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Wald-like formula for energy

Aaron J. Amsel*

Department of Physics and Beyond Center for Fundamental Concepts in Science, Arizona State University, Tempe, Arizona 85287, USA

Dan Gorbonos†

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We present a simple ''Wald-like'' formula for gravitational energy about a constant-curvature background spacetime. The formula is derived following the Abbott-Deser-Tekin approach for the definition of conserved asymptotic charges in higher-derivative gravity.

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I. INTRODUCTION

In this paper, we present a simple formula for the computation of global charges (in particular energy) in asymptotically constant-curvature spacetimes for general gravitational theories. This is of particular interest for solutions of higher-derivative theories that approach a constant curvature solution in the asymptotic region.

Our formula is similar to Wald's formula for entropy [1–3] in the sense that both formulas involve a derivative of the Lagrangian of the theory with respect to the Riemann tensor. The main difference is that the entropy is computed as an integral over the horizon, while the energy is computed in the asymptotic region (where we regard the solution as a perturbation of the background). Another difference is that Wald's formula involves only first derivatives with respect to curvature, whereas the formula for energy involves second derivatives. Nevertheless, the two formulas should be related by the first law of black hole thermodynamics and its integrated forms, i.e., Komar integrals and the Smarr formula [in some extended form to higher-derivative gravity and (A) dS spacetimes; see, e.g., Refs. [4,5]]. Proposals for similar Wald-like formulas for the shear viscosity were given in Refs. [6,7].

This approach gives us a new viewpoint on black hole thermodynamics. Wald's entropy formula was reinterpreted as the Bekenstein-Hawking entropy with an effective gravitational coupling [8] (see also Ref. [9]). This effective coupling comes from the coefficient of the kinetic term of a specific type of metric perturbation in the theory. Equivalently, this specific type of perturbation corresponds to the propagator for the exchange of a graviton between two covariantly conserved sources at the horizon, and the effective coupling comes from this propagator. Thus, Wald's formula not only has an advantage in the computational aspect, but also gives us a microscopic interpretation of the black hole entropy.

In a similar way, a Wald-like formula for energy can be naturally interpreted as giving the effective gravitational coupling in the asymptotic region, namely, on the background. This avenue was explored in Ref. [10] for cubic corrections and in Ref. [11] for Lanczos-Lovelock gravity, where the effective gravitational coupling was identified as coming from higher-derivative corrections to the tree-level scattering amplitude between two background covariantly conserved sources via the exchange of a graviton. This is the same effective coupling that appears in the Wald-like energy formula derived below. This viewpoint, for both the entropy and energy, gives us two sides of the microscopic interpretation (in terms of corrections to a graviton exchange amplitude) of higher-derivative corrections to black hole thermodynamics.

The derivation of the formula is based on the Abbot-Deser-Tekin (ADT) method for computing energy [12,13]. We basically reduce their general method to a single formula which only requires substitution of the Lagrangian and the background solution. The story of this method starts with the result of Arnowitt, Deser, and Misner [14] for energy in Einstein-Hilbert gravity with asymptotically flat boundary conditions. This was later generalized to spacetimes with a cosmological constant in Ref. [15]. These so-called ''Abbott-Deser (AD) charges'' were written in a manifestly covariant way and once again could be expressed as pure surface integrals. The method used to construct the Abbott-Deser charges was then further generalized to arbitrary higher-curvature theories in Refs. [12,13].

As we will see, the ADT method involves relatively little formalism and is computationally straightforward. In addition, this method has the advantage of not involving any explicit regularization or subtraction of infinities, as required in counterterm methods (see, e.g., Refs. [16,17]). Unlike Euclidean path integral techniques (e.g., Ref. [18]), the ADT framework naturally gives the gravitational mass as an integral at asymptotic infinity, without any need to identify a horizon in the interior. For perturbations that vanish sufficiently fast at asymptotic infinity, the ADT charges are exactly the same as the charges derived using

*aamsel@asu.edu
†dan.gorbonos@mail.huji.ac.il

the covariant phase space methods of Refs. [19–21], which in turn differ from the charges of Wald *et al.* [2,22,23] by a surface term proportional to the Killing equations.

This paper is organized as follows. In Sec. II, we present the Wald-like formula for energy and explain how to use it. We also give the energy of black hole solutions for two examples: Gauss-Bonnet gravity, and a theory with six-derivative corrections that was previously studied in Ref. [24]. We show that the calculation of energy becomes much shorter and simpler when using the formula. The rest of the paper is devoted to deriving this formula from the ADT method. After a short presentation of the ADT method in Sec. III, we discuss the general structure of the "effective" stress-energy tensor in higher-derivative gravity in Sec. IV. We then obtain an explicit formula for the stress-energy tensor for a flat background in Sec. V. This expression is generalized to a curved background in Sec. VI, using the effective quadratic curvature method [10,11,25–27]. Following Ref. [12], in Sec. VII we complete the computation by deriving the energy from the general expression of the stress-energy tensor. We conclude with a brief discussion of our results and future directions in Sec. VIII.

II. THE FORMULA FOR ENERGY

Here we present the formula for energy and explain how to use it. Let us consider a general d-dimensional theory of gravity whose action depends on the metric $g_{\mu\nu}$ and the curvature (through the Riemann tensor)

$$I = \int d^d x \sqrt{-g}\mathcal{L}(R_{\mu\nu\rho\sigma}, g_{\mu\nu}). \tag{2.1}$$

We will construct the energy for such theories following the approach of Refs. [12,13,15]. We assume that the action is invariant under diffeomorphisms. In order to define a gauge-invariant conserved charge we need the presence of an asymptotic Killing symmetry. The charge is then defined relative to a background solution, denoted as $\bar{g}_{\mu\nu}$, which admits a Killing vector $\bar{\xi}_\mu$. We assume that the background is a homogeneous solution, namely, described by an "effective" cosmological constant Λ, which can be negative, positive, or zero (which is the asymptotically flat case). In addition, the solutions are required to fall off sufficiently fast at infinity relative to the background.[1]

