qquad (4)$$

Hence, the two rightmost terms of (4) must vanish at equilibrium. Indeed, the surface term is supposed to vanish through a suitable choice of the field boundary conditions, while the third term vanishes in view of the constancy of $\beta^{\rm eq}$ and $\xi^{\rm eq}$. However, this is not the case for the most general form of equilibrium; in the most general form (see discussion in Ref. [11]), while the scalar $\xi^{\rm eq}$ stays constant, the four-vector β fulfills a Killing equation, whose solution is [12]

$$\beta^{\rm eq}_\nu(x) = b^{\rm eq}_\nu + \omega^{\rm eq}_{\nu\mu}x^\mu \qquad (5)$$

with both the four-vector $b^{\rm eq}$ and the antisymmetric tensor $\omega^{\rm eq}$ constant. Therefore,

$$\partial_\mu\beta^{\rm eq}_\nu = -\omega^{\rm eq}_{\mu\nu}$$

which, in general, is nonvanishing, so the third term on the right-hand side of Eq. (4) survives. For instance, for the

[1]Throughout the paper, the four-vector x implies the time t and position vector $\mathbf{x}$, i.e., $x = (t, \mathbf{x})$. The dependence of the stress-energy and spin tensor on x will always be understood.

thermodynamical equilibrium with rotation [11], the tensor ω turns out to be

$$\omega^{\rm eq}_{\lambda\nu} = \omega/T(\delta^1_\lambda \delta^2_\nu - \delta^2_\lambda \delta^1_\nu), \tag{6}$$

ω being the angular velocity and T the temperature measured by the inertial frame.

In order to find the appropriate generalization of the operator $\hat{\Upsilon}$, let us plug the formula (5) of general thermodynamical equilibrium into (4):

$$\begin{aligned}\hat{\Upsilon}^{\rm eq} = &\int {\rm d}^3{\rm x}(\hat{T}^{0\nu}\beta^{\rm eq}_\nu - \hat{j}^0\xi^{\rm eq}))\\ &+ \lim_{\varepsilon\to 0}\int_{-\infty}^{t'} {\rm d}t{\rm e}^{\varepsilon(t-t')}\int {\rm d}Sn_i(\hat{T}^{i\nu}(b^{\rm eq}_\nu + \omega^{\rm eq}_{\nu\mu}x^\mu)\\ &- \hat{j}^i\xi^{\rm eq})\\ &+ \lim_{\varepsilon\to 0}\int_{-\infty}^{t'} {\rm d}t{\rm e}^{\varepsilon(t-t')}\int {\rm d}^3{\rm x}\hat{T}^{\mu\nu}\omega^{\rm eq}_{\mu\nu},\end{aligned} \tag{7}$$

where $\partial_\mu \xi^{\rm eq} = 0$ has been taken into account. For a symmetric stress-energy tensor $\hat{T}$, the last term vanishes, but if a spin tensor is present $\hat{T}$ may have an antisymmetric part. Particularly, from the angular momentum continuity equation,

$$\hat{T}^{\mu\nu}\omega^{\rm eq}_{\mu\nu} = \frac{1}{2}(\hat{T}^{\mu\nu} - \hat{T}^{\nu\mu})\omega^{\rm eq}_{\mu\nu} = -\frac{1}{2}\partial_\lambda \hat{\mathcal{S}}^{\lambda,\mu\nu}\omega^{\rm eq}_{\mu\nu}, \tag{8}$$

so that the last term on the right-hand side of Eq. (7) can be rewritten as

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_{-\infty}^{t'} {\rm d}t{\rm e}^{\varepsilon(t-t')}\int {\rm d}^3{\rm x}\hat{T}^{\mu\nu}\omega^{\rm eq}_{\mu\nu}\\ &= -\frac{1}{2}\omega^{\rm eq}_{\mu\nu}\lim_{\varepsilon\to 0}\int_{-\infty}^{t'} {\rm d}t{\rm e}^{\varepsilon(t-t')}\int {\rm d}^3{\rm x}\partial_\lambda \hat{\mathcal{S}}^{\lambda,\mu\nu}\\ &= -\frac{1}{2}\omega^{\rm eq}_{\mu\nu}\lim_{\varepsilon\to 0}\int {\rm d}^3{\rm x}\int_{-\infty}^{t'} {\rm d}t{\rm e}^{\varepsilon(t-t')}\frac{\partial}{\partial t}\hat{\mathcal{S}}^{0,\mu\nu}\\ &\quad -\frac{1}{2}\omega^{\rm eq}_{\mu\nu}\lim_{\varepsilon\to 0}\int_{-\infty}^{t'} {\rm d}t{\rm e}^{\varepsilon(t-t')}\int {\rm d}Sn_i\hat{\mathcal{S}}^{i,\mu\nu}.\end{aligned} \tag{9}$$

The first term on the right-hand side of (9) can be integrated by parts, yielding

$$\begin{aligned}&-\frac{1}{2}\omega^{\rm eq}_{\mu\nu}\lim_{\varepsilon\to 0}\int {\rm d}^3{\rm x}\int_{-\infty}^{t'} {\rm d}t{\rm e}^{\varepsilon(t-t')}\frac{\partial}{\partial t}\hat{\mathcal{S}}^{0,\mu\nu}\\ &= -\frac{1}{2}\omega^{\rm eq}_{\mu\nu}\int {\rm d}^3{\rm x}\hat{\mathcal{S}}^{0,\mu\nu}(t',\mathbf{x})\\ &\quad +\frac{1}{2}\omega^{\rm eq}_{\mu\nu}\lim_{\varepsilon\to 0}\varepsilon\int_{-\infty}^{t'} {\rm d}t{\rm e}^{\varepsilon(t-t')}\int {\rm d}^3{\rm x}\hat{\mathcal{S}}^{0,\mu\nu}(x).\end{aligned} \tag{10}$$

Plugging Eq. (10) into (9), and this in turn into (7), we obtain

$$\begin{aligned}\hat{\Upsilon}^{\rm eq} = &\int {\rm d}^3{\rm x}\left(\hat{T}^{0\nu}\beta^{\rm eq}_\nu - \hat{j}^0\xi^{\rm eq} - \frac{1}{2}\omega^{\rm eq}_{\mu\nu}\hat{\mathcal{S}}^{0,\mu\nu}\right)\\ &+ \lim_{\varepsilon\to 0}\int_{-\infty}^{t'} {\rm d}t{\rm e}^{\varepsilon(t-t')}\bigg[b^{\rm eq}_\nu\int {\rm d}Sn_i\hat{T}^{i\nu}\\ &- \xi^{\rm eq}\int {\rm d}Sn_i\hat{j}^i - \frac{1}{2}\omega^{\rm eq}_{\mu\nu}\int {\rm d}Sn_i(x^\mu\hat{T}^{i\nu}\\ &- x^\nu\hat{T}^{\mu i} + \hat{\mathcal{S}}^{i,\mu\nu})\bigg]\\ &+ \frac{1}{2}\omega^{\rm eq}_{\mu\nu}\lim_{\varepsilon\to 0}\varepsilon\int_{-\infty}^{t'} {\rm d}t{\rm e}^{\varepsilon(t-t')}\int {\rm d}^3{\rm x}\hat{\mathcal{S}}^{0,\mu\nu}(x),\end{aligned} \tag{11}$$

where the surface term involving $\hat{T}$ in Eq. (7) has been rearranged, taking advantage of the antisymmetry of the ω tensor. The surface terms in the above equations are now manifestly the total momentum flux, the charge flux and the *total* angular momentum flux through the boundary. All of these terms are supposed to vanish at thermodynamical equilibrium through suitable conditions enforced on the field operators at the boundary, so (11) reduces to

$$\begin{aligned}\hat{\Upsilon}^{\rm eq} = &\int {\rm d}^3{\rm x}\left(\hat{T}^{0\nu}\beta^{\rm eq}_\nu - \hat{j}^0\xi^{\rm eq} - \frac{1}{2}\omega^{\rm eq}_{\mu\nu}\hat{\mathcal{S}}^{0,\mu\nu}\right)\\ &+ \frac{1}{2}\omega^{\rm eq}_{\mu\nu}\lim_{\varepsilon\to 0}\varepsilon\int_{-\infty}^{t'} {\rm d}t{\rm e}^{\varepsilon(t-t')}\int {\rm d}^3{\rm x}\hat{\mathcal{S}}^{0,\mu\nu}(x).\end{aligned} \tag{12}$$

The first term on the right-hand side just gives rise to the desired form of the equilibrium operator. For instance, for a rotating system with ω as in Eq. (6), one has [11]

$$\begin{aligned}&\int {\rm d}^3{\rm x}\left(\hat{T}^{0\nu}\beta^{\rm eq}_\nu - \hat{j}^0\xi^{\rm eq} - \frac{1}{2}\omega^{\rm eq}_{\mu\nu}\hat{\mathcal{S}}^{0,\mu\nu}\right)\\ &= \hat{H}/T - \mu\hat{Q}/T - \omega\hat{J}/T,\end{aligned}$$

$\hat{J}$ being the total angular momentum, which is the known form [13]. Nevertheless, the second term in Eq. (12) does not vanish and, thus, must be subtracted away with a suitable modification of the definition of the $\hat{\Upsilon}$ operator. The form of the unwanted term demands the following modification of (2):

$$\begin{aligned}\hat{\rho} &= \frac{1}{Z}\exp[-\hat{\Upsilon}]\\ &= \frac{1}{Z}\exp\bigg[-\lim_{\varepsilon\to 0}\varepsilon\int_{-\infty}^{t'} {\rm d}t{\rm e}^{\varepsilon(t-t')}\int {\rm d}^3{\rm x}\bigg(\hat{T}^{0\nu}\beta_\nu(x) - \hat{j}^0\xi(x)\\ &\quad -\frac{1}{2}\hat{\mathcal{S}}^{0,\mu\nu}\omega_{\mu\nu}(x)\bigg)\bigg],\end{aligned} \tag{13}$$

where $\omega_{\mu\nu}(x)$ is an antisymmetric tensor field which must reduce to the constant $\omega^{\rm eq}_{\mu\nu}$ tensor at equilibrium. It is easy to check, by tracing the previous calculations, that the equilibrium form of $\hat{\Upsilon}$ reduces to the desired form:

$$\hat{\Upsilon}^{\rm eq} = \int {\rm d}^3{\rm x}\left(\hat{T}^{0\nu}\beta^{\rm eq}_\nu - \hat{j}^0\xi^{\rm eq} - \frac{1}{2}\omega^{\rm eq}_{\mu\nu}\hat{\mathcal{S}}^{0,\mu\nu}\right),$$

as the spin tensor term in Eq. (12) cancels out. Therefore, the operator (13) is the only possible extension of the nonequilibrium density operator with a spin tensor.

The new operator $\hat{\Upsilon}$ can be worked out the same way as we have done when obtaining Eq. (3) from Eq. (2):

$$\begin{aligned}\hat{\Upsilon} = &\int \mathrm{d}^3\mathrm{x}\Big(\hat{T}^{0\nu}\beta_\nu(t',\mathbf{x}) - \hat{j}^0\xi(t',\mathbf{x}) - \frac{1}{2}\hat{\mathcal{S}}^{0,\mu\nu}\omega_{\mu\nu}(t',\mathbf{x})\Big)\\ &+ \lim_{\varepsilon\to 0}\int_{-\infty}^{t'} \mathrm{d}t e^{\varepsilon(t-t')}\int \mathrm{d}S n_i\Big(\hat{T}^{i\nu}\beta_\nu(x) - \hat{j}^i\xi(x) - \frac{1}{2}\hat{\mathcal{S}}^{i,\mu\nu}\omega_{\mu\nu}(x)\Big)\\ &-\frac{1}{2}\lim_{\varepsilon\to 0}\int_{-\infty}^{t'} \mathrm{d}t e^{\varepsilon(t-t')}\int \mathrm{d}^3\mathrm{x}(\hat{T}_S^{\mu\nu}(\partial_\mu\beta_\nu(x) + \partial_\mu\beta_\nu(x)) + \hat{T}_A^{\mu\nu}(\partial_\mu\beta_\nu(x) - \partial_\mu\beta_\nu(x) + 2\omega_{\mu\nu}(x))\\ &-\hat{\mathcal{S}}^{\lambda,\mu\nu}\partial_\lambda\omega_{\mu\nu}(x) - 2\hat{j}^\mu\partial_\mu\xi(x)),\end{aligned} \tag{14}$$

where

$$\hat{T}_S^{\mu\nu} = \frac{1}{2}(\hat{T}^{\mu\nu} + \hat{T}^{\nu\mu}) \qquad \hat{T}_A^{\mu\nu} = \frac{1}{2}(\hat{T}^{\mu\nu} - \hat{T}^{\nu\mu})$$

and the continuity equation for angular momentum has been used. The first term on the right-hand side is the new local thermodynamical term, while the third term can be further expanded to derive the relativistic Kubo formula of transport coefficients (see Appendix A).

III. NONEQUILIBRIUM DENSITY OPERATOR AND PSEUDO-GAUGE TRANSFORMATIONS

A natural requirement for the density operator (13) would be its independence of the particular couple of stress-energy and spin tensors, because one would like the mean value of any observable $\hat{O}$,

$$O \equiv \mathrm{tr}(\hat{\rho}\,\hat{O}),$$

to be an objective one.[2] In Ref. [5] we showed that, even at thermodynamical equilibrium with rotation, this is not the case for the components of the stress-energy and spin tensors themselves because they change through the pseudo-gauge transformation (1). However, at equilibrium, $\hat{\rho}$ itself is a function of just integral quantities (total energy, angular momentum, charge) which are invariant under a transformation (1) provided that boundary fluxes vanish, so a specific operator $\hat{O}$, including the components of a *specific* stress-energy tensor, does not change under (1). However, it is not obvious that this feature persists in a nonequilibrium case; in fact, we are going to show that, in general, this is not the case.

