

Tunneling during quantum collapse in AdS spacetimeCenalo Vaz^{1,*} and Kinjalk Lochan^{2,†}¹*Department of Physics, University of Cincinnati, Cincinnati, Ohio 45221-0011, USA*²*Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai 400 005, India*

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We extend previous results on the reflection and transmission of self-gravitating dust shells across the apparent horizon during quantum dust collapse to nonmarginally-bound dust collapse in arbitrary dimensions with a negative cosmological constant. We show that the Hawking temperature is independent of the energy function and that the wave functional describing the collapse is well behaved at the Hawking-Page transition point. Thermal radiation from the apparent horizon appears as a generic result of nonmarginal collapse in AdS spacetime owing to the singular structure of the Hamiltonian constraint at the apparent horizon.

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I. INTRODUCTION

A fundamental expectation of a quantum theory of gravity is that it will cure the problems that plague classical general relativity. One hopes, for example, that singularities get resolved in quantum gravity [1–5], that quantum gravity will provide the theoretical foundation for cosmic censorship [6] and that it will give a better understanding of the relationship, already predicted on the semiclassical level, between gravity and thermodynamics [7–10].

In the absence of a generally agreed upon framework for such a theory, a useful approach is to quantize simplified classical gravitational models using canonical techniques, in the expectation that this will lead to new ways to look at some of the issues raised above while, at the same time, pointing to what one may expect out of the full theory. In this spirit, we recently developed [11] a novel approach to Hawking evaporation taking place during the collapse of a self-gravitating dust ball. This approach, based on an exact canonical quantization of the nonrotating, marginally-bound gravity-dust system [12], exploited the matching conditions that must be satisfied at the apparent horizon by the wave functionals describing the collapse and differs from the traditional approach in which a preexisting black hole is imagined to be surrounded by a tenuous field and the Bogoliubov transformation of the field operators is computed in the black hole background [13–15].

The geometrodynamical constraints of all LeMaître-Tolman-Bondi (LTB) models in any dimension, with or without a cosmological constant, are expressible in terms of a canonical chart consisting of the area radius R , the dust proper time τ , the mass function F , and their conjugate momenta [16,17]. After a series of canonical transformations in the spirit of Kuchař [18], the Hamiltonian constraint can be shown to yield a Klein-Gordon-like Wheeler-DeWitt equation for the wave functional. This

equation can be solved by quadrature and, in the simplest cases, closed form solutions can be obtained after regularization is implemented on a lattice in a self-consistent manner. Self-consistency requires that the lattice decomposition is compatible with the diffeomorphism constraint [19]. In these models, the dust ball may be viewed as being made up of shells and the wave functional is described as the continuum limit of an infinite product over the shell wave functions.

For the special case of marginal collapse with a vanishing cosmological constant in $3 + 1$ dimensions, the Wheeler-DeWitt equation can be solved explicitly. We showed in Ref. [11] that matching the shell wave functions across the apparent horizon requires ingoing modes in the exterior to be accompanied by outgoing modes in the interior and, vice versa, ingoing modes in the interior to be accompanied by outgoing modes in the exterior. In each case the relative amplitude of the outgoing wave is suppressed by the square root of the Boltzmann factor at a “Hawking” temperature given by $T_H = (4\pi F)^{-1}$, where F represents twice the mass contained within the shell. Thus the temperature varies from shell to shell, decreasing from the interior to the exterior, but it has the Hawking form for any given shell.

Two separate solutions are possible: one in which there is a flow of matter toward the apparent horizon both in the exterior and in the interior, and another in which the flow is away from the horizon, again in both regions. Matter undergoing continual collapse across the apparent horizon is described by a linear superposition of these solutions and then, because ingoing waves in the interior are accompanied by outgoing waves in the exterior, the horizon appears, to the external observer with no access to the interior, to possess a reflectivity given by the Boltzmann factor at the above Hawking temperature. A different interpretation is also possible when the entire shell wave functions are taken into account. Ingoing waves in the exterior must be accompanied by outgoing waves in the interior, whose amplitude is also suppressed by the square root of the Boltzmann factor at the Hawking temperature.

*Cenalo.Vaz@UC.Edu

†kinjalk@tifr.res.in

We showed that the transmittance of the horizon is unity, whether for waves incident from the exterior or the interior. Thus this outgoing wave in the interior passes through the apparent horizon unhindered but, because its amplitude is suppressed by the Boltzmann factor at the Hawking temperature relative to the ingoing modes in the exterior, the emission probability of the horizon is given by the same factor. The net effect is therefore reminiscent of the quasiclassical tunneling of particles through the horizon in the semiclassical theory [20–24].