For a large class of solutions [which includes the asymptotically Schwarzschild-(A)dS (SdS) solutions defined below], we can write the energy of a generic higher-derivative theory in the same form as applies to Einstein-Hilbert gravity (with a cosmological constant), but with an overall multiplicative factor that depends on the higher-curvature terms. This will later serve as a basis for the interpretation of higher-derivative corrections as effective modifications of the gravitational coupling constant (Newton's constant). The Lagrangian of Einstein-Hilbert gravity with a cosmological constant is

$$\mathcal{L}_E = \frac{1}{2\kappa}(R - 2\Lambda_0), \tag{2.2}$$

where κ is the d-dimensional gravitational coupling constant. The ADT energy for solutions of this theory is denoted by E_0 and is given explicitly in Eq. (3.6). Then, the energy for the general Lagrangian (2.1) is

$$E = \left[P^{\mu\nu}_{\rho\sigma}\left(\frac{\partial \mathcal{L}}{\partial R^{\mu\nu}_{\rho\sigma}}\right)_{\bar{g}} - \frac{4\Lambda(d-3)}{(d-1)(d-2)} \times P_{(1)}{}^{\gamma\delta,\mu\nu}_{\alpha\beta\rho\sigma}\left(\frac{\partial^2 \mathcal{L}}{\partial R^{\gamma\delta}_{\alpha\beta}\partial R^{\mu\nu}_{\rho\sigma}}\right)_{\bar{g}}\right] 2\kappa E_0, \tag{2.3}$$

where

$$P^{\mu\nu}_{\rho\sigma} = \frac{2\delta^{\mu}_{[\rho}\delta^{\nu}_{\sigma]}}{d(d-1)}, \tag{2.4}$$

$$P_{(1)}{}^{\gamma\delta,\mu\nu}_{\alpha\beta\rho\sigma} = \frac{4}{d(d^2-1)(d-2)(d^2-2d+2)} \times \left[(d-1)^2\delta^{\mu}_{[\alpha}\delta^{\nu}_{\beta]}\delta^{[\gamma}_{\rho}\delta^{\delta]}_{\sigma} - \delta^{\gamma}_{[\alpha}\delta^{\delta}_{\beta]}\delta^{[\mu}_{\rho}\delta^{\nu]}_{\sigma} - (d-2)\delta^{\delta}_{[\beta}\delta^{\mu}_{\alpha]}\delta^{\nu}_{[\sigma}\delta^{\gamma}_{\rho]}\right]. \tag{2.5}$$

The complete energy formula for more general boundary conditions is given in Eq. (7.7).] Here Λ is the effective cosmological constant associated with the background solution $\bar{g}_{\mu\nu}$, which in general is distinct from the "bare" cosmological constant Λ_0 that may appear in the action.

The derivative of the Lagrangian with respect to $R^{\mu\nu}_{\rho\sigma}$ is performed formally, as if $R^{\mu\nu}_{\rho\sigma}$ and $g_{\mu\nu}$ are independent, and we impose the same tensor symmetries as $R^{\mu\nu}_{\rho\sigma}$. For example, in the case of Einstein gravity (2.2) we get

$$\frac{\partial \mathcal{L}_E}{\partial R^{\mu\nu}_{\rho\sigma}} = \frac{1}{2\kappa}\delta^{\rho}_{[\mu}\delta^{\sigma}_{\nu]}. \tag{2.6}$$

The $(\ldots)_{\bar{g}}$ notation indicates that the expression in parentheses is to be evaluated on the background spacetime $\bar{g}_{\mu\nu}$.

The expressions in Eqs. (2.4) and (2.5) are "projection" tensors that pick out certain coefficients to give the correct energy. [The subscript (1) will be explained later.] While the contractions with the projectors might appear complicated, their main use is to formally write the final formula (2.3). When we take a derivative with respect to the

[1] Here we mean that the perturbation about a solution falls off fast enough at infinity that the theory is asymptotically linear, i.e., that the linearized equations of motion are obeyed near infinity. In this case, the charges of the linearized theory can be used to obtain the charges of the nonlinear theory. For example, this condition holds for the case of standard asymptotically AdS boundary conditions [28].

Riemann tensor and evaluate on the (homogeneous) background, we always get an expression of the form

$$\left(\frac{\partial \mathcal{L}}{\partial R^{\mu\nu}_{\rho\sigma}}\right)_{\bar{g}} = N\delta^{\rho}_{[\mu}\delta^{\sigma}_{\nu]}, \tag{2.7}$$

where N is some constant coefficient. The projector (2.4) is defined to give precisely the coefficient N when acting on Eq. (2.7). Thus, in practice one often simply reads off the coefficient after computing the derivative on the background, rather than actually performing the contraction with $P^{\rho\sigma}_{\mu\nu}$.

Similarly, for the second derivative with respect to the Riemann tensor evaluated on the background, there are in general three terms,

$$\left(\frac{\partial^2 \mathcal{L}}{\partial R^{\gamma\delta}_{\alpha\beta}\partial R^{\mu\nu}_{\rho\sigma}}\right)_{\bar{g}} = N_1\delta^{[\alpha}_{\mu}\delta^{\beta]}_{\nu}\delta^{\rho}_{[\gamma}\delta^{\sigma}_{\delta]} + N_2\delta^{[\alpha}_{\gamma}\delta^{\beta]}_{\delta}\delta^{\rho}_{[\mu}\delta^{\sigma}_{\nu]} + N_3\delta^{[\beta}_{\delta}\delta^{\alpha]}_{\mu}\delta^{[\sigma}_{\nu}\delta^{\rho]}_{\gamma}, \tag{2.8}$$

where N_1, N_2, N_3 are constants. When the projector $P_{(1)\alpha\beta\rho\sigma}^{\ \ \gamma\delta,\mu\nu}$ acts on Eq. (2.8), it picks out the coefficient N_1, but again, we can also simply read off this coefficient by writing the expression for the second derivative in the above form. For example, if

$$\mathcal{L} = CR_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}, \tag{2.9}$$

then $N_1 = 2C$. In other words, the second derivative term in Eq. (2.3) roughly corresponds to coefficients of terms with the same type of contractions as in Eq. (2.9).

A typical example of solutions that fall off sufficiently fast at infinity are asymptotically SdS solutions. The asymptotic behavior of such solutions is

$$h_{tt} \approx \left(\frac{r_0}{r}\right)^{d-3}, \qquad h^{rr} \approx \left(\frac{r_0}{r}\right)^{d-3} + \cdots, \tag{2.10}$$

where r_0 is a constant. For these solutions, the energy in the case of Einstein gravity is given by

$$E_0 = \frac{(d-2)\mathrm{Vol}(S^{d-2})}{4\kappa} r_0^{d-3}, \tag{2.11}$$

and for a general theory we get

$$E = \frac{(d-2)\mathrm{Vol}(S^{d-2})}{2} r_0^{d-3}\left[P^{\mu\nu}_{\rho\sigma}\left(\frac{\partial \mathcal{L}}{\partial R^{\mu\nu}_{\rho\sigma}}\right)_{\bar{g}} - \frac{4\Lambda(d-3)}{(d-1)(d-2)}P_{(1)\alpha\beta\rho\sigma}^{\ \ \gamma\delta,\mu\nu}\left(\frac{\partial^2\mathcal{L}}{\partial R^{\gamma\delta}_{\alpha\beta}\partial R^{\mu\nu}_{\rho\sigma}}\right)_{\bar{g}}\right]. \tag{2.12}$$

Another important example is the energy density of black branes in AdS. When the asymptotic behavior of the black brane solution is

$$h_{tt} \approx \frac{A}{r^{d-3}}, \qquad h^{rr} \approx \frac{A}{r^{d-3}} + \cdots, \tag{2.13}$$

the energy density of the black brane is

$$E_0 = \frac{(d-2)A}{4\kappa}. \tag{2.14}$$

Let us now look at two examples.