Let us consider the operator $\hat{\Upsilon}$ in (13) and how it gets changed under a pseudo-gauge transformation (1) with $\hat{Z} = 0$. The new operator $\hat{\Upsilon}'$ reads

$$\hat{\Upsilon}' = \hat{\Upsilon} + \frac{1}{2}\lim_{\varepsilon\to 0}\varepsilon\int_{-\infty}^{t'} \mathrm{d}t e^{\varepsilon(t-t')} \times \int \mathrm{d}^3\mathrm{x}(\partial_\lambda\hat{\varphi}^{\lambda 0,\nu}\beta_\nu(x) + \hat{\Phi}^{0,\mu\nu}\omega_{\mu\nu}(x)), \tag{15}$$

where

$$\hat{\varphi}^{\lambda\mu,\nu} = \hat{\Phi}^{\lambda,\mu\nu} - \hat{\Phi}^{\mu,\lambda\nu} - \hat{\Phi}^{\nu,\lambda\mu} \tag{16}$$

is antisymmetric in the first two indices. We can rewrite Eq. (15) as

$$\begin{aligned}\hat{\Upsilon}' - \hat{\Upsilon} &= \frac{1}{2}\lim_{\varepsilon\to 0}\varepsilon\int_{-\infty}^{t'} \mathrm{d}t\int \mathrm{d}^3\mathrm{x} e^{\varepsilon(t-t')}[\partial_\lambda(\hat{\varphi}^{\lambda 0,\nu}\beta_\nu(x))\\ &\quad - \hat{\varphi}^{\lambda 0,\nu}\partial_\lambda\beta_\nu + \hat{\Phi}^{0,\mu\nu}\omega_{\mu\nu}(x)]\\ &= \frac{1}{2}\lim_{\varepsilon\to 0}\varepsilon\int_{-\infty}^{t'} \mathrm{d}t e^{\varepsilon(t-t')}\Big[\int \mathrm{d}S n_i\hat{\varphi}^{i0,\nu}\beta_\nu(x)\\ &\quad - \int \mathrm{d}^3\mathrm{x}(\hat{\varphi}^{\lambda 0,\nu}\partial_\lambda\beta_\nu - \hat{\Phi}^{0,\mu\nu}\omega_{\mu\nu}(x))\Big]\end{aligned} \tag{17}$$

after integration by parts. Let us now write the general fields β and ω as the sum of the equilibrium values and a perturbation, that is,

$$\beta(x) = \beta^{\mathrm{eq}}(x) + \delta\beta(x) \qquad \omega(x) = \omega^{\mathrm{eq}} + \delta\omega(x), \tag{18}$$

and first work out the equilibrium part off the right-hand side of Eq. (17). As $\partial_\lambda\beta_\nu^{\mathrm{eq}} = -\omega_{\lambda\nu}^{\mathrm{eq}}$ one has

$$\begin{aligned}(\hat{\Upsilon}' - \hat{\Upsilon})|_{\mathrm{eq}} &= \frac{1}{2}\lim_{\varepsilon\to 0}\varepsilon\int_{-\infty}^{t'} \mathrm{d}t e^{\varepsilon(t-t')}\Big[\int \mathrm{d}S n_i\hat{\varphi}^{i0,\nu}\beta_\nu^{\mathrm{eq}}(x)\\ &\quad + \int \mathrm{d}^3\mathrm{x}(\hat{\varphi}^{\lambda 0,\nu}\omega_{\lambda\nu}^{\mathrm{eq}} + \hat{\Phi}^{0,\mu\nu}\omega_{\mu\nu}^{\mathrm{eq}})\Big]\\ &= \frac{1}{2}\lim_{\varepsilon\to 0}\varepsilon\int_{-\infty}^{t'} \mathrm{d}t e^{\varepsilon(t-t')}\Big[\int \mathrm{d}S n_i\hat{\varphi}^{i0,\nu}\beta_\nu^{\mathrm{eq}}(x)\\ &\quad + \int \mathrm{d}^3\mathrm{x}(\hat{\Phi}^{\lambda,0\nu}\omega_{\lambda\nu}^{\mathrm{eq}} - \hat{\Phi}^{0,\lambda\nu}\omega_{\lambda\nu}^{\mathrm{eq}}\\ &\quad - \hat{\Phi}^{\nu,\lambda 0}\omega_{\lambda\nu}^{\mathrm{eq}} + \hat{\Phi}^{0,\mu\nu}\omega_{\mu\nu}^{\mathrm{eq}})\Big]\\ &= \frac{1}{2}\lim_{\varepsilon\to 0}\varepsilon\int_{-\infty}^{t'} \mathrm{d}t e^{\varepsilon(t-t')}\int \mathrm{d}S n_i\hat{\varphi}^{i0,\nu}\beta_\nu^{\mathrm{eq}}(x),\end{aligned} \tag{19}$$

[2] It should be pointed out that the mean values of operators involving quantum relativistic fields are generally divergent (e.g., T^{00} for a free field has an infinite zero point value). To remove the infinities, the mean values must be renormalized, which can be simply done for free fields by using normal ordering in all expressions, including the density operator itself. Henceforth, it will be understood that all the mean values of operators are the renormalized ones.

where we have used Eq. (16) and the antisymmetry of indices of the superpotential $\hat{\Phi}$. By using Eq. (5), the last expression can be rewritten as

$$\lim_{\varepsilon\to 0}\varepsilon \int_{-\infty}^{t'} \mathrm{d}t \mathrm{e}^{\varepsilon(t-t')}\Big[b_\nu^{\rm eq} \int \mathrm{d}S n_i \hat{\varphi}^{i0,\nu} + \frac{1}{2}\omega_{\nu\mu}^{\rm eq} \int \mathrm{d}S n_i (x^\mu \hat{\varphi}^{i0,\nu} - x^\nu \hat{\varphi}^{i0,\mu})\Big].$$

The two surface integrals above are the additional four-momentum and the additional *total* angular momentum, in the operator sense, after having made a pseudo-gauge transformation (1) of the stress-energy and spin tensors. If the boundary conditions ensure that the momentum and total angular momentum fluxes vanish (in order to have conserved energy and momentum operators) for any couple $(\hat{T}, \hat{\mathcal{S}})$ of tensors, then the two fluxes in the above equations must vanish as well. Therefore, we can conclude that

$$\hat{\Upsilon}'|_{\rm eq} = \hat{\Upsilon}|_{\rm eq}.$$

Now, let us focus on the nonequilibrium perturbation of the $\hat{\Upsilon}$ operator.

$$\begin{aligned}(\hat{\Upsilon}' - \hat{\Upsilon})|_{\rm non-eq} &= \frac{1}{2}\lim_{\varepsilon\to 0}\varepsilon \int_{-\infty}^{t'} \mathrm{d}t \mathrm{e}^{\varepsilon(t-t')}\Big[\int \mathrm{d}S n_i \hat{\varphi}^{i0,\nu}\delta\beta_\nu \\ &\quad - \int \mathrm{d}^3\mathrm{x}\,\hat{\varphi}^{\lambda 0,\nu}\partial_\lambda \delta\beta_\nu - \hat{\Phi}^{0,\mu\nu}\delta\omega_{\mu\nu}\Big] \\ &= \frac{1}{2}\lim_{\varepsilon\to 0}\varepsilon \int_{-\infty}^{t'} \mathrm{d}t \mathrm{e}^{\varepsilon(t-t')}\Big[\int \mathrm{d}S n_i \hat{\varphi}^{i0,\nu}\delta\beta_\nu \\ &\quad - \int \mathrm{d}^3\mathrm{x}(\hat{\Phi}^{\lambda,0\nu} - \hat{\Phi}^{0,\lambda\nu} - \hat{\Phi}^{\nu,\lambda 0})\partial_\lambda\delta\beta_\nu - \hat{\Phi}^{0,\mu\nu}\delta\omega_{\mu\nu}\Big] \\ &= \frac{1}{2}\lim_{\varepsilon\to 0}\varepsilon \int_{-\infty}^{t'} \mathrm{d}t \mathrm{e}^{\varepsilon(t-t')}\Big[\int \mathrm{d}S n_i \hat{\varphi}^{i0,\nu}\delta\beta_\nu \\ &\quad - \int \mathrm{d}^3\mathrm{x}\hat{\Phi}^{\lambda,0\nu}(\partial_\lambda\delta\beta_\nu + \partial_\nu\delta\beta_\lambda) \\ &\quad - \hat{\Phi}^{0,\lambda\nu}\Big(\frac{1}{2}(\partial_\lambda\delta\beta_\nu - \partial_\nu\delta\beta_\lambda) + \delta\omega_{\lambda\nu}\Big)\Big], \end{aligned} \tag{20}$$

where the dependence of $\delta\beta$ and $\delta\omega$ on x is now understood. It can be seen that it is impossible to make this difference vanishing in general. One can get rid of the surface term by choosing a perturbation which vanishes at the boundary and the last term by locking the perturbation of the tensor ω to that of the inverse temperature four-vector:

$$\delta\omega_{\lambda\nu}(x) = -\frac{1}{2}(\partial_\lambda\delta\beta_\nu(x) - \partial_\nu\delta\beta_\lambda(x)), \tag{21}$$

but it is impossible to cancel out the term

$$\delta\hat{\Upsilon} \equiv -\frac{1}{2}\lim_{\varepsilon\to 0}\varepsilon\int_{-\infty}^{t'} \mathrm{d}t \mathrm{e}^{\varepsilon(t-t')} \times \int \mathrm{d}^3\mathrm{x}\hat{\Phi}^{\lambda,0\nu}(\partial_\lambda\delta\beta_\nu(x) + \partial_\nu\delta\beta_\lambda(x)) \tag{22}$$

except in special cases, e.g., when the tensor $\hat{\Phi}$ is also antisymmetric in the first two indices.

We have thus come to the conclusion that the nonequilibrium density operator does depend, in general, on the particular choice of stress-energy and spin tensors of the quantum field theory under consideration. Therefore, the mean value of any observable in a nonequilibrium situation shall depend on that choice. It is worth stressing that this is a much deeper dependence on the stress-energy and spin tensors than what we showed in Ref. [5] for thermodynamical equilibrium with rotation. Therein, mean values of the angular momentum densities and momentum densities were found to be dependent on the pseudo-gauge transformation (1) because the relevant quantum operators could be varied, but not because the density operator $\hat{\rho}$ was dependent thereupon. In fact, at nonequilibrium, even $\hat{\rho}$ varies under a transformation (1). Note that, in principle, even the mean values of the total energy and momentum could be dependent on the quantum stress-energy tensor choice, although boundary conditions ensure, as we have assumed, that the total energy and momentum *operators* are invariant under a transformation (1). Again, this comes about because the density operator is not invariant under (1), in the following formula:

$$\mathrm{tr}(\hat{\rho}'\hat{P}'^\mu) = \mathrm{tr}(\hat{\rho}'\hat{P}^\mu) \neq \mathrm{tr}(\hat{\rho}\hat{P}^\mu).$$

It must be pointed out that the variation of the Zubarev nonequilibrium density operator (22) depends on the gradients of the four-temperature field, and it is thus a small one close to thermodynamical equilibrium. In the next section we will show in more detail how the mean values of observables change under a small change of the nonequilibrium density operator, or, in other words, when the system is close to thermodynamical equilibrium.

IV. VARIATION OF MEAN VALUES AND LINEAR RESPONSE

We will first study the general dependence of the mean value of an observable $\hat{O}$ on the spin tensor by denoting by $\delta\hat{\Upsilon}$ the supposedly small variation, under a transformation (1), of the operator $\hat{\Upsilon}$. This can be either the one in Eq. (22) or the more general one (only bulk terms) in Eq. (20). We have

$$\mathrm{tr}(\hat{\rho}'\hat{O}) = \frac{1}{Z'}\mathrm{tr}(\exp[-\hat{\Upsilon}']\hat{O}) = \frac{1}{Z'}\mathrm{tr}(\exp[-\hat{\Upsilon} - \delta\hat{\Upsilon}]\hat{O}), \tag{23}$$

with $Z' = \mathrm{tr}(\exp[-\hat{\Upsilon} - \delta\hat{\Upsilon}])$. We can expand in $\delta\hat{\Upsilon}$ at the first order (Zassenhaus formula):

$$Z' \simeq Z - \mathrm{tr}(\exp[-\hat{\Upsilon}]\delta\hat{\Upsilon})$$

$$\mathrm{tr}(\exp[-\hat{\Upsilon} - \delta\hat{\Upsilon}]\hat{O}) \simeq \mathrm{tr}(\exp[-\hat{\Upsilon}](I - \delta\hat{\Upsilon} + \frac{1}{2}[\hat{\Upsilon}, \delta\hat{\Upsilon}] - \frac{1}{6}[\hat{\Upsilon}, [\hat{\Upsilon}, \delta\hat{\Upsilon}]] + \cdots)\hat{O}); \tag{24}$$

hence, with $\langle\rangle = \mathrm{tr}(\hat\rho)$, at the first order in $\delta\hat\Upsilon$,

$$\begin{aligned}\mathrm{tr}(\hat\rho'\hat O) &\equiv \langle\hat O\rangle' \\ &\simeq \langle\hat O\rangle(1+\langle\delta\hat\Upsilon\rangle) - \langle\hat O\delta\hat\Upsilon\rangle + \frac{1}{2}\langle[\hat\Upsilon,\delta\hat\Upsilon]\hat O\rangle \\ &\quad - \frac{1}{6}\langle[\hat\Upsilon,[\hat\Upsilon,\delta\hat\Upsilon]]\hat O\rangle + \cdots,\end{aligned}$$

which makes the dependence of the mean value on the choice of the superpotential $\hat\Phi$ manifest.