The solutions just described relied on explicit solutions for the shell wave functions. These are available only in the case of the marginally bound models with a vanishing cosmological constant. Our aim here is to extend these results to nonmarginally-bound LTB models in arbitrary dimension with a negative cosmological constant. No explicit solutions can be given in this case. Nevertheless, we will show that the results mentioned in the previous paragraphs are indeed generic and that they are a consequence only of the essential singularity of the Klein-Gordon equation for shells at the apparent horizon. We will then discuss how diffeomorphism invariant wave functionals may be reconstructed out of the shell wave functions. Hawking radiation from the apparent horizon then appears as a consequence of the generic form of the Wheeler-DeWitt equation describing dust collapse and not of any particular solution discussed in the earlier work. This will provide a novel way to compute the entropy of the final state black hole.

The plan of this paper is as follows. In Sec. II we recall some key results for classical dust collapse with a negative cosmological constant in arbitrary dimensions. In Sec. III we present the exact wave functional that is factorizable on a lattice and solves the Wheeler-DeWitt equation for dust collapse. The (collapse) wave functional can be thought of as an infinite product of shell wave functions, each occupying a lattice site. Matching shell wave functions across the horizon by analytic continuation in Sec. IV, we argue that ingoing waves in one region must be accompanied by outgoing waves in the other. We superpose the two solutions to conserve the flux of shells across the horizon and then reconstruct the wave functional from the shell wave functions by going to the continuum limit. A consequence of matching shells across the apparent horizon is that the amplitude for outgoing waves relative to ingoing ones is given by $e^{-S/2}$, where S is the Bekenstein-Hawking entropy of the final state black hole. We close with a brief discussion of our results in Sec. V.

II. NONMARGINAL DUST COLLAPSE WITH $\Lambda \neq 0$ IN d DIMENSIONS

A. The classical models

Let us begin by briefly recalling some pertinent facts about spherical dust collapse in the presence of a negative cosmological constant (see, for example, Ref. [25] for details). The LTB models describe self-gravitating time-like dust whose

energy momentum tensor is $T_{\mu\nu} = \varepsilon(x)U_\mu U_\nu$, where $U^\mu(\tau, \rho)$ is the four velocity of the dust particles which are labeled by the ρ and with proper time τ . The line element can be taken to be

$$ds^2 = -g_{\mu\nu}dx^\mu dx^\nu = d\tau^2 - \frac{R^2(\tau, \rho)}{1 + 2E(\rho)}d\rho^2 + R^2(\tau, \rho)d\Omega_n^2, \quad (1)$$

where $R(\tau, \rho)$ is the area radius, $E(\rho)$ is an arbitrary function of the shell label coordinate, called the energy function, and Ω_n is the $n = d - 2$ dimensional solid angle. Einstein's equations in the presence of a negative cosmological constant, which we call $-\Lambda$,

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = -\kappa_d T_{\mu\nu} \quad (2)$$

yield one dynamical equation for the area radius,

$$R^{*2} = 2E(\rho) + \frac{F(\rho)}{R^{n-1}} - \frac{2\Lambda R^2}{n(n+1)}, \quad (3)$$

where the star refers to a derivative with respect to τ and $F(\rho)$ is a second arbitrary function of the shell label coordinate, called the mass function. Above, κ_d is given in terms of the d -dimensional gravitational constant G_d as $\kappa_d = 8\pi G_d$.

One also finds the energy density,

$$\varepsilon(\tau, \rho) = \frac{n}{2\kappa_d} \frac{\tilde{F}}{R^n \bar{R}}, \quad (4)$$

in terms of F , where the tilde refers to a derivative with respect to the label coordinate, ρ . Specific models are obtained by making choices of the mass and energy functions. For the solutions of (3) to describe gravitational collapse (as opposed to an expansion) one must impose the additional condition that $R^*(t, r) < 0$. The solutions to (3) have been explicitly given in Ref. [25] and analyzed in detail for the marginally-bound case, $E(\rho) = 0$. Each shell reaches a zero area radius, $R(\tau, \rho) = 0$, in a finite proper time, $\tau = \tau_s(\rho)$, which leads to a curvature singularity. Thus the proper time parameter lies in the interval $(-\infty, \tau_s]$. In general both naked singularities and black hole end states can form.