A. Example: Energy with a Gauss-Bonnet term

We start with the famous case of Gauss-Bonnet gravity. This term is topological (a total derivative) in four dimensions and leads to a ghost-free theory in any number of dimensions. The Lagrangain with the Gauss-Bonnet term reads

$$\mathcal{L}_{\rm GB} = \mathcal{L}_E + \frac{b_2}{2\kappa}(R^2_{\mu\nu\rho\sigma} - 4R^2_{\mu\nu} + R^2), \tag{2.15}$$

and we have

$$\frac{\partial \mathcal{L}_{\rm GB}}{\partial R^{\mu\nu}_{\rho\sigma}} = \frac{1}{2\kappa}(\delta^{\rho}_{[\mu}\delta^{\sigma}_{\nu]} + 2b_2R^{\rho\sigma}_{\mu\nu} - 8b_2\delta^{\rho}_{[\mu}R^{\sigma}_{\nu]} + 2b_2R\delta^{\rho}_{[\mu}\delta^{\sigma}_{\nu]}). \tag{2.16}$$

For a homogeneous background,

$$\bar{R}^{\rho\sigma}_{\mu\nu} = \frac{4\Lambda}{(d-1)(d-2)}\delta^{\rho}_{[\mu}\delta^{\sigma}_{\nu]}, \qquad \bar{R}^{\sigma}_{\nu} = \frac{2\Lambda}{d-2}\delta^{\sigma}_{\nu}, \qquad \bar{R} = \frac{2\Lambda d}{d-2}, \tag{2.17}$$

so

$$\left(\frac{\partial \mathcal{L}_{\rm GB}}{\partial R^{\mu\nu}_{\rho\sigma}}\right)_{\bar{g}} = \frac{1}{2\kappa}\left(1 + 4\Lambda b_2\frac{d-3}{d-1}\right)\delta^{\rho}_{[\mu}\delta^{\sigma}_{\nu]}, \tag{2.18}$$

and the projection $P^{\mu\nu}_{\rho\sigma}$ gives us the coefficient of $\delta^{\rho}_{[\mu}\delta^{\sigma}_{\nu]}$:

$$P^{\mu\nu}_{\rho\sigma}\left(\frac{\partial \mathcal{L}_{\rm GB}}{\partial R^{\mu\nu}_{\rho\sigma}}\right)_{\bar{g}} = \frac{1}{2\kappa}\left(1 + 4\Lambda b_2\frac{d-3}{d-1}\right). \tag{2.19}$$

The second derivative with respect to the Riemann tensor is

$$\frac{\partial^2 \mathcal{L}_{\rm GB}}{\partial R^{\gamma\delta}_{\alpha\beta}\partial R^{\mu\nu}_{\rho\sigma}} = \frac{b_2}{\kappa}(\delta^{[\alpha}_{\mu}\delta^{\beta]}_{\nu}\delta^{\rho}_{[\gamma}\delta^{\sigma}_{\delta]} + \delta^{[\alpha}_{\gamma}\delta^{\beta]}_{\delta}\delta^{\rho}_{[\mu}\delta^{\sigma}_{\nu]} - 4\delta^{[\beta}_{\delta}\delta^{\alpha]}_{\mu}\delta^{[\sigma}_{\nu}\delta^{\rho]}_{\gamma}). \tag{2.20}$$

Since we only require the coefficient of the first type of term, namely $\delta^{[\alpha}_{\mu}\delta^{\beta]}_{\nu}\delta^{\rho}_{[\gamma}\delta^{\sigma}_{\delta]}$, we get

$$P_{(1)\alpha\beta\rho\sigma}^{\ \ \gamma\delta,\mu\nu}\left(\frac{\partial^2 \mathcal{L}_{\rm GB}}{\partial R^{\gamma\delta}_{\alpha\beta}\partial R^{\mu\nu}_{\rho\sigma}}\right)_{\bar{g}} = \frac{b_2}{\kappa}. \tag{2.21}$$

Substituting Eqs. (2.19) and (2.21) into the formula for the energy (2.12), we obtain

$$E = \left(1 + \frac{4b_2\Lambda(d-4)(d-3)}{(d-2)(d-1)}\right)\frac{(d-2)\mathrm{Vol}(S^{d-2})}{4\kappa}r_0^{d-3}. \tag{2.22}$$

This agrees with the result in Ref. [13], which was obtained by a longer calculation.

B. Example: Energy with a six-derivative term

Here we will consider an example with six derivatives of the metric (cubic curvature),

$$\mathcal{L}_{I_1} = \mathcal{L}_E + \frac{c_1}{2\kappa} R^{\mu\nu}{}_{\alpha\beta} R^{\alpha\beta}{}_{\lambda\rho} R^{\lambda\rho}{}_{\mu\nu}. \tag{2.23}$$

The ADT energy of this theory was previously given in Ref. [24], but required a much lengthier calculation. In this example, the first derivative with respect to the Riemann tensor gives

$$\frac{\partial \mathcal{L}_{I_1}}{\partial R^{\mu\nu}_{\rho\sigma}} = \frac{1}{2\kappa}(\delta^{\rho}_{[\mu}\delta^{\sigma}_{\nu]} + 3c_1 R^{\rho\sigma}_{\lambda\epsilon} R^{\lambda\epsilon}_{\mu\nu}), \tag{2.24}$$

and substituting the background metric leads to

$$P^{\mu\nu}_{\rho\sigma}\left(\frac{\partial L_{I_1}}{\partial R^{\mu\nu}_{\rho\sigma}}\right)_{\bar{g}} = \frac{1}{2\kappa}\left(1 + \frac{48c_1\Lambda^2}{(d-1)^2(d-2)^2}\right). \tag{2.25}$$

The second derivative with respect to the Riemann tensor (which in this example is not just a constant) is

$$\frac{\partial^2 \mathcal{L}_{I_1}}{\partial R^{\gamma\delta}_{\alpha\beta}\partial R^{\mu\nu}_{\rho\sigma}} = \frac{1}{2\kappa}\cdot 3c_1(R^{\alpha\beta}_{\mu\nu}\delta^{\rho}_{[\gamma}\delta^{\sigma}_{\delta]} + R^{\rho\sigma}_{\gamma\delta}\delta^{\alpha}_{[\mu}\delta^{\beta}_{\nu]}). \tag{2.26}$$