As has been mentioned, close to thermodynamical equilibrium, the operator $\delta\hat\Upsilon$ is "small" and one can write an expansion of the mean value of the observable $\hat O$ in the gradients of the four-temperature field, according to relativistic linear response theory [9]. This method, just based on Zubarev's nonequilibrium density operator method, allows one to calculate the variation between the actual mean value of an operator and its value at local thermodynamical equilibrium for small deviations from it. In fact, it can be seen from Eq. (22) that the operator $\delta\hat\Upsilon$, from the linear response theory viewpoint, is an additional perturbation in the derivative of the four-temperature field, and therefore the difference between actual mean values at first order turns out be (see Appendix A for reference)

$$\begin{aligned}\Delta\langle\hat O\rangle \simeq -\lim_{\varepsilon\to 0}\frac{T}{2i}\int_{-\infty}^{t'}\mathrm{d}t e^{\varepsilon(t-t')}\int \mathrm{d}^3\mathrm{x}\langle[\hat\Phi^{\lambda,0\nu}(x),\hat O]\rangle_0 \\ \times(\partial_\lambda\delta\beta_\nu(x)+\partial_\nu\delta\beta_\lambda(x)),\end{aligned} \quad (25)$$

where $\langle\ldots\rangle_0$ stands for the expectation value calculated with the equilibrium density operator, that is,

$$\hat\rho_0 = \frac{1}{Z_0}\exp[-\hat H/T + \mu\hat Q/T]. \quad (26)$$

Since $\mathrm{tr}(\hat\rho_0[\hat\Phi^{\lambda,0\nu},\hat O]) = \mathrm{tr}(\hat\Phi^{\lambda,0\nu}[\hat O,\hat\rho_0])$ the right-hand side of (25) vanishes for all quantities commutating with the equilibrium density operator, notably total energy, momentum and angular momentum. Nevertheless, in principle, even the mean values of the conserved quantities are affected by the choice of a specific quantum stress-energy tensor, though at the second order in the perturbation $\delta\beta$.

We now set out to study the effect of the transformation (1) on the total entropy. In a nonequilibrium situation, entropy is usually defined as [13] the quantity maximizing $-\mathrm{tr}(\hat\rho\log\hat\rho)$ with the constraints of fixed mean conserved densities. The solution $\hat\rho_{\mathrm{LE}}$ of this problem is the local thermodynamical equilibrium operator, namely,

$$\hat\rho_{\mathrm{LE}}(t) = \frac{\exp[-\int\mathrm{d}^3\mathrm{x}(\hat T^{0\nu}\beta_\nu(x) - \hat j^0\xi(x) - \frac{1}{2}\hat{\mathcal S}^{0,\mu\nu}\omega_{\mu\nu}(x))]}{\mathrm{tr}(\exp[-\int\mathrm{d}^3\mathrm{x}(\hat T^{0\nu}\beta_\nu(x) - \hat j^0\xi(x) - \frac{1}{2}\hat{\mathcal S}^{0,\mu\nu}\omega_{\mu\nu}(x))])}, \quad (27)$$

which—as emphasized in the above equation—is explicitly dependent on time, unlike the Zubarev stationary nonequilibrium density operator (13); of course, the time dependence is crucial to make the entropy,

$$S = -\mathrm{tr}(\hat\rho_{\mathrm{LE}}\log\hat\rho_{\mathrm{LE}}), \quad (28)$$

increasing in a nonequilibrium situation. In order to study the effect of the transformation (1) on the entropy it is convenient to define

$$\hat\Upsilon_{\mathrm{LE}} = \int\mathrm{d}^3\mathrm{x}\left(\hat T^{0\nu}\beta_\nu(x) - \hat j^0\xi(x) - \frac{1}{2}\hat{\mathcal S}^{0,\mu\nu}\omega_{\mu\nu}(x)\right), \quad (29)$$

for which it can be shown that, with calculations similar to those in the previous section, the variation induced by the transformation (1) is

$$\begin{aligned}\delta\hat\Upsilon_{\mathrm{LE}} = \frac{1}{2}\Bigg\{\int \mathrm{d}Sn_i\hat\varphi^{i0,\nu}\delta\beta_\nu \\ -\int\mathrm{d}^3\mathrm{x}\Bigg[\hat\Phi^{\lambda,0\nu}(\partial_\lambda\delta\beta_\nu+\partial_\nu\delta\beta_\lambda) \\ -\hat\Phi^{0,\lambda\nu}\left(\frac{1}{2}(\partial_\lambda\delta\beta_\nu-\partial_\nu\delta\beta_\lambda)+\delta\omega_{\lambda\nu}\right)\Bigg]\Bigg\}.\end{aligned} \quad (30)$$

As has been mentioned, it is possible to get rid of the surface and the last term in the right hand side of above equation through a suitable choice of the perturbations, but not of the second term.

Since $\delta\hat\Upsilon_{\mathrm{LE}}$ is a small term compared to $\hat\Upsilon_{\mathrm{LE}}$ we can determine the variation of the entropy (28) with an expansion in $\delta\hat\Upsilon_{\mathrm{LE}}$ at first order. First, we observe that [see also Eq. (24)]

$$\begin{aligned}Z'_{\mathrm{LE}} &\equiv \mathrm{tr}(\exp[-\hat\Upsilon_{\mathrm{LE}}-\delta\hat\Upsilon_{\mathrm{LE}}]) \\ &\simeq \mathrm{tr}(\exp[-\hat\Upsilon_{\mathrm{LE}}](I-\delta\hat\Upsilon_{\mathrm{LE}})) = Z_{\mathrm{LE}}(1-\langle\delta\hat\Upsilon_{\mathrm{LE}}\rangle_{\hat\Upsilon}),\end{aligned}$$

where $\langle\rangle_{\hat\Upsilon}$ stands for the averaging with the original $\hat\Upsilon_{\mathrm{LE}}$ local equilibrium operator. Hence, the new entropy reads

$$\begin{aligned}S' &= \frac{1}{Z'_{\mathrm{LE}}}\mathrm{tr}(\exp[-\hat\Upsilon_{\mathrm{LE}}-\delta\hat\Upsilon_{\mathrm{LE}}](\hat\Upsilon_{\mathrm{LE}}+\delta\hat\Upsilon_{\mathrm{LE}})) + \log Z'_{\mathrm{LE}} \\ &\simeq \frac{1}{Z_{\mathrm{LE}}}(1+\langle\delta\hat\Upsilon_{\mathrm{LE}}\rangle_{\hat\Upsilon})\mathrm{tr}(\exp[-\hat\Upsilon_{\mathrm{LE}}-\delta\hat\Upsilon_{\mathrm{LE}}] \\ &\quad\times(\hat\Upsilon_{\mathrm{LE}}+\delta\hat\Upsilon_{\mathrm{LE}})) + \log Z_{\mathrm{LE}} + \log(1-\langle\delta\hat\Upsilon_{\mathrm{LE}}\rangle_{\hat\Upsilon}).\end{aligned} \quad (31)$$

We can now further expand the exponentials as we have done in Eq. (24). First,

$$\begin{aligned}&\mathrm{tr}(\exp[-\hat\Upsilon_{\mathrm{LE}}-\delta\hat\Upsilon_{\mathrm{LE}}]\hat\Upsilon_{\mathrm{LE}}) \\ &\simeq \mathrm{tr}\Big(\exp[-\hat\Upsilon_{\mathrm{LE}}](I-\delta\hat\Upsilon_{\mathrm{LE}}+\frac{1}{2}[\hat\Upsilon_{\mathrm{LE}},\delta\hat\Upsilon_{\mathrm{LE}}] \\ &\quad-\frac{1}{6}[\hat\Upsilon_{\mathrm{LE}},[\hat\Upsilon_{\mathrm{LE}},\delta\hat\Upsilon_{\mathrm{LE}}]]+\cdots)\hat\Upsilon_{\mathrm{LE}}\Big) \\ &= \mathrm{tr}(\exp[-\hat\Upsilon_{\mathrm{LE}}]\hat\Upsilon_{\mathrm{LE}}) - \mathrm{tr}(\exp[-\hat\Upsilon_{\mathrm{LE}}]\delta\hat\Upsilon_{\mathrm{LE}}\hat\Upsilon_{\mathrm{LE}}) \\ &= Z_{\mathrm{LE}}\langle\hat\Upsilon_{\mathrm{LE}}\rangle_{\hat\Upsilon} - Z_{\mathrm{LE}}\langle\delta\hat\Upsilon_{\mathrm{LE}}\hat\Upsilon_{\mathrm{LE}}\rangle_{\hat\Upsilon},\end{aligned} \quad (32)$$

where, in the second equality, we have taken advantage of commutativity and cyclicity of the trace. Then,

$$
\begin{aligned}
&\mathrm{tr}(\exp[-\hat{\Upsilon}_{\rm LE} - \delta\hat{\Upsilon}_{\rm LE}]\delta\hat{\Upsilon}_{\rm LE}) \\
&\quad \simeq \mathrm{tr}\Big(\exp[-\hat{\Upsilon}_{\rm LE}](I - \delta\hat{\Upsilon}_{\rm LE} + \frac{1}{2}[\hat{\Upsilon}_{\rm LE}, \delta\hat{\Upsilon}_{\rm LE}] \\
&\qquad - \frac{1}{6}[\hat{\Upsilon}_{\rm LE}, [\hat{\Upsilon}_{\rm LE}, \delta\hat{\Upsilon}_{\rm LE}]] + \cdots)\delta\hat{\Upsilon}_{\rm LE}\Big) \\
&\quad \simeq \mathrm{tr}(\exp[-\hat{\Upsilon}_{\rm LE}]\delta\hat{\Upsilon}_{\rm LE}) = Z_{\rm LE}\langle\delta\hat{\Upsilon}_{\rm LE}\rangle_{\hat{\Upsilon}}, \qquad (33)
\end{aligned}
$$

keeping only first order terms. Thus, Eq. (31) can be rewritten as

$$
\begin{aligned}
S' &\simeq \frac{1}{Z_{\rm LE}}(1 + \langle\delta\hat{\Upsilon}_{\rm LE}\rangle_{\hat{\Upsilon}})\mathrm{tr}(\exp[-\hat{\Upsilon}_{\rm LE} - \delta\hat{\Upsilon}_{\rm LE}] \\
&\quad \times (\hat{\Upsilon}_{\rm LE} + \delta\hat{\Upsilon}_{\rm LE})) + \log Z_{\rm LE} + \log(1 - \langle\delta\hat{\Upsilon}_{\rm LE}\rangle_{\hat{\Upsilon}}) \\
&\simeq \frac{1}{Z_{\rm LE}}(1 + \langle\delta\hat{\Upsilon}_{\rm LE}\rangle_{\hat{\Upsilon}})(Z_{\rm LE}\langle\hat{\Upsilon}_{\rm LE}\rangle_{\hat{\Upsilon}} - Z_{\rm LE}\langle\delta\hat{\Upsilon}_{\rm LE}\hat{\Upsilon}_{\rm LE}\rangle_{\hat{\Upsilon}} \\
&\quad + Z_{\rm LE}\langle\delta\hat{\Upsilon}_{\rm LE}\rangle_{\hat{\Upsilon}}) + \log Z_{\rm LE} + \log(1 - \langle\delta\hat{\Upsilon}_{\rm LE}\rangle_{\hat{\Upsilon}}) \\
&= (1 + \langle\delta\hat{\Upsilon}_{\rm LE}\rangle_{\hat{\Upsilon}})(\langle\hat{\Upsilon}_{\rm LE}\rangle_{\hat{\Upsilon}} - \langle\delta\hat{\Upsilon}_{\rm LE}\hat{\Upsilon}_{\rm LE}\rangle_{\hat{\Upsilon}} \\
&\quad + \langle\delta\hat{\Upsilon}_{\rm LE}\rangle_{\hat{\Upsilon}}) + \log Z_{\rm LE} + \log(1 - \langle\delta\hat{\Upsilon}_{\rm LE}\rangle_{\hat{\Upsilon}}). \qquad (34)
\end{aligned}
$$

Retaining only the first order terms in $\delta\hat{\Upsilon}_{\rm LE}$, expanding the logarithm for $\langle\delta\hat{\Upsilon}_{\rm LE}\rangle_{\rm LE} \ll 1$ and inserting the original expression of entropy, we obtain

$$
S' \simeq S - \langle\delta\hat{\Upsilon}_{\rm LE}\hat{\Upsilon}_{\rm LE}\rangle_{\hat{\Upsilon}} + \langle\delta\hat{\Upsilon}_{\rm LE}\rangle_{\hat{\Upsilon}}\langle\hat{\Upsilon}_{\rm LE}\rangle_{\hat{\Upsilon}}. \qquad (35)
$$

Therefore, the variation of the total entropy is, to the lowest order, proportional to the correlation between $\hat{\Upsilon}$ and $\delta\hat{\Upsilon}$, which is generally nonvanishing.