Trapped surfaces occur when

$$\mathcal{F} \stackrel{\text{def}}{=} 1 - \frac{F}{R^{n-1}} + \frac{2\Lambda R^2}{n(n+1)} = 0, \quad (5)$$

which determines the physical radius, R_h , of the apparent horizon. \mathcal{F} is positive outside, i.e., when $R > R_h$, and negative inside, when $R < R_h$.

B. The canonical formulation

To develop a canonical formulation of the LTB models, one begins with the spherically symmetric Arnowitt-Deser-Misner (ADM) metric

$$ds^2 = N^2 dt^2 - L^2(dr + N^r dt)^2 - R^2 d\Omega_n^2, \quad (6)$$

where $N(t, r)$ and $N^r(t, r)$ are, respectively, the lapse and shift functions and the Einstein-Hilbert action for a self-gravitating dust ball

$$S_{\text{EH}} = \frac{1}{2\kappa_d} \int d^d x \sqrt{-g} ({}^d\mathcal{R} + 2\Lambda) - \frac{1}{2} \int d^d x \sqrt{-g} (g_{\alpha\beta} U^\alpha U^\beta + 1), \quad (7)$$

where $U_\alpha = -\tau_{,\alpha}$ for nonrotating dust, where τ is the dust proper time. The phase space consists of the dust proper time, $\tau(t, r)$, the area radius, $R(t, r)$, the radial function, $L(t, r)$, and their conjugate momenta, respectively $P_\tau(t, r)$, $P_R(t, r)$, and $P_L(t, r)$.

When the ADM metric is embedded in the spacetime described by (1) it becomes possible, through a series of canonical transformations described in detail in Ref. [17], to reexpress the canonical constraints in terms of a new canonical chart consisting of the dust proper time, the area radius and the mass density function, $\Gamma(r)$, defined by

$$F(r) = \frac{2\kappa_d}{n\Omega_n} \left[M_0 + \int_0^r \Gamma(r') dr' \right] \quad (8)$$

and new conjugate momenta, $P_\tau(t, r)$, $\bar{P}_R(t, r)$, and $P_\Gamma(t, r)$. The energy function is expressible in this chart as

$$\frac{1}{\sqrt{1+2E}} = \frac{2P_\tau}{\Gamma}, \quad (9)$$

and the transformations also absorb a boundary term, which is present in the original chart. The constraints for the dust-gravity system in any dimension are

$$\mathcal{H}^s = P_\tau^2 + \mathcal{F} \bar{P}_R^2 - \frac{\Gamma^2}{\mathcal{F}} \approx 0, \quad (10)$$

$$\mathcal{H}_r = \tau' P_\tau + R' \bar{P}_R - \Gamma P'_\Gamma \approx 0,$$

where the prime denotes a derivative with respect to the ADM label coordinate, r . The Hamiltonian constraint in (10) will be seen to contain no derivative terms, which makes it easier to quantize. However, the Poisson brackets of the Hamiltonian with itself vanishes, indicating that the Hamiltonian constraint does not generate hypersurface deformations. Rather, the transformations generated by the Hamiltonian constraint act along the dust flow lines.

Of importance in what follows will be the following relationship between the dust proper time and the remaining canonical variables (see, for example, Ref. [26]):

$$\tau' = \frac{2P'_\Gamma}{a} \pm R' \frac{\sqrt{1-a^2\mathcal{F}}}{a\mathcal{F}}, \quad (11)$$

where $a = 1/\sqrt{1+2E}$.¹ The positive sign describes a collapsing dust cloud in the exterior and an expanding dust cloud in the interior whereas the negative sign

describes an expanding cloud in the exterior and a collapsing cloud in the interior. Integrating on a hypersurface of constant t we have the formal solution

$$\tau = 2 \int_{t=\text{const}} \frac{dP_\Gamma}{a} \pm \int_{t=\text{const}}^R dR \frac{\sqrt{1-a^2\mathcal{F}}}{a\mathcal{F}} + \tilde{\tau}(t), \quad (12)$$

where $\tilde{\tau}(t)$ is undetermined. This integral can be difficult to solve for arbitrary (r dependent) mass and energy functions, but when they are both constant beyond some boundary, r_b , then the solution may be expressed as

$$a\tau = 2P_\Gamma \pm \int^R dR \frac{\sqrt{1-a^2\mathcal{F}}}{\mathcal{F}} + \tilde{\tau}(t). \quad (13)$$

In this case we are dealing with the static Schwarzschild-AdS geometry for which $2P_\Gamma$ may be associated with the Killing time [18], so (13) gives the relationship between Schwarzschild-AdS and Painlevé-Gullstrand time [27]. Solutions in the absence of a cosmological constant have been given in Refs. [19,26].