When we substitute the background metric, the second derivative becomes proportional only to the first term in Ref. (2.8),

$$\left(\frac{\partial^2 \mathcal{L}_{I_1}}{\partial R^{\gamma\delta}_{\alpha\beta}\partial R^{\mu\nu}_{\rho\sigma}}\right)_{\bar{g}} = \frac{1}{2\kappa}\cdot\frac{24\Lambda c_1}{(d-1)(d-2)}\delta^{\alpha}_{[\mu}\delta^{\beta}_{\nu]}\delta^{\rho}_{[\gamma}\delta^{\sigma}_{\delta]}, \tag{2.27}$$

so the coefficient is just N_1, and $N_2 = N_3 = 0$. Substituting this and Eq. (2.25) into the energy formula (2.12), we finally obtain

$$E = \left(1 - \frac{48(2d-7)c_1\Lambda^2}{(d-2)^2(d-1)^2}\right)\frac{(d-2)\mathrm{Vol}(S^{d-2})}{4\kappa} r_0^{d-3}, \tag{2.28}$$

which agrees with the result in Ref. [24].

III. THE ADT METHOD

In this section we give a brief review of the ADT method. The ADT method is similar in spirit to the Landau-Lifshitz pseudotensor method for calculating energy [29] in asymptotically flat curved spacetime. In particular, one proceeds by linearizing the equations of motion with respect to a background spacetime. This leads to an effective stress-energy tensor that consists of matter sources and terms higher-order in the perturbation. This tensor turns out to be covariantly conserved and can thus be used to construct a conserved charge associated with an isometry of the background.

Let us consider some arbitrary gravitational theory with equations of motion of the form

$$\Phi_{\mu\nu}(g, R, \nabla R, R^2, \ldots) = \kappa\tau_{\mu\nu}, \tag{3.1}$$

where κ is the gravitational coupling and $\tau_{\mu\nu}$ is the matter stress-energy tensor. The symmetric tensor $\Phi_{\mu\nu}$, which is the analogue of the Einstein tensor, may depend on the metric, the curvature, derivatives of the curvature, and various combinations thereof. Assuming that the action is invariant under diffeomorphisms, we obtain the geometric identity $\nabla^\mu\Phi_{\mu\nu} = 0$ (the generalized Bianchi identity) and the covariant conservation of the stress tensor $\nabla^\mu\tau_{\mu\nu} = 0$.

Now, we further assume that there exists a background solution $\bar{g}_{\mu\nu}$ to the equations (3.1) with $\tau_{\mu\nu} = 0$. Then we decompose the metric as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \tag{3.2}$$

where we note that the deviation $h_{\mu\nu}$ is not necessarily infinitesimal, but it is required to fall off sufficiently fast at infinity. Asymptotically SdS spacetimes are a typical example meeting this requirement. By expanding the left-hand side of Eq. (3.1) in $h_{\mu\nu}$, the equations of motion may be expressed as

$$\phi^{(1)}_{\mu\nu} = \kappa\tau_{\mu\nu} - \phi^{(2)}_{\mu\nu} - \phi^{(3)}_{\mu\nu}\ldots \equiv \kappa T_{\mu\nu}, \tag{3.3}$$

where $\phi^{(i)}_{\mu\nu}$ denotes all terms in the expansion of $\Phi_{\mu\nu}$ involving i powers of $h_{\mu\nu}$, and we have defined the effective stress-tensor $T_{\mu\nu}$. It then follows from the Bianchi identity of the full theory that $\bar{\nabla}^\mu\phi^{(1)}_{\mu\nu} = 0 = \bar{\nabla}^\mu T_{\mu\nu}$.

Suppose that the background spacetime admits a time-like Killing vector $\bar{\xi}^\mu$, and let Σ be a constant-time hypersurface with unit normal n^μ. Then we can construct a conserved energy in the standard way:

$$E = \int_\Sigma d^{d-1}x\sqrt{\bar{g}_\Sigma}n_\mu T^{\mu\nu}\bar{\xi}_\nu, \tag{3.4}$$

where $\bar{g}_\Sigma$ denotes the determinant of the induced metric on Σ. Because $\bar{\nabla}^\mu(T_{\mu\nu}\bar{\xi}^\nu) = 0$, it follows that $T_{\mu\nu}\bar{\xi}^\nu = \bar{\nabla}^\nu\mathcal{F}_{\nu\mu}$ for some antisymmetric tensor $\mathcal{F}_{\nu\mu}$. The bulk integral (3.4) can therefore be rewritten as a surface integral over the boundary $\partial\Sigma$:

$$E = \int_{\partial\Sigma} d^{d-2}x\sqrt{\bar{g}_{\partial\Sigma}}n_\mu r_\nu\mathcal{F}^{\nu\mu}, \tag{3.5}$$

where r_μ is the unit normal to the boundary. For example, for the Einstein-Hilbert theory (2.2), the explicit expression for the energy is

$$E_0 = \frac{1}{4\kappa}\int_{\partial\Sigma} d^{d-2}x\sqrt{\bar{g}_{\partial\Sigma}}n_\mu r_\nu[\bar{\xi}_\lambda\bar{\nabla}^\mu h^{\nu\lambda} - \bar{\xi}_\lambda\bar{\nabla}^\nu h^{\mu\lambda} + \bar{\xi}^\mu\bar{\nabla}^\nu h - \bar{\xi}^\nu\bar{\nabla}^\mu h + h^{\mu\lambda}\bar{\nabla}^\nu\bar{\xi}_\lambda - h^{\nu\lambda}\bar{\nabla}^\mu\bar{\xi}_\lambda + \bar{\xi}^\nu\bar{\nabla}_\lambda h^{\mu\lambda} - \bar{\xi}^\mu\bar{\nabla}_\lambda h^{\nu\lambda} + h\bar{\nabla}^\mu\bar{\xi}^\nu]. \quad (3.6)$$

In summary, to apply the ADT method, one linearizes the equations of motion to obtain the stress-energy tensor, and then expresses the conserved current $T^{\mu\nu}\bar{\xi}_\nu$ as a total derivative to find the "potential" $\mathcal{F}^{\nu\mu}$. Note that by construction, the background spacetime $\bar{g}_{\mu\nu}$ has $E = 0$.