We can expand the above correlation to gain further insight. For the $\delta\hat{\Upsilon}_{\rm LE}$, let us keep only the second term of the right-hand side of Eq. (30):

$$
\delta\hat{\Upsilon}_{\rm LE} = -\frac{1}{2}\int \mathrm{d}^3\mathrm{x}\hat{\Phi}^{\lambda,0\nu}(\partial_\lambda\delta\beta_\nu + \partial_\nu\delta\beta_\lambda). \qquad (36)
$$

By using (29), (36), and (35) can be rewritten as

$$
\begin{aligned}
S'(t) \simeq S(t) &+ \frac{1}{2}\int \mathrm{d}^3\mathrm{x}\int \mathrm{d}^3\mathrm{x}'(\langle\hat{\Phi}^{\lambda,0\nu}(x)\hat{T}^{0\mu}(x')\rangle_{\hat{\Upsilon}} - \langle\hat{\Phi}^{\lambda,0\nu}(x)\rangle_{\hat{\Upsilon}}\langle\hat{T}^{0\mu}(x')\rangle_{\hat{\Upsilon}})\beta_\mu(x')(\partial_\lambda\delta\beta_\nu(x) + \partial_\nu\delta\beta_\lambda(x)) \\
&- \frac{1}{2}\int \mathrm{d}^3\mathrm{x}\int \mathrm{d}^3\mathrm{x}'(\langle\hat{\Phi}^{\lambda,0\nu}(x)\hat{j}^0(x')\rangle_{\hat{\Upsilon}} - \langle\hat{\Phi}^{\lambda,0\nu}(x)\rangle_{\hat{\Upsilon}}\langle\hat{j}^0(x')\rangle_{\hat{\Upsilon}})\xi(x')(\partial_\lambda\delta\beta_\nu(x) + \partial_\nu\delta\beta_\lambda(x)) \\
&- \frac{1}{4}\int \mathrm{d}^3\mathrm{x}\int \mathrm{d}^3\mathrm{x}'(\langle\hat{\Phi}^{\lambda,0\nu}(x)\hat{\mathcal{S}}^{0,\rho\sigma}(x')\rangle_{\hat{\Upsilon}} - \langle\hat{\Phi}^{\lambda,0\nu}(x)\rangle_{\hat{\Upsilon}}\langle\hat{\mathcal{S}}^{0,\rho\sigma}(x')\rangle_{\hat{\Upsilon}})\omega_{\rho\sigma}(x')(\partial_\lambda\delta\beta_\nu(x) + \partial_\nu\delta\beta_\lambda(x)), \qquad (37)
\end{aligned}
$$

where x and x' have equal times. The above expression could be further simplified by e.g., approximating the local equilibrium mean $\langle\rangle_{\hat{\Upsilon}}$ with the global equilibrium one $\langle\rangle_0$, but this does not lead to further conceptual insight. The physical meaning of Eq. (37) is that the entropy difference depends on the correlation between local operators in two different space points multiplied by a factor which is at most of the second order in the perturbation $\delta\beta$. This kind of expression resembles the product of transport coefficients expressed by a Kubo formula times the squared gradient of the perturbation field. Therefore, the difference between entropies suggests that the introduction of a superpotential may lead to a modification of the transport coefficients. We will show this in detail in the next section.

V. TRANSPORT COEFFICIENTS: SHEAR VISCOSITY AS AN EXAMPLE

As has been mentioned, a remarkable consequence of the transformation (1) is a difference in the predicted values of transport coefficients calculated with the relativistic Kubo formula, which is obtained by working out the mean value of the stress-energy tensor itself with the linear response theory and the nonequilibrium density operator in Eq. (2). For this purpose, the derivation in Ref. [9] must be extended to the most general expression of the nonequilibrium density operator including a spin tensor, that is, Eq. (13); it can be found in Appendix A.

Equation (25), which yields the difference of mean values of a general observable under a transformation (1), cannot be straightforwardly used to calculate the mean value of the stress-energy tensor setting $\hat{O} = \hat{T}^{\mu\nu}(y)$ because $\hat{T}^{\mu\nu}(y)$ gets transformed itself. It is therefore more convenient to work out the general expression of the Kubo formula and study how it is modified by (1) thereafter.

We will take shear viscosity as an example; the transformation of other transport coefficients can be obtained with the same reasoning. Shear viscosity, in the Kubo formula, is related to the spatial components of the symmetric part of the stress-energy tensor. It is worth pointing out that, since a nonvanishing spin tensor can make the stress-energy tensor nonsymmetric, there might be a new transport coefficient related to the antisymmetric part of the stress-energy tensor.

For the symmetric part of the stress-energy tensor $T_S^{\mu\nu} \equiv (1/2)(T^{\mu\nu} + T^{\nu\mu})$, using the general formula of relativistic linear response theory [Eq. (A14) of Appendix A], the difference $\delta T_S^{\mu\nu}(y)$ between the actual mean value and the local equilibrium value reads, at the lowest order in gradients,

$$\delta T_S^{\mu\nu}(y) = \lim_{\varepsilon\to 0}\frac{T}{i}\int_{-\infty}^{t'} \mathrm{d}t \frac{1-\mathrm{e}^{\varepsilon(t-t')}}{\varepsilon}\int \mathrm{d}^3\mathrm{x}\langle[\hat{T}^{\rho\sigma}(x), \hat{T}_S^{\mu\nu}(y)]\rangle_0 \partial_\rho \delta\beta_\sigma(x) - \frac{1}{2}\lim_{\varepsilon\to 0}\frac{T}{i}\times\int_{-\infty}^{t'} \mathrm{d}t \mathrm{e}^{\varepsilon(t-t')}\int \mathrm{d}^3\mathrm{x}\langle[\hat{\mathcal{S}}^{0,\rho\sigma}(x), \hat{T}_S^{\mu\nu}(y)]\rangle_0 \delta\omega_{\rho\sigma}(x) - \frac{1}{2}\lim_{\varepsilon\to 0}\frac{T}{i}\times\int_{-\infty}^{t'} \mathrm{d}t \mathrm{e}^{\varepsilon(t-t')}\int_{-\infty}^{t}\mathrm{d}\tau\int \mathrm{d}^3\mathrm{x}\langle[\hat{\mathcal{S}}^{0,\rho\sigma}(\tau,\mathbf{x}), \hat{T}_S^{\mu\nu}(y)]\rangle_0 \frac{\partial}{\partial t}\delta\omega_{\rho\sigma}(x). \quad (38)$$

In order to obtain transport coefficients, a suitable perturbation must be chosen which can be eventually taken out of the integral. Physically, this corresponds to enforcing a particular hydrodynamical motion and observing the response of the stress-energy tensor to infer the dissipative coefficient. The perturbation $\delta\beta = 1/T\delta u$ is taken to be a stationary one and nonvanishing only within a finite region V, at whose boundary it goes to zero in a continuous and derivable fashion. The perturbation $\delta\omega$ is also taken to be stationary, and it can be chosen either to vanish or to be as in Eq. (21); in both cases, one gets the same final result.

Let us then set $\delta\omega = 0$ and expand the perturbation $\delta\beta = (0, 0, \delta\beta^2(x^1), 0)$ dependent on x^1 in a Fourier series (it vanishes at some large, yet finite boundary). Since we want the higher order gradients of the perturbation to be negligibly small (the so-called hydrodynamic limit), the Fourier components with short wavelengths must be correspondingly suppressed. The component with the longest wavelength will then be much larger than any other, and, therefore, $\delta\beta^2$ can be approximately written, at least far from the boundary, as $A\sin(\pi x^1/L)$, where L is the size of the region V in the x^1 direction and A is a constant. The derivative of this perturbation reads

$$\partial_1\delta\beta_2(\mathbf{x}) = \frac{\pi}{L}A\cos(\pi x^1/L) = \partial_1\delta\beta_2(\mathbf{0})\cos(\pi x^1/L) \equiv \partial_1\delta\beta_2(\mathbf{0})\cos(kx^1),$$

where $k \equiv \pi/L$. Therefore, by defining $\mathbf{k} = (k, 0, 0)$ and using the last equation in Eq. (38),

$$\delta T_S^{\mu\nu}(y) = \lim_{\varepsilon\to 0}\frac{T}{i}\partial_1\delta\beta_2(\mathbf{0})\int_{-\infty}^{t'}\mathrm{d}t\frac{1-\mathrm{e}^{\varepsilon(t-t')}}{\varepsilon}\times\int_V \mathrm{d}^3\mathrm{x}\cos\mathbf{k}\cdot\mathbf{x}\langle[\hat{T}^{12}(x), \hat{T}_S^{\mu\nu}(y)]\rangle_0 = \lim_{\varepsilon\to 0}T\partial_1\delta\beta_2(\mathbf{0})\mathrm{Im}\int_{-\infty}^{t'}\mathrm{d}t\frac{1-\mathrm{e}^{\varepsilon(t-t')}}{\varepsilon}\times\int_V \mathrm{d}^3\mathrm{x}\mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}}\langle[\hat{T}^{12}(x), \hat{T}_S^{\mu\nu}(y)]\rangle_0, \quad (39)$$

taking into account that the commutator is purely imaginary. To extract shear viscosity we have to evaluate the stress-energy tensor in $\mathbf{y} = 0$ to make it proportional to the derivative of the four-temperature field in the same point, and we have to take the limit $L \to \infty$, which implies $V \to \infty$ and $\mathbf{k} \to 0$ at the same time:

$$\delta T_S^{\mu\nu}(t_y, \mathbf{0}) = \lim_{\varepsilon\to 0}\lim_{\mathbf{k}\to 0}T\partial_1\delta\beta_2(\mathbf{0})\mathrm{Im}\int_{-\infty}^{t'}\mathrm{d}t\frac{1-\mathrm{e}^{\varepsilon(t-t')}}{\varepsilon}\times\int \mathrm{d}^3\mathrm{x}\mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}}\langle[\hat{T}^{12}(x), \hat{T}_S^{\mu\nu}(t_y, \mathbf{0})]\rangle_0, \quad (40)$$

where we assume that the integration domain goes to its thermodynamic limit independently of the integrand. Because of the time-translation symmetry of the equilibrium density operator $\hat{\rho}_0$, the mean value in the integral only depends on the time difference $t - t_y$. Thus, choosing the arbitrary time $t' = t_y$ and redefining the integration variables, Eq. (40) can be rewritten as

$$\delta T_S^{\mu\nu}(t_y, \mathbf{0}) = \lim_{\varepsilon\to 0}\lim_{\mathbf{k}\to 0}T\partial_1\delta\beta_2(\mathbf{0})\mathrm{Im}\int_{-\infty}^{0}\mathrm{d}t\frac{1-\mathrm{e}^{\varepsilon t}}{\varepsilon}\times\int \mathrm{d}^3\mathrm{x}\mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}}\langle[\hat{T}^{12}(x), \hat{T}_S^{\mu\nu}(0)]\rangle_0, \quad (41)$$

which shows that the mean value $\delta T_S^{\mu\nu}(t_y, \mathbf{0})$ is indeed independent of t_y, which is expected as $\delta\beta$ is stationary.

We can now take advantage of the well-known Curie symmetry "principle" which states that tensors belonging to some irreducible representation of the rotation group will only respond to perturbations belonging to the same representation and with the same components.[3] In our case the Curie principle implies that only the same component of the symmetric part of the stress-energy tensor, i.e., $\hat{T}_S^{12}$, will give a nonvanishing value:

$$\delta T_S^{12}(t_y, \mathbf{0}) = \lim_{\varepsilon\to 0}\lim_{\mathbf{k}\to 0}T\partial_1\delta\beta_2(\mathbf{0})\mathrm{Im}\int_{-\infty}^{0}\mathrm{d}t\frac{1-\mathrm{e}^{\varepsilon t}}{\varepsilon}\times\int \mathrm{d}^3\mathrm{x}\mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}}\langle[\hat{T}_S^{12}(x), \hat{T}_S^{12}(0)]\rangle_0. \quad (42)$$

From the above expression, a Kubo formula for shear viscosity can be extracted, setting $\delta\beta = (1/T)\delta u$,

$$\eta = \lim_{\varepsilon\to 0}\lim_{\mathbf{k}\to 0}\mathrm{Im}\int_{-\infty}^{0}\mathrm{d}t\frac{1-\mathrm{e}^{\varepsilon t}}{\varepsilon}\times\int \mathrm{d}^3\mathrm{x}\mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}}\langle[\hat{T}_S^{12}(x), \hat{T}_S^{12}(0)]\rangle_0 \quad (43)$$

which, after a little algebra, can be shown to be the same expression obtained in Ref. [9]. Because of the rotational

[3] This is true provided that the right-hand side of Eq. (41) is a continuous function of $\mathbf{k}$ for $\mathbf{k} = 0$ or that its limit for $\mathbf{k} \to 0$ exists; i.e., it is independent of the direction of $\mathbf{k}$.

invariance of the equilibrium density operator, shear viscosity is independent of the particular couple (1, 2) of chosen indices. It is worth pointing out that, had we started from Eq. (A15) instead of Eq. (A14), choosing $\delta\omega = 0$ or like in Eq. (21), we would have come to the same formula for shear viscosity; in the latter case, the third contributing term in Eq. (A15) would have been of higher order in derivatives of $\delta\beta$, and hence negligible.