III. QUANTUM STATES IN A LATTICE DECOMPOSITION

When Dirac's quantization condition is used to raise the classical constraints to operator constraints, which act on a wave functional, the Hamiltonian constraint turns into the Wheeler-DeWitt equation and the momentum constraint imposes spatial diffeomorphism invariance on the wave functional. One sees that the second is solved automatically by a wave functional of the form

$$\Psi[\tau, R, F] = U \left[\int_{-\infty}^{\infty} dr \Gamma(r) \mathcal{W}(\tau(r), R(r), F(r)) \right] \quad (14)$$

provided that \mathcal{W} contains no explicit dependence on r and where $U: \mathbb{R} \rightarrow \mathbb{C}$ is an arbitrary differentiable function of its argument.

The wave functional is factorizable on a lattice placed on the real line (e.g., see Ref. [11]) if U is chosen to be the exponential map for then, taking the lattice spacing to be σ , (14) can be written as

$$\Psi[\tau, R, F] = \lim_{\sigma \rightarrow 0} \prod_i \psi_i(\tau_i, R_i, F_i), \quad (15)$$

where

$$\psi_i(\tau_i, R_i, F_i) = e^{\sigma \Gamma_i \mathcal{W}(\tau_i, R_i, F_i)} \quad (16)$$

and we have used $X_i = X(r_i)$. Thus we can think of $\Psi[\tau, R, F]$ as an infinite product of shell wave functions, each occupying a lattice site. Each shell wave function satisfies

$$\left[\hbar^2 \left(\frac{\partial^2}{\partial \tau_i^2} + \mathcal{F}_i \frac{\partial^2}{\partial R_i^2} + A_i \frac{\partial}{\partial R_i} + B_i \right) + \frac{\sigma^2 \Gamma_i^2}{\mathcal{F}_i} \right] \psi_i = 0, \quad (17)$$

where $A_i = A(R_i, F_i)$ and $B_i = B(R_i, F_i)$ are functions capturing the factor ordering ambiguities that are always present in the canonical approach. They can be uniquely

¹Einstein's equations guarantee that $1 - a^2\mathcal{F} > 0$ for $0 < a < 1$, which corresponds to the case $E > 0$.

determined by requiring the above equation to be independent of the lattice spacing; one finds the general positive energy solutions [17]

$$\psi_i = e^{\omega_i b_i} \times \exp \left\{ -\frac{i\omega_i}{\hbar} \left[a_i \tau_i \pm \int^{R_i} dR_i \frac{\sqrt{1 - a_i^2 \mathcal{F}_i}}{\mathcal{F}_i} \right] \right\}, \quad (18)$$

where $\omega_i = \sigma \Gamma_i / 2$ and the factor $e^{\omega_i b_i}$ amounts to a normalization. Note that shell “ i ” crosses the apparent horizon when $\mathcal{F}_i = 0$, which is an essential singularity of the wave equation. From the shell wave functions in (18) one reconstructs the wave functionals with the ansatz (14) and $U = \exp$ as

$$\Psi[\tau, R, \Gamma] = e^{\frac{i}{2} \int dr \Gamma b(F(r))} \exp \left\{ -\frac{i}{2\hbar} \int dr \Gamma \left[a(F(r)) \tau \pm \int_{r=\text{const}}^R dR \frac{\sqrt{1 - a^2(F(r)) \mathcal{F}}}{\mathcal{F}} \right] \right\}, \quad (19)$$

where we have set $a(r) = a(F(r))$ and $b(r) = b(F(r))$ as is required for diffeomorphism invariance.

IV. TUNNELING

The wave functions of the previous section are defined in the interior as well as the exterior of the apparent horizon, but the Wheeler-DeWitt equation has an essential singularity at the apparent horizon along the path of integration. In order to match interior to exterior solutions, it is necessary to deform the path in the complex R -plane. The direction of the deformation is chosen so that positive energy solutions decay. While this deformed path does not correspond to the trajectory of any classical particle it represents a tunneling of s -waves across the gravitational barrier represented by the apparent horizon. This is analogous to the quasiclassical tunneling approach employed in semiclassical analyses [20–24].