IV. THE GENERAL STRUCTURE OF THE STRESS TENSOR

In Ref. [12], it was shown that the most general quadratic curvature theory has a stress tensor that is schematically of the form

$$T_{\mu\nu} = \alpha_1\mathcal{G}^L_{\mu\nu} + \alpha_2 H^{(1)}_{\mu\nu} + \alpha_3 H^{(2)}_{\mu\nu}, \quad (4.1)$$

where

$$\mathcal{G}^L_{\mu\nu} = R^L_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}R_L - \frac{2\Lambda}{d-2}h_{\mu\nu}, \quad (4.2)$$

$$H^{(1)}_{\mu\nu} = \left(\bar{g}_{\mu\nu}\bar{\Box} - \bar{\nabla}_\mu\bar{\nabla}_\nu + \frac{2\Lambda}{d-2}\bar{g}_{\mu\nu}\right)R_L, \quad (4.3)$$

$$H^{(2)}_{\mu\nu} = \bar{\Box}\mathcal{G}^L_{\mu\nu} - \frac{2\Lambda}{d-2}\bar{g}_{\mu\nu}R_L, \quad (4.4)$$

and the α_i are constants. Here $R^L_{\mu\nu}$ is the linearized Ricci tensor

$$R^L_{\mu\nu} = R_{\mu\nu} - \bar{R}_{\mu\nu} = \frac{1}{2}(-\bar{\Box}h_{\mu\nu} - \bar{\nabla}_\mu\bar{\nabla}_\nu h + \bar{\nabla}^\sigma\bar{\nabla}\nu h_{\sigma\mu} + \bar{\nabla}^\sigma\bar{\nabla}_\mu h_{\sigma\nu}), \quad (4.5)$$

and R_L is the linearized Ricci scalar

$$R_L = \bar{\nabla}^\sigma\bar{\nabla}^\mu h_{\sigma\mu} - \bar{\Box}h - \frac{2\Lambda}{d-2}h. \quad (4.6)$$

Note that the tensors $\mathcal{G}^L_{\mu\nu}$, $H^{(i)}_{\mu\nu}$ are *each* divergence free:

$$\bar{\nabla}^\mu\mathcal{G}^L_{\mu\nu} = \bar{\nabla}^\mu H^{(1)}_{\mu\nu} = \bar{\nabla}^\mu H^{(2)}_{\mu\nu} = 0. \quad (4.7)$$

It was later found in Ref. [24] that the stress tensor for a certain cubic curvature theory took exactly the same form (the only modification was to the values of the coefficients α_i) and it was suggested that this observation might hold more generally.[2] In this section, we will argue that this is indeed the case for any theory of the form (2.1).

The basic idea is as follows. We saw in the previous section that the ADT stress tensor is given by the linearized (in h) equations of motion. This means that $T_{\mu\nu}$ only depends on the action to $O(h^2)$. Now, expanding the general action (2.1) to $O(h^2)$ involves expanding the Riemann tensor, which contains terms of the form $\bar{\nabla}\bar{\nabla}h$. Hence, this can yield terms of *at most* four derivatives. This suggests quite generally that the basic form of the stress tensor does not change from Eq. (4.1) even if the action contains more than two powers of the Riemann tensor.[3] We will show that there exists a basis of three different components for the stress-energy tensor [to $O(h^2)$ and up to four derivatives], and that this basis can be chosen to correspond to $\mathcal{G}^L_{\mu\nu}$, $H^{(1)}_{\mu\nu}$, and $H^{(2)}_{\mu\nu}$.

To demonstrate this claim in more detail, we first consider the most general $O(h^2)$ action with two derivatives:

$$I_2 = \int d^dx(\beta_1\partial^\rho h^{\mu\nu}\partial_\rho h_{\mu\nu} + \beta_2\partial^\mu h^{\nu\rho}\partial_\nu h_{\mu\rho} + \beta_3\partial^\mu h\partial^\lambda h_{\lambda\mu} + \beta_4\partial^\mu h\partial_\mu h). \quad (4.8)$$

For simplicity we work in the case of a flat background, $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$. The generalization to a curved background will be discussed later. Varying I_2 with respect to $h_{\mu\nu}$ yields the stress tensor

$$T_{\alpha\beta} = -\frac{\delta I_2}{\delta h^{\alpha\beta}} = 2\beta_1\bar{\Box}h^{\alpha\beta} + 2\beta_2\partial_\nu\partial^{(\alpha}h^{\beta\nu)} + \beta_3\eta^{\alpha\beta}\partial^\mu\partial^\lambda h_{\lambda\mu} + \beta_3\partial^\alpha\partial^\beta h + 2\beta_4\eta^{\alpha\beta}\bar{\Box}h. \quad (4.9)$$

If we impose conservation of the stress tensor, we obtain

$$0 = \partial_\alpha T^{\alpha\beta} = (2\beta_1 + \beta_2)\bar{\Box}\partial_\alpha h^{\alpha\beta} + (\beta_2 + \beta_3)\partial^\beta\partial_\nu\partial_\alpha h^{\nu\alpha} + (\beta_3 + 2\beta_4)\bar{\Box}\partial^\beta h, \quad (4.10)$$

so equating the coefficients to zero gives

$$\beta_2 = -2\beta_1, \qquad \beta_3 = 2\beta_1, \qquad \beta_4 = -\beta_1. \quad (4.11)$$

Substituting these relations into Eq. (4.9) gives $T_{\mu\nu} = -4\beta_1\mathcal{G}^L_{\mu\nu}$. It follows that the Lagrangian

$$L_\mathcal{G} = \frac{1}{4}(2\partial^\mu h^{\nu\rho}\partial_\nu h_{\mu\rho} + \partial^\mu h\partial_\mu h - 2\partial^\mu h\partial^\lambda h_{\lambda\mu} - \partial^\rho h^{\mu\nu}\partial_\rho h_{\mu\nu}), \quad (4.12)$$

yields a conserved stress tensor that is precisely $\mathcal{G}^L_{\mu\nu}$ (in a flat background). Note that this is also just the famous Fierz-Pauli Lagrangian [30].

Now let us repeat the same procedure for the most general $O(h^2)$ action with four derivatives:

[2] This general claim was also stated in Ref. [10].