Now, the question we want to answer is whether Eq. (43) is invariant by a pseudo-gauge transformation (1), which turns the symmetric part of the stress-energy tensor into

$$\hat{T}_S'^{\mu\nu} = \hat{T}_S^{\mu\nu} - \frac{1}{2}\partial_\lambda(\hat{\Phi}^{\mu,\lambda\nu} + \hat{\Phi}^{\nu,\lambda\mu}) = \hat{T}_S^{\mu\nu} - \partial_\lambda \hat{\Xi}^{\lambda\mu\nu}, \tag{44}$$

where

$$\frac{1}{2}(\hat{\Phi}^{\mu,\lambda\nu} + \hat{\Phi}^{\nu,\lambda\mu}) \equiv \hat{\Xi}^{\lambda\mu\nu}, \tag{45}$$

$\hat{\Xi}$ being symmetric in the last two indices. We will study the effect of the transformation on the mean value of the stress-energy tensor in the point $y = 0$ starting from Eq. (A15) instead of Eq. (A14), with $\delta\omega = 0$ or like in Eq. (21), which allows us to retain only the first contributing term to $\delta T_S^{12}(0)$. The perturbation $\delta\beta$ is taken to be stationary and t' is set to be equal to $t_y = 0$. Eventually, the appropriate limits will be calculated to get the new shear viscosity. Thus,

$$\begin{aligned}\delta T_S'^{12}(0) = {}& \delta T_S^{12}(0) + \lim_{\varepsilon\to 0}\int_{-\infty}^0 \mathrm{d}t \frac{1-\mathrm{e}^{\varepsilon t}}{\varepsilon}\int \mathrm{d}^3\mathrm{x}\langle[\partial_\alpha \hat{\Xi}^{\alpha 12}(x), \partial_\beta \hat{\Xi}^{\beta 12}(0)]\rangle_0(\partial_1\delta\beta_2(\mathbf{x}) + \partial_2\delta\beta_1(\mathbf{x}))\\ &- \lim_{\varepsilon\to 0}\int_{-\infty}^0 \mathrm{d}t \frac{1-\mathrm{e}^{\varepsilon t}}{\varepsilon}\int \mathrm{d}^3\mathrm{x}(\langle[\partial_\alpha \hat{\Xi}^{\alpha 12}(x), \hat{T}_S^{12}(0)]\rangle_0 + \langle[\hat{T}_S^{12}(t,\mathbf{x}), \partial_\alpha \hat{\Xi}^{\alpha 12}(0)]\rangle_0)(\partial_1\delta\beta_2(\mathbf{x}) + \partial_2\delta\beta_1(\mathbf{x})).\end{aligned} \tag{46}$$

We can simplify the above formula by noting that the mean value of two operators at equilibrium can only depend on the difference of the coordinates, so

$$\langle[\hat{O}_1(y), \partial_\mu \hat{O}_2(x)]\rangle_0 = \frac{\partial}{\partial x^\mu}\langle[\hat{O}_1, \hat{O}_2]\rangle_0(y-x) = -\frac{\partial}{\partial y^\mu}\langle[\hat{O}_1, \hat{O}_2]\rangle_0(y-x).$$

Hence, Eq. (46) can be rewritten as

$$\begin{aligned}\delta T_S'^{12}(0) = {}& \delta T_S^{12}(0) - \lim_{\varepsilon\to 0}\int_{-\infty}^0 \mathrm{d}t \frac{1-\mathrm{e}^{\varepsilon t}}{\varepsilon}\int \mathrm{d}^3\mathrm{x}\frac{\partial^2}{\partial x^\alpha \partial x^\beta}\langle[\hat{\Xi}^{\alpha 12}(x), \hat{\Xi}^{\beta 12}(0)]\rangle_0(\partial_1\delta\beta_2(\mathbf{x}) + \partial_2\delta\beta_1(\mathbf{x}))\\ &- \lim_{\varepsilon\to 0}\int_{-\infty}^0 \mathrm{d}t \frac{1-\mathrm{e}^{\varepsilon t}}{\varepsilon}\int \mathrm{d}^3\mathrm{x}\frac{\partial}{\partial x^\alpha}(\langle[\hat{\Xi}^{\alpha 12}(x), \hat{T}_S^{12}(0)]\rangle_0 - \langle[\hat{T}_S^{12}(x), \hat{\Xi}^{\alpha 12}(0)]\rangle_0)(\partial_1\delta\beta_2(\mathbf{x}) + \partial_2\delta\beta_1(\mathbf{x})).\end{aligned} \tag{47}$$

We are now going to inspect the two terms on the right-hand side of the above equation. If the Hamiltonian is time-reversal invariant, it can be shown (see Appendix B) that

$$\langle[\hat{T}_S^{ij}(t,\mathbf{x}), \hat{\Xi}^{\alpha ij}(0,\mathbf{0})]\rangle_0 = (-1)^{n_0}\langle[\hat{\Xi}^{\alpha ij}(0,\mathbf{0}), \hat{T}_S^{ij}(-t,\mathbf{x})]\rangle_0 = (-1)^{n_0}\langle[\hat{\Xi}^{\alpha ij}(t,-\mathbf{x}), \hat{T}_S^{ij}(0,\mathbf{0})]\rangle_0,$$

where n_0 is the total number of time indices among those in the above expression. Similarly, if the Hamiltonian is parity invariant, then

$$\langle[\hat{\Xi}^{\alpha ij}(t,-\mathbf{x}), \hat{T}_S^{ij}(0,\mathbf{0})]\rangle_0 = (-1)^{n_s}\langle[\hat{\Xi}^{\alpha ij}(t,\mathbf{x}), \hat{T}_S^{ij}(0,\mathbf{0})]\rangle_0,$$

where n_s is the total number of space indices. Using the last two equations to work out the last term of Eq. (47), one gets

$$\begin{aligned}\delta T_S'^{12}(0) = {}& \delta T_S^{12}(0) - \lim_{\varepsilon\to 0}\int_{-\infty}^0 \mathrm{d}t \frac{1-\mathrm{e}^{\varepsilon t}}{\varepsilon}\int_V \mathrm{d}^3\mathrm{x}\frac{\partial^2}{\partial x^\alpha \partial x^\beta}\langle[\hat{\Xi}^{\alpha 12}(t,\mathbf{x}), \hat{\Xi}^{\beta 12}(0,\mathbf{0})]\rangle_0(\partial_1\delta\beta_2(\mathbf{x}) + \partial_2\delta\beta_1(\mathbf{x}))\\ &- 2\lim_{\varepsilon\to 0}\int_{-\infty}^0 \mathrm{d}t \frac{1-\mathrm{e}^{\varepsilon t}}{\varepsilon}\int_V \mathrm{d}^3\mathrm{x}\frac{\partial}{\partial x^\alpha}\langle[\hat{\Xi}^{\alpha 12}(t,\mathbf{x}), \hat{T}_S^{12}(0,\mathbf{0})]\rangle_0(\partial_1\delta\beta_2(\mathbf{x}) + \partial_2\delta\beta_1(\mathbf{x})).\end{aligned} \tag{48}$$

Now, the two terms on the right-hand side of (48) can be worked out separately. Using invariance by time-reversal and parity, one has

$$\begin{aligned}\langle[\hat{\Xi}^{\alpha ij}(t,\mathbf{x}), \hat{\Xi}^{\beta ij}(0,\mathbf{0})]\rangle_0 &= (-1)^{n_0}\langle[\hat{\Xi}^{\beta ij}(0,\mathbf{0}), \hat{\Xi}^{\alpha ij}(-t,\mathbf{x})]\rangle_0 = (-1)^{n_0}\langle[\hat{\Xi}^{\beta ij}(t,-\mathbf{x}), \hat{\Xi}^{\alpha ij}(0,\mathbf{0})]\rangle_0\\ &= (-1)^{n_0+n_s}\langle[\hat{\Xi}^{\beta ij}(t,\mathbf{x}), \hat{\Xi}^{\alpha ij}(0,\mathbf{0})]\rangle_0 = \langle[\hat{\Xi}^{\beta ij}(t,\mathbf{x}), \hat{\Xi}^{\alpha ij}(0,\mathbf{0})]\rangle_0,\end{aligned} \tag{49}$$

with $n_0 + n_s = 6$. Hence, the first term on the right-hand side of (48) can be decomposed as

$$-\lim_{\varepsilon\to 0}\int_{-\infty}^{0}\mathrm{d}t\frac{1-\mathrm{e}^{\varepsilon t}}{\varepsilon}\int_V \mathrm{d}^3\mathrm{x}\bigg(\frac{\partial^2}{\partial t^2}\langle[\hat{\Xi}^{0ij}(x),\hat{\Xi}^{0ij}(0)]\rangle_0 + 2\frac{\partial}{\partial t}\frac{\partial}{\partial x^k}\langle[\hat{\Xi}^{kij}(x),\hat{\Xi}^{0ij}(0)]\rangle_0 + \frac{\partial}{\partial x^k}\frac{\partial}{\partial x^l}\langle[\hat{\Xi}^{kij}(x),\hat{\Xi}^{lij}(0)]\rangle_0\bigg) \times(\partial_i\delta\beta_j(\mathbf{x}) + \partial_j\delta\beta_i(\mathbf{x})), \tag{50}$$

and, similarly, the second term can be decomposed as

$$-2\lim_{\varepsilon\to 0}\int_{-\infty}^{0}\mathrm{d}t\frac{1-\mathrm{e}^{\varepsilon t}}{\varepsilon}\int_V \mathrm{d}^3\mathrm{x}\frac{\partial}{\partial t}\langle[\hat{\Xi}^{012}(x),\hat{T}_S^{12}(0)]\rangle_0 + \frac{\partial}{\partial x^k}\langle[\hat{\Xi}^{k12}(t,\mathbf{x}),\hat{T}_S^{12}(0)]\rangle_0(\partial_1\delta\beta_2(\mathbf{x}) + \partial_2\delta\beta_1(\mathbf{x})). \tag{51}$$

All terms in Eqs. (50) and (51) with a space derivative do not yield any contribution to first-order transport coefficients. This can be shown by, first, integrating by parts and generating two terms, one of which is a total derivative and the second involves the second derivative of the perturbation $\delta\beta$. The total derivative term can be transformed into a surface integral on the boundary of V which vanishes because therein the perturbation $\delta\beta$ is supposed to vanish along with its first-order derivatives. The second term, involving higher order derivatives, does not give a contribution to transport coefficients at first order in the derivative expansion. Altogether, Eq. (48) turns into

$$\delta T_S'^{12}(0) = \delta T_S^{12}(0) - \lim_{\varepsilon\to 0}\int_{-\infty}^{0}\mathrm{d}t\frac{1-\mathrm{e}^{\varepsilon t}}{\varepsilon}\int_V \mathrm{d}^3\mathrm{x}\partial_t^2\langle[\hat{\Xi}^{012}(x),\hat{\Xi}^{012}(0)]\rangle_0(\partial_1\delta\beta_2(\mathbf{x}) + \partial_2\delta\beta_1(\mathbf{x})) - 2\lim_{\varepsilon\to 0}\int_{-\infty}^{0}\mathrm{d}t\frac{1-\mathrm{e}^{\varepsilon t}}{\varepsilon}\int_V \mathrm{d}^3\mathrm{x}\partial_t\langle[\hat{\Xi}^{012}(x),\hat{T}_S^{12}(0)]\rangle_0(\partial_1\delta\beta_2(\mathbf{x}) + \partial_2\delta\beta_1(\mathbf{x})) + \mathcal{O}(\partial^2\delta\beta), \tag{52}$$

which can be further integrated by parts in the time t, yielding

$$\delta T_S'^{12}(0) = \delta T_S^{12}(0) - \lim_{\varepsilon\to 0}\int_{-\infty}^{0}\mathrm{d}t(\delta(t) - \varepsilon\mathrm{e}^{\varepsilon t})\int_V \mathrm{d}^3\mathrm{x}\langle[\hat{\Xi}^{012}(x),\hat{\Xi}^{012}(0)]\rangle_0(\partial_1\delta\beta_2(\mathbf{x}) + \partial_2\delta\beta_1(\mathbf{x})) - 2\lim_{\varepsilon\to 0}\int_{-\infty}^{0}\mathrm{d}t\mathrm{e}^{\varepsilon t}\int_V \mathrm{d}^3\mathrm{x}\langle[\hat{\Xi}^{012}(x),\hat{T}_S^{12}(0)]\rangle_0(\partial_1\delta\beta_2(\mathbf{x}) + \partial_2\delta\beta_1(\mathbf{x})) + \mathcal{O}(\partial^2\delta\beta), \tag{53}$$

provided that, for general space-time dependent operators $\hat{O}_1$ and $\hat{O}_2$,

$$\lim_{t\to-\infty}\int_V \mathrm{d}^3\mathrm{x}\mathrm{e}^{n\varepsilon t}\frac{\partial}{\partial t}\langle[\hat{O}_1(t,\mathbf{x}),\hat{O}_2(0,\mathbf{0})]\rangle_0 = 0 \qquad \lim_{t\to-\infty}\int_V \mathrm{d}^3\mathrm{x}\mathrm{e}^{n\varepsilon t}\langle[\hat{O}_1(t,\mathbf{x}),\hat{O}_2(0,\mathbf{0})]\rangle_0 = 0$$

with $n = 0, 1$, which is reasonable because thermodynamical correlations are expected to vanish exponentially as a function of time for fixed points in space.[4]