A. Shell wave functions

From the expression for the phase in (18),

$$\mathcal{W}_i^{(\pm)} = \frac{\omega_i}{\hbar} \left[a_i \tau_i \pm \int^{R_i} dR_i \frac{\sqrt{1 - a_i^2 \mathcal{F}_i}}{\mathcal{F}_i} \right], \quad (20)$$

the phase velocity of the i th shell wave function is given by

$$\dot{R}_i = \mp \frac{a_i \mathcal{F}_i}{\sqrt{1 - a_i^2 \mathcal{F}_i}}. \quad (21)$$

Thus the positive sign in (20) describes ingoing waves in the exterior ($\mathcal{F}_i > 0$), whereas it describes outgoing waves in the interior ($\mathcal{F}_i < 0$) and, likewise, the negative sign describes outgoing waves in the exterior and ingoing waves in the interior.

A closed form solution for the integrals appearing in (19) and (12) cannot be given when the mass function and/or the energy function and/or the cosmological constant are nonvanishing. We may however analyze their properties near the apparent horizon in the following way. Noting that $\mathcal{F}_i = 0$, equivalently $R_i = R_{i,h}$, is a singularity of the integral appearing in the phase, \mathcal{W}_i , we define the integral by analytically continuing to the complex plane and deforming the integration path so as to go around the pole at $R_{i,h}$ in a semicircle of radius ϵ drawn in the upper half plane. Let L_ϵ denote the deformed path and let S_ϵ denote the semicircle of radius ϵ around $R_{i,h}$, then

$$\int^{R_i} dR_i \frac{\sqrt{1 - a_i^2 \mathcal{F}_i}}{\mathcal{F}_i} \stackrel{\text{def}}{=} \lim_{\epsilon \rightarrow 0} \int_{L_\epsilon}^{R_i} dR_i \frac{\sqrt{1 - a_i^2 \mathcal{F}_i}}{\mathcal{F}_i}. \quad (22)$$

Performing the integration from left to right,² for $R_i = R_{i,h} + \epsilon$ we have

$$\begin{aligned} \int_{L_\epsilon}^{R_{i,h} + \epsilon} dR_i \frac{\sqrt{1 - a_i^2 \mathcal{F}_i}}{\mathcal{F}_i} &= \int_{L_\epsilon}^{R_{i,h} - \epsilon} dR_i \frac{\sqrt{1 - a_i^2 \mathcal{F}_i}}{\mathcal{F}_i} \\ &+ \int_{S_\epsilon} dR_i \frac{\sqrt{1 - a_i^2 \mathcal{F}_i}}{\mathcal{F}_i} \end{aligned} \quad (23)$$

and, for the integral over the semicircle, a Laurent series expansion about $\mathcal{F}_i = 0$ gives to lowest order

$$\int_{S_\epsilon} dR_i \frac{\sqrt{1 - a_i^2 \mathcal{F}_i}}{\mathcal{F}_i} = \frac{1}{\mathcal{F}'_i(R_{i,h})} \int_{S_\epsilon} \frac{dR_i}{R_i - R_{i,h}}. \quad (24)$$

This integral is half the integral over a complete circle taken in a clockwise manner,

$$\begin{aligned} \int_{S_\epsilon} dR_i \frac{\sqrt{1 - a_i^2 \mathcal{F}_i}}{\mathcal{F}_i} &= \frac{1}{2\mathcal{F}'_i(R_{i,h})} \oint_{C_\epsilon} \frac{dR_i}{R_i - R_{i,h}} \\ &= -\frac{i\pi}{2g_{i,h}}, \end{aligned} \quad (25)$$

where $2g_{i,h} = \mathcal{F}'_i(R_{i,h})$ is the surface gravity of the horizon. Therefore we find

$$\int_{L_\epsilon}^{R_{i,h} + \epsilon} dR_i \frac{\sqrt{1 - a_i^2 \mathcal{F}_i}}{\mathcal{F}_i} = \int_{L_\epsilon}^{R_{i,h} - \epsilon} dR_i \frac{\sqrt{1 - a_i^2 \mathcal{F}_i}}{\mathcal{F}_i} - \frac{i\pi}{2g_{i,h}}. \quad (26)$$

The expression defining the proper time in (12) involves a similar integral, but taken over a spatial slice. Even so, the same argument can be made for this integral (see Refs. [28,29] for an analogous argument in the semiclassical context). If we assume, moreover, that $a(F(r))$ and $P_\Gamma(r)$ are both regular across the horizon, the net result is

²The same result is obtained if the integration is performed from right to left.