[3] For a flat background, it is clear that no term with more than two powers of the Riemann tensor can contribute to the $O(h^2)$ part of the action, since $\bar{R}_{\mu\nu\rho\sigma} = 0$.

$$I_4 = \int d^d x(\beta_5 \partial_\alpha \partial_\beta h^{\alpha\beta} \partial_\gamma \partial_\delta h^{\gamma\delta} + \beta_6 \bar{\Box} h^{\beta\gamma} \partial_\alpha \partial_\gamma h^\alpha_\beta + \beta_7 \bar{\Box} h^{\alpha\beta} \partial_\alpha \partial_\beta h + \beta_8 \bar{\Box} h_{\beta\gamma} \bar{\Box} h^{\beta\gamma} + \beta_9 \bar{\Box} h \bar{\Box} h). \quad (4.13)$$

The corresponding stress tensor is

$$T_{\mu\nu} = -\frac{\delta I_4}{\delta h^{\mu\nu}} = -(2\beta_5 \partial_\mu \partial_\nu \partial_\gamma \partial_\delta h^{\gamma\delta} + 2\beta_6 \bar{\Box} \partial_\alpha \partial_{(\mu} h^\alpha_{\nu)} + \beta_7 (\bar{\Box} \partial_\mu \partial_\nu h + \eta_{\mu\nu} \bar{\Box} \partial_\alpha \partial_\beta h^{\alpha\beta}) + 2\beta_8 \bar{\Box}^2 h_{\mu\nu} + 2\beta_9 \eta_{\mu\nu} \bar{\Box}^2 h), \quad (4.14)$$

and imposing $\partial^\mu T_{\mu\nu} = 0$ gives

$$\beta_7 = -\beta_6 - 2\beta_5, \qquad \beta_8 = -\frac{\beta_6}{2}, \qquad \beta_9 = \beta_5 + \frac{\beta_6}{2}. \quad (4.15)$$

We see that there are two independently conserved tensors, and substituting these relations into Eq. (4.14) gives $T_{\mu\nu} = -2\beta_5 H^{(1)}_{\mu\nu} + 2\beta_6 H^{(2)}_{\mu\nu}$. It follows that the Lagrangian

$$L_{H^{(1)}} = \frac{1}{2}(\partial_\alpha \partial_\beta h^{\alpha\beta} \partial_\gamma \partial_\delta h^{\gamma\delta} - 2\bar{\Box} h^{\alpha\beta} \partial_\alpha \partial_\beta h + \bar{\Box} h \bar{\Box} h), \quad (4.16)$$

yields a conserved stress tensor that is precisely $H^{(1)}_{\mu\nu}$ and the Lagrangian

$$L_{H^{(2)}} = -\frac{1}{4}(2\bar{\Box} h^{\beta\gamma} \partial_\alpha \partial_\gamma h^\alpha_\beta - 2\bar{\Box} h^{\alpha\beta} \partial_\alpha \partial_\beta h - \bar{\Box} h_{\beta\gamma} \bar{\Box} h^{\beta\gamma} + \bar{\Box} h \bar{\Box} h), \quad (4.17)$$

yields a conserved stress tensor that is precisely $H^{(2)}_{\mu\nu}$. Thus, we see that, in a flat background, there are at most two possible conserved combinations of terms with four derivatives in the Lagrangian.

The above calculation can in principle be repeated for the case of a curved background spacetime, but it becomes complicated by the fact that the covariant derivatives no longer commute. The key point, however, is that commuting two derivatives in a given expression only produces extra terms of lower differential order. Thus, the expressions for $L_{H^{(i)}}$ now must include two-derivative and zero-derivative terms, but the highest derivative order (four) terms are the same as in a flat background. Once these are fixed, the two-derivative and zero-derivative terms are chosen by requiring that each $H^{(i)}_{\mu\nu}$ is separately conserved. A similar argument shows that the unique conserved quantity consisting of two-derivative and zero-derivative terms is $\mathcal{G}^L_{\mu\nu}$. In other words, the terms with the highest derivatives (four derivatives in $H^{(i)}_{\mu\nu}$ and two derivatives in $\mathcal{G}^L_{\mu\nu}$) are the same for curved and flat backgrounds.

In summary, we have just seen that any $O(h^2)$ action with no more than four derivatives produces a conserved stress tensor with the same structure as Eq. (4.1). Combining this with the argument that any theory $\mathcal{L} = \mathcal{L}(R^{\mu\nu}_{\rho\sigma}, g_{\mu\nu})$ expanded to $O(h^2)$ cannot have terms with more than four derivatives, we conclude that any such theory also has a stress tensor of the form (4.1).

V. THE GENERAL FORMULA FOR THE STRESS TENSOR (FLAT BACKGROUND)

In this section we derive an efficient method to extract the coefficients α_i in the stress tensor (4.1) given a Lagrangian $\mathcal{L} = \mathcal{L}(R^{\mu\nu}_{\rho\sigma}, g_{\mu\nu})$ for a flat background. The generalization for a curved background will be done in the next section.

We wish to expand the action $\sqrt{-g}\mathcal{L}$ to second order in the perturbation h. The Lagrangian can be expanded as

$$\delta\mathcal{L} = \left(\frac{\partial\mathcal{L}}{\partial R^{\mu\nu}_{\rho\sigma}}\right)_{\bar{g}} \delta R^{\mu\nu}_{\rho\sigma} + \frac{1}{2}\left(\frac{\partial^2\mathcal{L}}{\partial R^{\mu\nu}_{\rho\sigma} \partial R^{\gamma\delta}_{\alpha\beta}}\right)_{\bar{g}} \delta R^{\mu\nu}_{\rho\sigma} \delta R^{\gamma\delta}_{\alpha\beta} + O(h^3). \quad (5.1)$$

The variation of the Riemann tensor is

$$\delta R^\rho{}_{\mu\lambda\nu} = R^\rho{}_{\mu\lambda\nu} - \bar{R}^\rho{}_{\mu\lambda\nu} = \bar{\nabla}_\lambda \delta\Gamma^\rho_{\nu\mu} - \bar{\nabla}_\nu \delta\Gamma^\rho_{\lambda\mu} + \delta\Gamma^\rho_{\lambda\delta}\delta\Gamma^\delta_{\mu\nu} - \delta\Gamma^\rho_{\delta\nu}\delta\Gamma^\delta_{\mu\lambda}, \quad (5.2)$$

where

$$\delta\Gamma^\rho_{\mu\nu} \equiv \frac{1}{2} g^{\rho\kappa}(\bar{\nabla} g_{\nu\kappa} + \bar{\nabla} g_{\mu\kappa} - \bar{\nabla}_\kappa g_{\mu\nu}) \quad (5.3)$$

$$= \Upsilon^\rho_{\mu\nu} - h^{\rho\kappa}\Upsilon_{\mu\nu\kappa} + O(h^3) \quad (5.4)$$

and

$$\Upsilon_{\alpha\beta\gamma} = \frac{1}{2}(\bar{\nabla}_\alpha h_{\beta\gamma} + \bar{\nabla}_\beta h_{\alpha\gamma} - \bar{\nabla}_\gamma h_{\alpha\beta}). \quad (5.5)$$

By convention, indices of $\Upsilon_{\alpha\beta\gamma}$ are raised/lowered with the background metric or its inverse. Note that each factor of $\delta R^{\mu\nu}_{\rho\sigma}$ contributes *at least* one $h_{\mu\nu}$ and two derivatives.