From Eq. (53) the variation of the shear viscosity can be inferred with the very same reasoning that led us to formula (43), that is,

$$\Delta\eta = \eta' - \eta = -\lim_{\varepsilon\to 0}\lim_{k\to 0}\mathrm{Im}\int_{-\infty}^{0}\mathrm{d}t(\delta(t) - \varepsilon\mathrm{e}^{\varepsilon t})\int \mathrm{d}^3\mathrm{x}\mathrm{e}^{ikx^1}\langle[\hat{\Xi}^{012}(t,\mathbf{x}),\hat{\Xi}^{012}(0,\mathbf{0})]\rangle_0 - 2\lim_{\varepsilon\to 0}\lim_{k\to 0}\mathrm{Im}\int_{-\infty}^{0}\mathrm{d}t\mathrm{e}^{\varepsilon t}\int \mathrm{d}^3\mathrm{x}\mathrm{e}^{ikx^1}\langle[\hat{\Xi}^{012}(t,\mathbf{x}),\hat{T}_S^{12}(0,\mathbf{0})]\rangle_0. \tag{54}$$

If the first integral is regular, then the $\varepsilon\to 0$ limit kills one term and the (54) reduces to

$$\Delta\eta = \eta' - \eta = -\lim_{k\to 0}\int_V \mathrm{d}^3\mathrm{x}\cos kx^1\langle[\hat{\Xi}^{012}(0,\mathbf{x}),\hat{\Xi}^{012}(0,\mathbf{0})]\rangle_0 - 2\lim_{\varepsilon\to 0}\lim_{k\to 0}\mathrm{Im}\int_{-\infty}^{0}\mathrm{d}t\mathrm{e}^{\varepsilon t}\int \mathrm{d}^3\mathrm{x}\mathrm{e}^{ikx^1}\langle[\hat{\Xi}^{012}(x),\hat{T}_S^{12}(0,\mathbf{0})]\rangle_0. \tag{55}$$

In general, this difference is nonvanishing, leading to the conclusion that the specific form of the stress-energy tensor and, possibly, the existence of a spin tensor in the underlying quantum field theory affect the value of transport coefficients. The relative difference of those values depends on the particular transformation (1), and hence on the particular stress-energy tensor. In the next section a specific instance will be presented and discussed.

An important point to make is that the found dependence of the transport coefficients on the particular set of stress-energy and spin tensors of the theory is indeed physically meaningful. This means that the variation of some coefficient is not compensated by a corresponding variation of another coefficient so as to eventually leave measurable

[4]There might be singularities on the light cone; however, for fixed $\mathbf{x}$ and $\mathbf{0}$ and integration over a finite region V, in the limit $t\to-\infty$ the light cone is not involved.

quantities unchanged. This has been implicitly proved in Sec. IV where it was shown that total entropy itself undergoes a variation under a transformation of the stress-energy and spin tensors [see Eq. (35)].

VI. DISCUSSION AND CONCLUSIONS

As a first point, we would like to emphasize that, in our arguments, space-time curvature and gravitational coupling have been disregarded. On one hand, this shows that the nature of the stress-energy tensor and, possibly, the existence of a fundamental spin tensor could, at least in principle, be demonstrated independently of gravity. On the other hand, for each stress-energy tensor created with the transformation (1), it should be shown that an extension of general relativity exists, having it as a source, which is not always possible.

An important question is whether a concrete physical system indeed exists for which the transformation (1) leads to actually different values for e.g., transport coefficients, entropy production rate or other quantities in nonequilibrium situations. For this purpose, we discuss a specific instance regarding spinor electrodynamics. Starting from the symmetrized gauge-invariant Belinfante tensor of the coupled Dirac and electromagnetic fields, with associated $\hat{\mathcal{S}} = 0$,

$$\hat{T}^{\mu\nu} = \frac{i}{4}(\bar{\Psi}\gamma^{\mu}\overleftrightarrow{\nabla}^{\nu}\Psi + \bar{\Psi}\gamma^{\nu}\overleftrightarrow{\nabla}^{\mu}\Psi) + \hat{F}^{\mu}{}_{\lambda}\hat{F}^{\lambda\nu} + \frac{1}{4}g^{\mu\nu}\hat{F}^2, \tag{56}$$

where $\nabla_\mu = \partial_\mu - ieA_\mu$ is the gauge covariant derivative, one can generate other stress-energy tensors with suitable rank three tensors and then set $\hat{\Phi} = -\hat{\mathcal{S}}'$ where $\hat{\mathcal{S}}'$ is the new spin tensor, according to (1). One of the best known is the *canonical* Dirac spin tensor:

$$\hat{\Phi}^{\lambda,\mu\nu} = -\frac{i}{8}\bar{\Psi}\{\gamma^{\lambda},[\gamma^{\mu},\gamma^{\nu}]\}\Psi$$

({} stands for the anticommutator) which is gauge invariant and transforms the Belinfante tensor (56) back to the canonical one obtained from the spinor electrodynamics Lagrangian (see also Ref. [5] for a detailed discussion). However, this is totally antisymmetric in the three indices λ, μ, ν, and thus the variation of the $\hat{\Upsilon}$ operator [see Eq. (22)] as well as transport coefficients, which depend on the symmetrized $\hat{\Xi}$ tensor (45), vanish. Nevertheless, other gauge-invariant $\hat{\Phi}$-like tensors can be found. For instance, one could employ a superpotential:

$$\begin{aligned}\hat{\Phi}^{\lambda,\mu\nu} &= \frac{1}{8m}\bar{\Psi}(\gamma^{\mu}\overleftrightarrow{\nabla}^{\nu} - \gamma^{\nu}\overleftrightarrow{\nabla}^{\mu})\gamma^{\lambda}\Psi + \text{H.c}\\ &= \frac{1}{8m}\bar{\Psi}([\gamma^{\mu},\gamma^{\lambda}]\overleftrightarrow{\nabla}^{\nu} - [\gamma^{\nu},\gamma^{\lambda}]\overleftrightarrow{\nabla}^{\mu})\Psi,\end{aligned}$$

which is the gauge-invariant version of the one used in Ref. [12] to obtain a conserved spin current. This superpotential gives rise to a nonvanishing spin tensor as well as a $\hat{\Xi}$ tensor [see Eq. (45)]:

$$\hat{\Xi}^{\lambda\mu\nu} = \frac{1}{16m}\bar{\Psi}([\gamma^{\lambda},\gamma^{\mu}]\overleftrightarrow{\nabla}^{\mu} + [\gamma^{\lambda},\gamma^{\nu}]\overleftrightarrow{\nabla}^{\mu})\Psi$$

and hence a variation of thermodynamics. By noting that the structure of the above tensor is very similar to the Belinfante stress-energy tensor (56), it is not difficult to find a rough estimate of the variation of e.g., shear viscosity induced by the transformation. Looking at Eq. (55) we note that $\hat{\Xi}^{012}$ mainly differs from $\hat{T}^{012}$ in Eq. (56) by the factor $1/m$. The last term on the right-hand side of Eq. (56) tells us that the dimension of $\hat{\Xi}$ is that of a stress-energy tensor multiplied by a time, and therefore this term must be of the order of $\eta\hbar/mc^2\tau$ where τ is the microscopic correlation time scale of the original stress-energy tensor or the collisional time scale in the kinetic language and η the shear viscosity obtained from the original stress-energy tensor. Thus, the expected relative variation of shear viscosity from Eq. (55) in this case is of the order

$$\frac{\Delta\eta}{\eta} \approx \mathcal{O}\left(\frac{\hbar}{mc^2\tau}\right),$$

which is (as is expected) a quantum relativistic correction governed by the ratio $(\lambda_c/c)/\tau$, λ_c being the Compton wavelength. For the electron, the ratio $\lambda_c/c \approx 10^{-21}$ sec, which is a very small time scale compared to the usual kinetic time scales, yet it could be detectable for particular systems with very low shear viscosity.

It is also interesting to note that the "improved" stress-energy tensor by Callan, Coleman, and Jackiw [14], with renormalizable matrix elements at all orders of perturbation theory, is obtained from Belinfante's symmetrized one in Eq. (56) with a transformation of the kind (1), setting (for the Dirac field and vanishing constants [14])

$$\hat{Z}^{\alpha\lambda,\mu\nu} = -\frac{1}{6}(g^{\alpha\mu}g^{\lambda\nu} - g^{\alpha\nu}g^{\lambda\mu})\bar{\Psi}\Psi$$

and requiring $\hat{\mathcal{S}}' = \hat{\mathcal{S}} = 0$ so that $\hat{\Phi}^{\lambda,\mu\nu} = \partial_\alpha\hat{Z}^{\alpha\lambda,\mu\nu}$; hence,

$$\begin{aligned}\hat{\Phi}^{\lambda,\mu\nu} &= -\frac{1}{6}(g^{\lambda\nu}\partial^{\mu} - g^{\lambda\mu}\partial^{\nu})\bar{\Psi}\Psi\\ \hat{\Xi}^{\lambda\mu\nu} &= \frac{1}{2}(\hat{\Phi}^{\mu,\lambda\nu} + \hat{\Phi}^{\nu,\lambda\mu})\\ &= -\frac{1}{6}\left[g^{\mu\nu}\partial^{\lambda} - \frac{1}{2}(g^{\lambda\nu}\partial^{\mu} + g^{\lambda\mu}\partial^{\nu})\right]\bar{\Psi}\Psi\\ \hat{T}'^{\mu\nu} &= \hat{T}^{\mu\nu} - \partial_\lambda\hat{\Xi}^{\lambda\mu\nu} = \hat{T}^{\mu\nu} + \frac{1}{6}(g^{\mu\nu}\Box - \partial^{\mu}\partial^{\nu})\bar{\Psi}\Psi,\end{aligned} \tag{57}$$

which is just the improved stress-energy tensor [14]. It is likely (though not verified) that the aforementioned modified stress-energy tensors imply a different thermodynamics with respect to the original Belinfante symmetrized tensor. This problem has been recently pointed out in Ref. [15].

To summarize, we have concluded that different quantum stress-energy tensors imply different values of nonequilibrium thermodynamical quantities like transport coefficients and entropy production rate. This reinforces our previous similar conclusion concerning differences of momentum and angular momentum densities in rotational equilibrium [5]. The existence of a fundamental spin tensor thus has an impact on the microscopic number of degrees of freedom and on how quickly macroscopic information is converted into microscopic. The difference of transport coefficients depends on the particular form of the tensors, and in the examined case, it scales like a quantum relativistic effect with $\hbar/c$. Therefore, at least in principle, it is possible to disprove a supposed stress-energy tensor with a suitably designed thermodynamical experiment.

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APPENDIX A: RELATIVISTIC LINEAR RESPONSE THEORY WITH A SPIN TENSOR

We extend the relativistic linear response theory in Zubarev's approach to the case of a nonvanishing spin tensor. The (stationary) nonequilibrium density operator is written in Eq. (13), with $\hat{\Upsilon}$ expanded as in Eq. (14). As has been shown in Sec. II, at equilibrium, only the first term of the $\hat{\Upsilon}$ operator survives in Eq. (14); therefore, one can rewrite that equation using the perturbations $\delta\beta$, $\delta\xi$ and $\delta\omega$ which are defined as the difference between the actual value and their value at thermodynamical equilibrium,

$$\begin{aligned}\hat{\Upsilon} &= \int \mathrm{d}^3\mathrm{x}\Big(\hat{T}^{0\nu}\beta_\nu(t',\mathbf{x}) - \hat{j}^0\xi(t',\mathbf{x}) - \frac{1}{2}\hat{\mathcal{S}}^{0,\mu\nu}\omega_{\mu\nu}(t',\mathbf{x})\Big)\\ &\quad + \lim_{\varepsilon\to 0}\int_{-\infty}^{t'} \mathrm{d}t e^{\varepsilon(t-t')}\int \mathrm{d}S n_i\Big(\hat{T}^{i\nu}\delta\beta_\nu(x) - \hat{j}^i\delta\xi(x) - \frac{1}{2}\hat{\mathcal{S}}^{i,\mu\nu}\delta\omega_{\mu\nu}(x)\Big)\\ &\quad - \frac{1}{2}\lim_{\varepsilon\to 0}\int_{-\infty}^{t'}\mathrm{d}t\int \mathrm{d}^3\mathrm{x} e^{\varepsilon(t-t')}(\hat{T}_S^{\mu\nu}(\partial_\mu\delta\beta_\nu(x) + \partial_\mu\delta\beta_\nu(x)) + \hat{T}_A^{\mu\nu}(\partial_\mu\delta\beta_\nu(x) - \partial_\mu\delta\beta_\nu(x)\\ &\quad + 2\delta\omega_{\mu\nu}(x)) - \hat{\mathcal{S}}^{\lambda,\mu\nu}\partial_\lambda\delta\omega_{\mu\nu}(x) - 2\hat{j}^\mu\partial_\mu\delta\xi(x)),\end{aligned} \tag{A1}$$

where it is understood that $x = (t, \mathbf{x})$.