that the phases in the exterior get matched to the phases in the interior by the addition of a constant imaginary term,

$$\mathcal{W}_{\text{out}}^{(\pm)}(\tau_i, R_i, F_i) = \mathcal{W}_{\text{in}}^{(\pm)}(\tau_i, R_i, F_i) \mp \frac{i\pi\omega_i}{\hbar g_{i,h}}. \quad (27)$$

Thus we can give two independent solutions with support everywhere in the spacetime: an ingoing wave in the exterior that is matched to an outgoing wave in the interior

$$\psi_i^{(1)}(\tau_i, R_i, F_i) = \begin{cases} e^{\omega_i b_i} \times \exp\left[-\frac{i\omega_i}{\hbar} \left[a_i \tau_i + \int^{R_i} dR_i \frac{\sqrt{1-a_i^2 \mathcal{F}_i}}{\mathcal{F}_i} \right] \right] & \mathcal{F}_i > 0 \\ e^{-\frac{\pi\omega_i}{\hbar g_{i,h}}} \times e^{\omega_i b_i} \times \exp\left[-\frac{i\omega_i}{\hbar} \left[a_i \tau_i + \int^{R_i} dR_i \frac{\sqrt{1-a_i^2 \mathcal{F}_i}}{\mathcal{F}_i} \right] \right] & \mathcal{F}_i < 0 \end{cases}, \quad (28)$$

and an outgoing wave in the exterior that is matched to an ingoing wave in the interior according to

$$\psi_i^{(2)}(\tau_i, R_i, F_i) = \begin{cases} e^{-\frac{\pi\omega_i}{\hbar g_{i,h}}} \times e^{\omega_i b_i} \times \exp\left[-\frac{i\omega_i}{\hbar} \left[a_i \tau_i - \int^{R_i} dR_i \frac{\sqrt{1-a_i^2 \mathcal{F}_i}}{\mathcal{F}_i} \right] \right] & \mathcal{F}_i > 0 \\ e^{\omega_i b_i} \times \exp\left[-\frac{i\omega_i}{\hbar} \left[a_i \tau_i - \int^{R_i} dR_i \frac{\sqrt{1-a_i^2 \mathcal{F}_i}}{\mathcal{F}_i} \right] \right] & \mathcal{F}_i < 0 \end{cases}. \quad (29)$$

The first of these solutions represents a flow towards the apparent horizon both in the exterior as well as in the interior whereas the second represents a flow away from the apparent horizon in both regions. While each solution is self-consistent, neither wave function accurately reflects the physical situation one expects from semiclassical collapse, in which an ingoing shell proceeds all the way to the center and the horizon emits thermal radiation into the exterior at the Hawking temperature.

To recover this picture we consider a linear superposition of the two wave functions

$$\psi_i = \psi_i^{(1)} + A_i \psi_i^{(2)}, \quad (30)$$

where A_i are complex valued constants. Following Ref. [11], we fix these constants by requiring that the current density is constant across the horizon. This implies that $|A_i|^2 = 1$ and we take $A_i = 1$ for every shell, which gives an absorption probability of unity for a shell to cross the apparent horizon from the exterior. We therefore have

$$\psi_i = \begin{cases} e^{\omega_i b_i} \times \exp\left[-\frac{i\omega_i}{\hbar} \left[a_i \tau_i + \int^{R_i} dR_i \frac{\sqrt{1-a_i^2 \mathcal{F}_i}}{\mathcal{F}_i} \right] \right] + e^{-\frac{\pi\omega_i}{\hbar g_{i,h}}} \times e^{\omega_i b_i} \times \exp\left[-\frac{i\omega_i}{\hbar} \left[a_i \tau_i - \int^{R_i} dR_i \frac{\sqrt{1-a_i^2 \mathcal{F}_i}}{\mathcal{F}_i} \right] \right] & \mathcal{F}_i > 0 \\ e^{-\frac{\pi\omega_i}{\hbar g_{i,h}}} \times e^{\omega_i b_i} \times \exp\left[-\frac{i\omega_i}{\hbar} \left[a_i \tau_i + \int^{R_i} dR_i \frac{\sqrt{1-a_i^2 \mathcal{F}_i}}{\mathcal{F}_i} \right] \right] + e^{\omega_i b_i} \times \exp\left[-\frac{i\omega_i}{\hbar} \left[a_i \tau_i - \int^{R_i} dR_i \frac{\sqrt{1-a_i^2 \mathcal{F}_i}}{\mathcal{F}_i} \right] \right] & \mathcal{F}_i < 0 \end{cases}. \quad (31)$$

The second term in the expression for ψ_i in the exterior ($\mathcal{F}_i > 0$) is an outgoing wave that, to an external observer, would represent a reflection with relative probability

$$\frac{P_{\text{ref},i}}{P_{\text{abs},i}} = e^{-\frac{2\pi\omega_i}{\hbar g_{i,h}}}. \quad (32)$$

This is precisely the Boltzmann factor for the shell at temperature $T_{i,H} = \frac{\hbar g_{i,h}}{2\pi k_B}$, where $g_{i,h}$ is the surface gravity of the apparent horizon.