In the remainder of this section we restrict to the case of a flat background, $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$. Now, the terms in the action with two derivatives and two h's can only arise from expanding the term with one $\delta R^{\mu\nu}_{\rho\sigma}$, that is

$$\left(\frac{\partial L}{\partial R^{\mu\nu}_{\rho\sigma}}\right)_{\bar{g}} \sqrt{-g}\,\delta R^{\mu\nu}_{\rho\sigma}. \quad (5.6)$$

Since $\mathcal{L}$ is a function only of $g_{\mu\nu}$ and $R^{\mu\nu}_{\rho\sigma}$, it follows that $\partial\mathcal{L}/\partial R^{\mu\nu}_{\rho\sigma}$ evaluated on a homogeneous background can only be a function of $\bar{g}_{\mu\nu}$. Furthermore, this quantity has the same symmetries as the Riemann tensor, so it must take the general form

$$\left(\frac{\partial\mathcal{L}}{\partial R^{\mu\nu}_{\rho\sigma}}\right)_{\bar{g}} = N\delta^{[\rho}_\mu \delta^{\sigma]}_\nu \quad (5.7)$$

for some constant N. Formally, this constant can be expressed as a "projection"

$$N = P^{\mu\nu}_{\rho\sigma}\left(\frac{\partial \mathcal{L}}{\partial R^{\mu\nu}_{\rho\sigma}}\right)_{\bar{g}} \tag{5.8}$$

for the projection tensor

$$P^{\mu\nu}_{\rho\sigma} = \frac{2\delta^{\mu}_{[\rho}\delta^{\nu}_{\sigma]}}{d(d-1)}. \tag{5.9}$$

Inserting Eq. (5.7) into Eq. (5.6), we see that we simply need the expansion of $N\sqrt{-g}R$. This is of course just the Einstein-Hilbert action (up to the overall factor N), whose expansion is well-known to give the Fierz-Pauli action (see, e.g., Ref. [31]), $NL_{\mathcal{G}}$.

The $O(h^2)$ terms in the action with four derivatives can only arise from the term

$$\sqrt{-\bar{g}}\left(\frac{\partial^2 L}{\partial R^{\mu\nu}_{\rho\sigma}\partial R^{\gamma\delta}_{\alpha\beta}}\right)_{\bar{g}} \delta R^{\mu\nu}_{\rho\sigma}\delta R^{\gamma\delta}_{\alpha\beta}, \tag{5.10}$$

and for this we just need the linear term in $\delta R^{\mu\nu}_{\rho\sigma}$,

$$(\delta R^{\rho\sigma}_{\lambda\nu})^L = \partial_\lambda \Upsilon_\nu{}^{\sigma\rho} - \partial_\nu \Upsilon_\lambda{}^{\sigma\rho}. \tag{5.11}$$

Now, the second derivative evaluated on a homogeneous background can only be a function of $\bar{g}_{\mu\nu}$, and this quantity must have the same index symmetries as the product of two Riemann tensors. Hence, there can be three independent contributions

$$\left(\frac{\partial^2 \mathcal{L}}{\partial R^{\gamma\delta}_{\alpha\beta}\partial R^{\mu\nu}_{\rho\sigma}}\right)_{\bar{g}} = N_1\delta^{[\alpha}_{\mu}\delta^{\beta]}_{\nu}\delta^{\rho}_{[\gamma}\delta^{\sigma}_{\delta]} + N_2\delta^{[\alpha}_{\gamma}\delta^{\beta]}_{\delta}\delta^{\rho}_{[\mu}\delta^{\sigma}_{\nu]} + N_3\delta^{[\beta}_{\delta}\delta^{\alpha]}_{\mu}\delta^{[\sigma}_{\nu}\delta^{\rho]}_{\gamma} \tag{5.12}$$

for some constants N_i. This is similar to the statement that there are only three independent curvature invariants formed from contracting two Riemann tensors. Formally, these constants can be expressed by acting with projectors

$$N_i = P_{(i)\alpha\beta\rho\sigma}^{\gamma\delta,\mu\nu}\left(\frac{\partial^2 \mathcal{L}}{\partial R^{\gamma\delta}_{\alpha\beta}\partial R^{\mu\nu}_{\rho\sigma}}\right)_{\bar{g}}, \tag{5.13}$$

where

$$P_{(i)\alpha\beta\rho\sigma}^{\gamma\delta,\mu\nu} = a_i\delta^{\mu}_{[\alpha}\delta^{\nu}_{\beta]}\delta^{[\gamma}_{\rho}\delta^{\delta]}_{\sigma} + b_i\delta^{\gamma}_{[\alpha}\delta^{\delta}_{\beta]}\delta^{[\mu}_{\rho}\delta^{\nu]}_{\sigma} + c_i\delta^{\delta}_{[\beta}\delta^{\mu}_{\alpha]}\delta^{\nu}_{[\sigma}\delta^{\gamma}_{\rho]}. \tag{5.14}$$

The coefficients are

$$a_1 = b_2 = (d-1)^3 p, \tag{5.15}$$

$$a_2 = b_1 = -(d-1)p, \tag{5.16}$$

$$a_3 = b_3 = c_1 = c_2 = -(d-2)(d-1)p, \tag{5.17}$$

$$c_3 = (d^2 - d + 2)(d-2)p, \tag{5.18}$$

with

$$p \equiv \frac{4}{d(d^2-1)(d-1)(d-2)(d^2-2d-2)}. \tag{5.19}$$

The next step is to insert Eq. (5.12) into Eq. (5.10) and use Eq. (5.11). We treat the three contractions separately. The first is analogous to $R^2_{\mu\nu\rho\sigma}$ and gives

$$N_1\delta^{[\alpha}_{\mu}\delta^{\beta]}_{\nu}\delta^{\rho}_{[\gamma}\delta^{\sigma}_{\delta]}(\delta R^{\mu\nu}_{\rho\sigma})^L(\delta R^{\gamma\delta}_{\alpha\beta})^L = 2N_1 L_{H^{(1)}} + 4N_1 L_{H^{(2)}}. \tag{5.20}$$

The second contraction is analogous to R^2 and gives

$$N_2\delta^{[\alpha}_{\gamma}\delta^{\beta]}_{\delta}\delta^{\rho}_{[\mu}\delta^{\sigma}_{\nu]}(\delta R^{\mu\nu}_{\rho\sigma})^L(\delta R^{\gamma\delta}_{\alpha\beta})^L = 2N_2 L_{H^{(1)}}. \tag{5.21}$$