In fact, we will use a rearrangement of the right-hand-side expression which is more convenient if one wants to work with an unspecified, yet small, $\delta\omega$. Therefore, the above equation is rewritten as

$$\begin{aligned}\hat{\Upsilon} &= \int \mathrm{d}^3\mathrm{x}\Big(\hat{T}^{0\nu}\beta_\nu(t',\mathbf{x}) - \hat{j}^0\xi(t',\mathbf{x}) - \frac{1}{2}\hat{\mathcal{S}}^{0,\mu\nu}\omega_{\mu\nu}(t',\mathbf{x})\Big)\\ &\quad - \lim_{\varepsilon\to 0}\int_{-\infty}^{t'}\mathrm{d}t e^{\varepsilon(t-t')}\frac{\partial}{\partial t}\int \mathrm{d}^3\mathrm{x}\Big(\hat{T}^{0\nu}\delta\beta_\nu(x)\\ &\quad - \frac{1}{2}\hat{\mathcal{S}}^{0,\mu\nu}\delta\omega_{\mu\nu}(x) - \hat{j}^0\delta\xi(x)\Big),\end{aligned} \tag{A2}$$

which can be easily obtained from Eq. (13) integrating by parts in time.

For the sake of simplicity we calculate the linear response with $\xi_{\rm eq} = \delta\xi = 0$, but it can be shown that our final expressions hold for $\xi_{\rm eq} \neq 0$ (in other words, with a nonvanishing chemical potential $\mu \neq 0$). Let us now define

$$\hat{A} = -\int \mathrm{d}^3\mathrm{x}\Big(\hat{T}^{0\nu}\beta_\nu(t',\mathbf{x}) - \frac{1}{2}\hat{\mathcal{S}}^{0,\mu\nu}\omega_{\mu\nu}(t',\mathbf{x})\Big)$$

and

$$\hat{B} = \lim_{\varepsilon\to 0}\int_{-\infty}^{t'}\mathrm{d}t e^{\varepsilon(t-t')}\frac{\partial}{\partial t}\int \mathrm{d}^3\mathrm{x}\Big(\hat{T}^{0\nu}\delta\beta_\nu(x) - \frac{1}{2}\hat{\mathcal{S}}^{0,\mu\nu}\delta\omega_{\mu\nu}(x)\Big)$$

so that

$$\hat{\rho} = \frac{1}{Z}\exp[-\hat{\Upsilon}] = \frac{1}{Z}\exp[\hat{A} + \hat{B}] \tag{A3}$$

with $Z = \mathrm{tr}(\exp[\hat{A} + \hat{B}])$.

The operator $\hat{B}$ is the small term in which $\hat{\rho}$ is to be expanded, according to the linear response theory. It can be rewritten in a way which will be useful later on. Since

$$\begin{aligned}\int \mathrm{d}^3\mathrm{x}\frac{\partial}{\partial t}(\hat{T}^{0\nu}(x)\delta\beta_\nu(x)) &= \int \mathrm{d}^3\mathrm{x}\,\partial_\mu(\hat{T}^{\mu\nu}(x)\delta\beta_\nu(x)) - \int \mathrm{d}^3\mathrm{x}\,\partial_i\hat{T}^{i\nu}(x)\delta\beta_\nu(x)\\ &= \int \mathrm{d}^3\mathrm{x}\,\hat{T}^{\mu\nu}(x)\partial_\mu\delta\beta_\nu(x) - \int_{\partial V}\mathrm{d}S\hat{n}_i\hat{T}^{i\nu}(x)\delta\beta_\nu(x),\end{aligned}$$

then

$$\begin{aligned}\hat{B} &= \lim_{\varepsilon\to 0}\int_{-\infty}^{t'}\mathrm{d}t e^{\varepsilon(t-t')}\int \mathrm{d}^3\mathrm{x}(\hat{T}^{\mu\nu}\partial_\mu\delta\beta_\nu(x) - \frac{1}{2}\frac{\partial}{\partial t}\\ &\quad \times(\hat{\mathcal{S}}^{0,\mu\nu}\delta\omega_{\mu\nu}(x))) - \int_{\partial V}\mathrm{d}S\hat{n}_i\hat{T}^{i\nu}(x)\delta\beta_\nu(x).\end{aligned}$$

The perturbation $\delta\beta$ must be chosen such that $\delta\beta|_{\partial V} = 0$ so that only the bulk term survives in the above equation:

$$\begin{aligned}\hat{B} &= \lim_{\varepsilon\to 0}\int_{-\infty}^{t'}\mathrm{d}t e^{\varepsilon(t-t')}\int \mathrm{d}^3\mathrm{x}\Big(\hat{T}^{\mu\nu}\partial_\mu\delta\beta_\nu(x)\\ &\quad - \frac{1}{2}\frac{\partial}{\partial t}(\hat{\mathcal{S}}^{0,\mu\nu}\delta\omega_{\mu\nu}(x))\Big).\end{aligned} \tag{A4}$$

At the lowest order in $\hat{B}$,

$$Z = \mathrm{tr}(\mathrm{e}^{\hat{A}+\hat{B}}) \simeq \mathrm{tr}(\mathrm{e}^{\hat{A}}[1+\hat{B}]) = Z_{\mathrm{LE}}(1+\langle\hat{B}\rangle_{\mathrm{LE}})$$
$$\Rightarrow \frac{1}{Z} \simeq \frac{1}{Z_{\mathrm{LE}}}(1-\langle\hat{B}\rangle_{\mathrm{LE}}) \tag{A5}$$

and, according to the Kubo identity,

$$\mathrm{e}^{\hat{A}+\hat{B}} = \left[1+\int_0^1 \mathrm{d}z \mathrm{e}^{z(\hat{A}+\hat{B})}\hat{B}\mathrm{e}^{-z\hat{A}}\right]\mathrm{e}^{\hat{A}}$$
$$\simeq \left[1+\int_0^1 \mathrm{d}z \mathrm{e}^{z\hat{A}}\hat{B}\mathrm{e}^{-z\hat{A}}\right]\mathrm{e}^{\hat{A}}, \tag{A6}$$

where the subscript LE stands for local equilibrium and implies the calculation of mean values with the local equilibrium density operator (see Sec. IV). Thereby, putting together (A5) and (A6) and retaining only first-order terms in $\hat{B}$,

$$\hat{\rho} \simeq (1-\langle\hat{B}\rangle_{\mathrm{LE}})\hat{\rho}_{\mathrm{LE}} + \int_0^1 \mathrm{d}z \mathrm{e}^{z\hat{A}}\hat{B}\mathrm{e}^{-z\hat{A}}\hat{\rho}_{\mathrm{LE}},$$

and hence the mean value of an operator $\hat{O}(y)$ becomes

$$\langle\hat{O}(y)\rangle \simeq (1-\langle\hat{B}\rangle_{\mathrm{LE}})\langle\hat{O}(y)\rangle_{\mathrm{LE}} + \left\langle \hat{O}(y)\int_0^1 \mathrm{d}z \mathrm{e}^{z\hat{A}}\hat{B}\mathrm{e}^{-z\hat{A}}\right\rangle. \tag{A7}$$

Let us focus on the last term, which, by virtue of (A4), contains expressions of this sort:

$$\langle\hat{O}(y)\hat{X}'(z,t,\mathbf{x})\rangle_{\mathrm{LE}} \equiv \langle\hat{O}(y)\mathrm{e}^{z\hat{A}}\hat{X}(t,\mathbf{x})\mathrm{e}^{-z\hat{A}}\rangle_{\mathrm{LE}},$$

where $\hat{X}$ stands for components of either $\hat{T}$ or $\hat{\mathcal{S}}$ or $\partial_0\hat{\mathcal{S}}$. From the identity

$$\langle\hat{O}(y)\hat{X}'(z,t,\mathbf{x})\rangle_{\mathrm{LE}} = \int_{-\infty}^t \mathrm{d}\tau\langle\hat{O}(y)\partial_\tau\hat{X}'(z,\tau,\mathbf{x})\rangle_{\mathrm{LE}} + \lim_{\tau\to-\infty}\langle\hat{O}(y)\hat{X}'(z,\tau,\mathbf{x})\rangle_{\mathrm{LE}},$$

and the observation that correlations vanish for very distant times (check footnote [4]), one obtains

$$\langle\hat{O}(y)\hat{X}'(z,t,\mathbf{x})\rangle_{\mathrm{LE}} = \int_{-\infty}^t \mathrm{d}\tau\langle\hat{O}(y)\partial_\tau\hat{X}'(z,\tau,\mathbf{x})\rangle_{\mathrm{LE}} + \lim_{\tau\to-\infty}\langle\hat{O}(y)\rangle_{\mathrm{LE}}\langle\hat{X}(\tau,\mathbf{x})\rangle_{\mathrm{LE}}, \tag{A8}$$

where we have also taken advantage of the commutation between $\exp[\hat{A}]$ and $\exp[\pm z\hat{A}]$.

We now approximate [9] the local equilibrium density operator with the nearest equilibrium operator $\hat{\rho}_0$ in Eq. (26), which also implies that

$$\hat{A} \simeq -\hat{H}/T,$$

where $\hat{H}$ is the Hamiltonian operator (which ought to exist given the chosen boundary conditions). The straightforward consequence of this approximation is that the second term on the right-hand side in Eq. (A8) can be written as

$$\langle\hat{X}(-\infty,\mathbf{x})\rangle_{\mathrm{LE}} \simeq \langle\hat{X}(-\infty,\mathbf{x})\rangle_0 = \langle\hat{X}(t,\mathbf{x})\rangle_0$$

because the mean value is stationary under the equilibrium distribution. Therefore, Eq. (A8) can be approximated as

$$\langle\hat{O}(y)\hat{X}'(z,t,\mathbf{x})\rangle_{\mathrm{LE}} \simeq \int_{-\infty}^t \mathrm{d}\tau\langle\hat{O}(y)\partial_\tau\hat{X}'(z,\tau,\mathbf{x})\rangle_0 + \langle\hat{O}(y)\rangle_0\langle\hat{X}(t,\mathbf{x})\rangle_0, \tag{A9}$$

and (A7) as

$$\langle\hat{O}(y)\rangle \simeq (1-\langle\hat{B}\rangle_0)\langle\hat{O}(y)\rangle_0 + \int_0^1 \mathrm{d}z\langle\hat{O}(y)\mathrm{e}^{-z\hat{H}/T}\hat{B}\mathrm{e}^{z\hat{H}/T}\rangle_0. \tag{A10}$$

Once integrated, the second term in (A9) gives rise to a term which cancels out $\langle\hat{B}\rangle_0\langle\hat{O}(y)\rangle_0$ exactly in the equation above, which then becomes

$$\langle\hat{O}(y)\rangle \simeq \langle\hat{O}(y)\rangle_0 + \int_0^1 \mathrm{d}z\int_{-\infty}^t \mathrm{d}\tau\langle\hat{O}(y)\partial_\tau\hat{X}'(z,\tau,\mathbf{x})\rangle_0. \tag{A11}$$

Let us now integrate the last term on the right-hand side in z:

$$\int_0^1 \mathrm{d}z\int_{-\infty}^t \mathrm{d}\tau\langle\hat{O}(y)\partial_\tau\hat{X}'(z,\tau,\mathbf{x})\rangle_0 = \frac{1}{\bar{\beta}}\int_0^{\bar{\beta}}\mathrm{d}u\int_{-\infty}^t \mathrm{d}\tau\langle\hat{O}(y)\partial_\tau\mathrm{e}^{-u\hat{H}}\hat{X}(\tau,\mathbf{x})\mathrm{e}^{u\hat{H}}\rangle_0,$$

where $\bar{\beta} = 1/T$ and $\bar{\beta}z = u$. As $\hat{H}$ is the generator of time translations,

$$\begin{aligned}
\frac{1}{\bar{\beta}}\int_0^{\bar{\beta}}\mathrm{d}u\int_{-\infty}^t \mathrm{d}\tau\langle\hat{O}(y)\partial_\tau\mathrm{e}^{-u\hat{H}}\hat{X}(\tau,\mathbf{x})\mathrm{e}^{u\hat{H}}\rangle_0
&= \frac{1}{\bar{\beta}}\int_0^{\bar{\beta}}\mathrm{d}u\int_{-\infty}^t \mathrm{d}\tau\langle\hat{O}(y)\partial_\tau\hat{X}(\tau+iu,\mathbf{x})\rangle_0 \\
&= \frac{1}{i\bar{\beta}}\int_0^{\bar{\beta}}\mathrm{d}u\int_{-\infty}^t \mathrm{d}\tau\langle\hat{O}(y)\frac{\partial}{\partial u}\hat{X}(\tau+iu,\mathbf{x})\rangle_0 \\
&= \frac{1}{i\bar{\beta}}\int_0^{\bar{\beta}}\mathrm{d}u\int_{-\infty}^t \mathrm{d}\tau\frac{\partial}{\partial u}(\langle\hat{O}(y)\hat{X}(\tau+iu,\mathbf{x})\rangle_0) \\
&= \frac{1}{i\bar{\beta}}\int_{-\infty}^t \mathrm{d}\tau\int_0^{\bar{\beta}}\mathrm{d}u\frac{\partial}{\partial u}(\langle\hat{O}(y)\hat{X}(\tau+iu,\mathbf{x})\rangle_0) \\
&= \frac{1}{i\bar{\beta}}\int_{-\infty}^t(\langle\hat{O}(y)\hat{X}(\tau+i\bar{\beta},\mathbf{x})\rangle_0 - \langle\hat{O}(y)\hat{X}(\tau,\mathbf{x})\rangle_0).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\langle\hat{O}(y)\hat{X}(\tau+i\bar{\beta},\mathbf{x})\rangle_0 &= \mathrm{tr}(\hat{\rho}_0\hat{O}(y)\mathrm{e}^{-\bar{\beta}\hat{H}}\hat{X}(\tau,\mathbf{x})\mathrm{e}^{+\bar{\beta}\hat{H}}) \\
&= \frac{1}{Z_0}\mathrm{tr}(\mathrm{e}^{-\bar{\beta}\hat{H}}\hat{O}(y)\mathrm{e}^{-\bar{\beta}\hat{H}}\hat{X}(\tau,\mathbf{x})\mathrm{e}^{\bar{\beta}\hat{H}}) \\
&= \frac{1}{Z_0}\mathrm{tr}(\hat{O}(y)\mathrm{e}^{-\bar{\beta}\hat{H}}\hat{X}(\tau,\mathbf{x})) \\
&= \mathrm{tr}(\hat{X}(\tau,\mathbf{x}))\hat{\rho}_0\hat{O}(y)) \\
&= \langle\hat{X}(\tau,\mathbf{x})\hat{O}(y)\rangle_0.
\end{aligned}$$