This reflected piece is a purely quantum effect, necessitated by the existence of an ingoing wave in the interior, i.e., by requiring the continued collapse of the shell beyond its apparent horizon. This continued collapse is represented by the second term in the expression for ψ_i in the interior ($\mathcal{F}_i < 0$). On the other hand, the ingoing wave in the exterior, represented by the first term in the expression for ψ_i when $\mathcal{F}_i > 0$, is necessarily accompanied by an outgoing wave in the interior occurring with a relative

amplitude of $e^{-\frac{\pi\omega_i}{\hbar g_{i,h}}}$, which is equal to the amplitude for “reflection” at the apparent horizon. This leads to the alternate picture mentioned in the Introduction, in which the Hawking process can be viewed as an effective emission from the apparent horizon.

B. Phase transition

We can use our results above to examine what happens near the Hawking-Page transition point [30]. It is worth noting that the Hawking temperature is independent of the energy function. This was first noted in Ref. [26] in connection with nonmarginal LTB models without a cosmological constant. Here we have shown that the result is robust, holding even in the presence of a negative cosmological constant.

Now the apparent horizon is given for each shell as the solution of the equation

$$2x_{i,h}^{n+1} + n(n+1)x_{i,h}^{n-1} - n(n+1)F_i\Lambda^{\frac{n-1}{2}} = 0, \quad (33)$$

where $x_i = R_i\sqrt{\Lambda}$ is dimensionless, and it is straightforward to show that the surface gravity for each shell is

$$g_{i,h} = \frac{\sqrt{\Lambda}}{2} \left[\frac{2x_{i,h}}{n} + \frac{n-1}{x_{i,h}} \right]. \quad (34)$$

For fixed n and Λ ,

$$\frac{dg_{i,h}}{dF_i} = \frac{\sqrt{\Lambda}}{2} \frac{dx_{i,h}}{dF_i} \left[\frac{2}{n} - \frac{n-1}{x_{i,h}^2} \right], \quad (35)$$

so using the fact that $dx_{i,h}/dF_i > 0$, which follows directly from (33), we find that $dg_{i,h}/dF > 0$ when $2x_{i,h}^2 > n(n-1)$. This is the condition for positive specific heat. When $2x_{i,h}^2 < n(n-1)$ the specific heat is negative and the two regimes are separated by the Hawking-Page (phase) transition [30], which occurs at $2x_{i,h}^2 = n(n+1)$. The wave

functions of collapse in (31) are well behaved at the transition point.

C. Wave functionals

We now turn to the question of how the collapsing shell wave functions described in the previous section may be combined to yield wave functionals. Obviously the superposed wave functions of (31) cannot be directly used for this purpose as they are not the simple exponentials required for diffeomorphism invariance by (14). Instead, we take the continuum limit of the product of the shell wave functions $\psi_i^{(1)}$ and $\psi_i^{(2)}$ separately to form two corresponding diffeomorphism invariant wave functionals, Ψ_1 and Ψ_2 . Then, we take a linear combination of these wave functionals to form the full wave functional describing the collapse [11].

Accordingly, the functional equivalent of the superposed wave functions in (31) is $\Psi = \Psi_1 + \Psi_2$, or

$$\Psi = \begin{cases} e^{\frac{1}{2} \int dr \Gamma b} \times \exp \left\{ -\frac{i}{2\hbar} \int^R dr \Gamma \left[a\tau + \int_{r=\text{const}}^R dR \frac{\sqrt{1-a^2\mathcal{F}}}{\mathcal{F}} \right] \right\} + e^{-S/2} \times e^{\frac{1}{2} \int dr \Gamma b} \\ \quad \times \exp \left\{ -\frac{i}{2\hbar} \int dr \Gamma \left[a\tau - \int_{r=\text{const}}^R dR \frac{\sqrt{1-a^2\mathcal{F}}}{\mathcal{F}} \right] \right\} & \mathcal{F} > 0 \\ e^{-S/2} \times e^{\frac{1}{2} \int dr \Gamma b} \times \exp \left\{ -\frac{i}{2\hbar} \int dr \Gamma \left[a\tau + \int_{r=\text{const}}^R dR \frac{\sqrt{1-a^2\mathcal{F}}}{\mathcal{F}} \right] \right\} + e^{\frac{1}{2} \int dr \Gamma b} \\ \quad \times \exp \left\{ -\frac{i}{2\hbar} \int dr \Gamma \left[a\tau - \int_{r=\text{const}}^R dR \frac{\sqrt{1-a^2\mathcal{F}}}{\mathcal{F}} \right] \right\} & \mathcal{F} < 0 \end{cases}. \quad (36)$$

When use is made of (33) and (34), the relative amplitude, $e^{-\frac{\pi}{2\hbar} \int dr \Gamma / g_h}$, works out to precisely $e^{-S/2}$, where $S = A_h/4\hbar G_d$ is the Bekenstein-Hawking entropy of the black hole. Thus the ratio of the reflection probability to the probability for absorption is determined only by the entropy of the black hole,

$$\frac{P_{\text{ref}}}{P_{\text{abs}}} = e^{-S} \quad (37)$$

and we have recovered the results of Ref. [11] in the more general setting of d -dimensional, nonmarginal collapse and in the presence of a (negative) cosmological constant.

V. CONCLUSIONS

In this paper we have used the wave functionals of an exact midi-superspace quantization of the nonmarginally-bound LTB models in the presence of a cosmological constant and in an arbitrary number of spatial dimensions to study the Hawking evaporation process. As in previous works, regularization was performed on a lattice and the wave functionals were shown to be constructed out of wave functions describing individual shells of collapsing dust. The apparent horizon is an essential singularity of the Wheeler-DeWitt equation and the solutions of the latter

could only be given by quadrature separately in the exterior and in the interior of the apparent horizon. The central issue discussed here was how the interior and exterior solutions can be matched across the horizon. To accomplish the required matching we defined the integrals appearing in the solution of the Wheeler-DeWitt equation by deforming the integration path in the complex plane to go around the pole at the apparent horizon. This implied that crossing the horizon involved a rotation of the dust proper time in the complex plane and had the effect of introducing an imaginary constant into the phase of the outgoing wave functions. We were then able to show that an ingoing shell wave function in one region is required to be accompanied by an outgoing shell wave function in the other region. The relative amplitude of the outgoing wave function in each case was shown to be given by the square root of the Boltzmann factor at the Hawking temperature appropriate to the shell.

The approach in this paper enjoys several advantages over the approach via Bogoliubov coefficients, while also producing an alternative and attractive view of the evaporation process. In the first place, no near horizon expansion of the wave functional is necessary. Second, the approach via Bogoliubov coefficients does not get much beyond the semiclassical level because it is necessary to approximate

the mass function in such a way that it represents a massive black hole surrounded by tenuous dust. Hence one is effectively looking at the semiclassical radiation from the event horizon of a static black hole. By contrast, no such approximation to the mass function is necessary here, so we are genuinely examining the radiation from the apparent horizon *during* collapse. Third, the inner product used in the calculation of the Bogoliubov coefficient is not the one that is uniquely determined by the lattice regularization (see Appendix B of Ref. [19]) but one that is determined from the DeWitt supermetric. This can be justified only in the approximation described above because, for this case alone, no measure is uniquely determined by the regularization scheme [31]. In all other cases, the measure is uniquely determined and different from that provided by the DeWitt supermetric. The results we report here are independent of the inner product.

If the matter is assumed to undergo continued collapse we showed that the relative probability for the shell wave function to cross the apparent horizon is unity, whether it is incident from the interior or the exterior. This was argued to lead to two pictures of the evaporation process. In the first picture, one takes the point of view of an external observer with no access to the interior. To this observer the horizon

appears to possess a nonzero reflectivity. On the other hand, the observer who has access to the entire wave function sees the outgoing exterior portion of the shell wave function as a transmission of an outgoing interior wave across the horizon, which exists because of the collapsing exterior. Thus the horizon can also be thought of as an emitter.

We showed that the Hawking temperature is independent of the energy function, i.e., of the initial velocity distribution of the shells, and that the shell wave functions are well behaved at the Hawking-Page transition point during the collapse. Moreover, the relative amplitude for outgoing wave functionals is $e^{-S/2}$, where S is the Bekenstein-Hawking entropy of the final state black hole even in the presence of the cosmological constant. This generalizes [11], for which closed form solutions were available. The results are therefore generic to dust collapse.

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