Hence, putting the last three equations together, we have

$$\int_0^1 \mathrm{d}z \int_{-\infty}^{t} \mathrm{d}\tau \langle \hat{O}(y) \partial_\tau \hat{X}'(z,\tau,\mathbf{x})\rangle_0 = \frac{1}{i\bar{\beta}} \int_{-\infty}^{t} \mathrm{d}\tau \langle [\hat{X}(\tau,\mathbf{x}), \hat{O}(y)]\rangle_0. \tag{A12}$$

Substituting now $\hat{X}$ with its specific operators, Eq. (A11) can be expanded as

$$\begin{aligned}
\delta\langle\hat{O}(y)\rangle &= \langle\hat{O}(y)\rangle - \langle\hat{O}(y)\rangle_0 \\
&\simeq \lim_{\varepsilon\to 0} \frac{1}{i\bar{\beta}} \int_{-\infty}^{t'} \mathrm{d}t e^{\varepsilon(t-t')} \int_{-\infty}^{t} \mathrm{d}\tau \int \mathrm{d}^3\mathrm{x} \langle [\hat{T}^{\mu\nu}(\tau,\mathbf{x}), \hat{O}(y)]\rangle_0 \partial_\mu \delta\beta_\nu(x) \\
&\quad - \frac{1}{2} \lim_{\varepsilon\to 0} \frac{1}{i\bar{\beta}} \int_{-\infty}^{t'} \mathrm{d}t e^{\varepsilon(t-t')} \frac{\partial}{\partial t} \int_{-\infty}^{t} \mathrm{d}\tau \int \mathrm{d}^3\mathrm{x} \langle [\hat{\mathcal{S}}^{0,\mu\nu}(\tau,\mathbf{x}), \hat{O}(y)]\rangle_0 \delta\omega_{\mu\nu}(x) \\
&= \lim_{\varepsilon\to 0} \frac{1}{i\bar{\beta}} \int_{-\infty}^{t'} \mathrm{d}t e^{\varepsilon(t-t')} \int_{-\infty}^{t} \mathrm{d}\tau \int \mathrm{d}^3\mathrm{x} \langle [\hat{T}^{\mu\nu}(\tau,\mathbf{x}), \hat{O}(y)]\rangle_0 \partial_\mu \delta\beta_\nu(x) \\
&\quad - \frac{1}{2} \lim_{\varepsilon\to 0} \frac{1}{i\bar{\beta}} \int_{-\infty}^{t'} \mathrm{d}t e^{\varepsilon(t-t')} \int \mathrm{d}^3\mathrm{x} \langle [\hat{\mathcal{S}}^{0,\mu\nu}(t,\mathbf{x}), \hat{O}(y)]\rangle_0 \delta\omega_{\mu\nu}(x) \\
&\quad - \frac{1}{2} \lim_{\varepsilon\to 0} \frac{1}{i\bar{\beta}} \int_{-\infty}^{t'} \mathrm{d}t e^{\varepsilon(t-t')} \int_{-\infty}^{t} \mathrm{d}\tau \int \mathrm{d}^3\mathrm{x} \langle [\hat{\mathcal{S}}^{0,\mu\nu}(\tau,\mathbf{x}), \hat{O}(y)]\rangle_0 \frac{\partial}{\partial t} \delta\omega_{\mu\nu}(x).
\end{aligned} \tag{A13}$$

The first term on the right-hand side of the above equation can be integrated by parts using

$$\begin{aligned}
\int_{-\infty}^{t'} \mathrm{d}t e^{\varepsilon(t-t')} \int_{-\infty}^{t} \mathrm{d}\tau f(\tau) &= \int_{-\infty}^{t'} \mathrm{d}t \frac{\partial}{\partial t}\left(\frac{e^{\varepsilon(t-t')}}{\varepsilon}\right) \int_{-\infty}^{t} \mathrm{d}\tau f(\tau) = \frac{1}{\varepsilon}\int_{-\infty}^{t'} \mathrm{d}\tau f(\tau) - \int_{-\infty}^{t'} \mathrm{d}t \frac{e^{\varepsilon(t-t')}}{\varepsilon} f(t) \\
&= \int_{-\infty}^{t'} \mathrm{d}t \frac{1 - e^{\varepsilon(t-t')}}{\varepsilon} f(t)
\end{aligned}$$

so that Eq. (A13) can finally be written

$$\begin{aligned}
\delta\langle\hat{O}(y)\rangle &= \lim_{\varepsilon\to 0} \frac{1}{i\bar{\beta}} \int_{-\infty}^{t'} \mathrm{d}t \frac{1 - e^{\varepsilon(t-t')}}{\varepsilon} \int \mathrm{d}^3\mathrm{x} \langle [\hat{T}^{\mu\nu}(x), \hat{O}(y)]\rangle_0 \partial_\mu \delta\beta_\nu(x) \\
&\quad - \frac{1}{2} \lim_{\varepsilon\to 0} \frac{1}{i\bar{\beta}} \int_{-\infty}^{t'} \mathrm{d}t e^{\varepsilon(t-t')} \int \mathrm{d}^3\mathrm{x} \langle [\hat{\mathcal{S}}^{0,\mu\nu}(x), \hat{O}(y)]\rangle_0 \delta\omega_{\mu\nu}(x) \\
&\quad - \frac{1}{2} \lim_{\varepsilon\to 0} \frac{1}{i\bar{\beta}} \int_{-\infty}^{t'} \mathrm{d}t e^{\varepsilon(t-t')} \int_{-\infty}^{t} \mathrm{d}\tau \int \mathrm{d}^3\mathrm{x} \langle [\hat{\mathcal{S}}^{0,\mu\nu}(\tau,\mathbf{x}), \hat{O}(y)]\rangle_0 \frac{\partial}{\partial t} \delta\omega_{\mu\nu}(x).
\end{aligned} \tag{A14}$$

Another useful (equivalent) expression of $\delta\langle\hat{O}(y)\rangle$ can be obtained starting from the expression (14) of $\hat{\Upsilon}$, where the continuity equation for angular momentum is used from the beginning. Repeating the same reasoning as above, it can be shown that one gets

$$\begin{aligned}
\delta\langle\hat{O}(y)\rangle &= \lim_{\varepsilon\to 0} \frac{1}{2i\bar{\beta}} \int_{-\infty}^{t'} \mathrm{d}t \frac{1 - e^{\varepsilon(t-t')}}{\varepsilon} \int \mathrm{d}^3\mathrm{x} \langle [\hat{T}_S^{\mu\nu}(x), \hat{O}(y)]\rangle_0 (\partial_\mu \delta\beta_\nu(x) + \partial_\nu \delta\beta_\mu(x)) \\
&\quad + \lim_{\varepsilon\to 0} \frac{1}{2i\bar{\beta}} \int_{-\infty}^{t'} \mathrm{d}t \frac{1 - e^{\varepsilon(t-t')}}{\varepsilon} \int \mathrm{d}^3\mathrm{x} \langle [\hat{T}_A^{\mu\nu}(x), \hat{O}(y)]\rangle_0 (\partial_\mu \delta\beta_\nu(x) - \partial_\nu \delta\beta_\mu(x) + 2\delta\omega_{\mu\nu}(x)) \\
&\quad - \frac{1}{2} \lim_{\varepsilon\to 0} \frac{1}{i\bar{\beta}} \int_{-\infty}^{t'} \mathrm{d}t e^{\varepsilon(t-t')} \int_{-\infty}^{t} \mathrm{d}\tau \int \mathrm{d}^3\mathrm{x} \langle [\hat{\mathcal{S}}^{\lambda,\mu\nu}(\tau,\mathbf{x}), \hat{O}(y)]\rangle_0 \partial_\lambda \delta\omega_{\mu\nu}(x).
\end{aligned} \tag{A15}$$

As we have pointed out, these expressions hold when $\hat{\rho}_0$ has a nonvanishing chemical potential.

APPENDIX B: COMMUTATORS AND DISCRETE SYMMETRIES

We want to study the effect of space inversion and time reversal on the mean value of commutators like

$$\langle [\hat{O}_1^{\mu_1\cdots\mu_m}(t,\mathrm{x}), \hat{O}_2^{\nu_1\cdots\nu_n}(0,\mathbf{0})]\rangle_0,$$

where $\hat{O}_1$ and $\hat{O}_2$ are physical tensor densities of rank m and n, respectively.

The equilibrium density operator $\hat{\rho} = \exp[-\hat{H}/T]/Z$ is symmetric for space-time translations and rotations, as well as time reversal and parity, if the Hamiltonian is itself parity and time-reversal invariant. The symmetry under this class of transformations allows one to simplify the above expression. For any linear unitary transformation $\hat{\mathsf{U}}$ which commutes with $\hat{\rho}$, one has

$$\begin{aligned}
\langle\hat{O}\rangle_0 &= \mathrm{tr}(\hat{\rho}_0\hat{O}) = \mathrm{tr}(\hat{\mathsf{U}}^{-1}\hat{\rho}_0\hat{\mathsf{U}}\,\hat{O}) = \mathrm{tr}(\hat{\rho}_0\hat{\mathsf{U}}\,\hat{O}\,\hat{\mathsf{U}}^{-1}) \\
&= \langle\hat{\mathsf{U}}\,\hat{O}\,\hat{\mathsf{U}}^{-1}\rangle_0.
\end{aligned}$$

Taking $\hat{\mathsf{U}} = \hat{\mathsf{T}}(a)$, with $\hat{\mathsf{T}}(a)$ a general translation operator,

$$\langle[\hat{O}_1^{\mu_1\cdots\mu_n}(t,\mathbf{x}),\hat{O}_2^{\nu_1\cdots\nu_n}(0,\mathbf{0})]\rangle_0 = \langle[\hat{O}_1^{\mu_1\cdots\mu_n}(t+a^0,\mathbf{x}+\mathbf{a}),\hat{O}_2^{\nu_1\cdots\nu_n}(a^0,\mathbf{a})]\rangle_0$$

and so, setting $(a^0,\mathbf{a}) = (-t,-\mathbf{x})$,

$$\langle[\hat{O}_1^{\mu_1\cdots\mu_m}(t,\mathbf{x}),\hat{O}_2^{\nu_1\cdots\nu_n}(0,\mathbf{0})]\rangle_0 = \langle[\hat{O}_1^{\mu_1\cdots\mu_m}(0,\mathbf{0}),\hat{O}_2^{\nu_1\cdots\nu_n}(-t,-\mathbf{x})]\rangle_0.$$

Similarly, for a space inversion,

$$\langle[\hat{O}_1^{\mu_1\cdots\mu_m}(t,\mathbf{x}),\hat{O}_2^{\nu_1\cdots\nu_n}(0,\mathbf{0})]\rangle_0 = (-1)^{n_s+m_s}\langle[\hat{O}_1^{\mu_1\cdots\mu_m}(t,-\mathbf{x}),\hat{O}_2^{\nu_1\cdots\nu_n}(0,\mathbf{0})]\rangle_0,$$

where m_s and n_s are the number of space indices among $\mu_1,\cdots\mu_m$ and $\nu_1,\cdots\nu_n$, respectively.

The time-reversal operator $\hat{\Theta}$ is antiunitary; thus a point-dependent physical scalar operator $\hat{A}(t,\mathbf{x})$ transforms as follows:

$$\hat{\Theta}\hat{A}(t,\mathbf{x})\hat{\Theta}^{-1} = \hat{A}^\dagger(-t,\mathbf{x}).$$

Hence, for commutators

$$\hat{\Theta}[\hat{A}(t,\mathbf{x}),\hat{B}(t,\mathbf{x})]\hat{\Theta}^{-1} = [\hat{B}^\dagger(-t,\mathbf{x}),\hat{A}^\dagger(-t,\mathbf{x})].$$

Then, for Hermitian operators, what gets changed is the order of the operators besides their time argument. For tensor Hermitian observables and a time-reversal symmetric Hamiltonian, one obtains

$$\langle[\hat{O}_1^{\mu_1\cdots\mu_m}(t,\mathbf{x}),\hat{O}_2^{\nu_1\cdots\nu_n}(0,\mathbf{0})]\rangle_0 = (-1)^{m_0+n_0}\langle[\hat{O}_2^{\nu_1\cdots\nu_n}(0,\mathbf{0}),\hat{O}_1^{\mu_1\cdots\mu_m}(-t,\mathbf{x})]\rangle_0,$$

where m_0 and n_0 are the number of time indices among $\mu_1,\cdots\mu_m$ and $\nu_1,\cdots\nu_n$, respectively.

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