The last contraction is analogous to $R^2_{\mu\nu}$ and gives

$$N_3\delta^{[\beta}_{\delta}\delta^{\alpha]}_{\mu}\delta^{[\sigma}_{\nu}\delta^{\rho]}_{\gamma}(\delta R^{\mu\nu}_{\rho\sigma})^L(\delta R^{\gamma\delta}_{\alpha\beta})^L = N_3 L_{H^{(1)}} + N_3 L_{H^{(2)}}. \tag{5.22}$$

Thus the relevant part of the expanded action is

$$\delta(\sqrt{-g}\mathcal{L}) = NL_{\mathcal{G}} + \left(N_1 + N_2 + \frac{N_3}{2}\right)L_{H^{(1)}} + 2\left(N_1 + \frac{N_3}{4}\right)L_{H^{(2)}} + \cdots, \tag{5.23}$$

and the corresponding stress tensor is

$$T_{\mu\nu} = N\mathcal{G}^L_{\mu\nu} + \left(N_1 + N_2 + \frac{1}{2}N_3\right)H^{(1)}_{\mu\nu} + \left(2N_1 + \frac{1}{2}N_3\right)H^{(2)}_{\mu\nu}. \tag{5.24}$$

VI. THE GENERAL FORMULA FOR THE STRESS TENSOR (CURVED BACKGROUND)

The procedure described previously for a flat background should in principle generalize to a curved background. The calculation becomes cumbersome, however, since the covariant derivatives no longer commute. Instead, we shall adopt a different approach that turns out to be much more straightforward.

It was argued in Refs. [10,11,25–27] that any higher-curvature theory which is polynomial in the Riemann tensor and its contractions can be reduced to an "effective quadratic curvature" action with the same propagator. Since the propagator also only depends on the action up to order h^2, we can adapt this procedure to determine the ADT stress-tensor for a general theory.

Consider expanding the Lagrangian of a generic higher-curvature theory of the form $\mathcal{L} = \mathcal{L}(R^{\mu\nu}_{\rho\sigma})$,

$$\mathcal{L} = \mathcal{L}(\bar{R}^{\mu\nu}_{\rho\sigma}) + \left(\frac{\partial \mathcal{L}}{\partial R^{\mu\nu}_{\rho\sigma}}\right)_{\bar{g}} (R^{\mu\nu}_{\rho\sigma} - \bar{R}^{\mu\nu}_{\rho\sigma}) + \frac{1}{2}\left(\frac{\partial^2 \mathcal{L}}{\partial R^{\mu\nu}_{\rho\sigma}\partial R^{\gamma\delta}_{\alpha\beta}}\right)_{\bar{g}} (R^{\mu\nu}_{\rho\sigma} - \bar{R}^{\mu\nu}_{\rho\sigma})(R^{\gamma\delta}_{\alpha\beta} - \bar{R}^{\gamma\delta}_{\alpha\beta}) + \cdots. \tag{6.1}$$

Here the dots represent terms which are necessarily of order h^3 and therefore are not relevant to the ADT energy. Next we substitute the general expressions for the derivatives of the Lagrangian with respect to the Riemann tensor evaluated on the background, which were previously given in Eqs. (5.7) and (5.12). Using Eq. (2.17) and collecting coefficients of the full Riemann tensor terms, we obtain the effective quadratic theory

$$L_{\text{eff}} = \frac{1}{2\tilde{\kappa}}(R - 2\Lambda_0^{\text{eff}}) + \alpha R^2 + \beta R^2_{\mu\nu} + \gamma(R^2_{\mu\nu\rho\sigma} - 4R^2_{\mu\nu} + R^2). \tag{6.2}$$

Here we have defined

$$\frac{1}{2\tilde{\kappa}} \equiv N - \frac{2\Lambda d}{d-2}N_2 - \frac{4\Lambda}{(d-1)(d-2)}N_1 - \frac{2\Lambda}{d-2}N_3, \tag{6.3}$$

$$\alpha \equiv \frac{1}{2}(N_2 - N_1), \tag{6.4}$$

$$\beta \equiv \frac{1}{2}(N_3 + 4N_1), \tag{6.5}$$

$$\gamma \equiv \frac{1}{2}N_1, \tag{6.6}$$

and the "bare" cosmological constant for the effective theory is

$$\Lambda_0^{\text{eff}} = -\tilde{\kappa}\left(\mathcal{L}(\bar{R}^{\mu\nu}_{\rho\sigma}) - \frac{2\Lambda d}{d-2}N + \frac{2\Lambda^2 d^2}{(d-2)^2}N_2 + \frac{4\Lambda^2 d}{(d-2)^2(d-1)}N_1 + \frac{2d\Lambda^2}{(d-2)^2}N_3\right). \tag{6.7}$$

Now, the most general quadratic curvature theory has already been treated in Ref. [13]. The result is that the stress tensor is

$$T_{\mu\nu} = \left(\frac{1}{2\tilde{\kappa}} + \frac{4d\Lambda}{d-2}\alpha + \frac{4\Lambda}{d-1}\beta + \frac{4(d-3)(d-4)\Lambda}{(d-2)(d-1)}\gamma\right)\mathcal{G}^L_{\mu\nu} + (2\alpha + \beta)H^{(1)}_{\mu\nu} + \beta H^{(2)}_{\mu\nu}, \tag{6.8}$$

where $\mathcal{G}^L_{\mu\nu}$, $H^{(1)}_{\mu\nu}$, and $H^{(2)}_{\mu\nu}$ were given in Eqs. (4.2), (4.3), and (4.4). Furthermore, the effective cosmological constant Λ is fixed by evaluating the equation of motion on the background solution:

$$\left[\frac{(d-4)(d\alpha + \beta)}{(d-2)^2} + \frac{(d-3)(d-4)\gamma}{(d-1)(d-2)}\right]\Lambda^2 + \frac{\Lambda - \Lambda_0^{\text{eff}}}{4\tilde{\kappa}} = 0. \tag{6.9}$$

Substituting the above expressions for $\tilde{\kappa}$, α, β, γ into Eq. (6.8) yields

$$T_{\mu\nu} = \left[N - \frac{4\Lambda}{d-2}N_1 - \frac{2\Lambda}{(d-1)(d-2)}N_3\right]\mathcal{G}^L_{\mu\nu} + \left[N_1 + N_2 + \frac{1}{2}N_3\right]H^{(1)}_{\mu\nu} + \left[2N_1 + \frac{1}{2}N_3\right]H^{(2)}_{\mu\nu}. \tag{6.10}$$

Note that for $\Lambda = 0$, this agrees with the result of the previous section.

A. Examples

Let us look at some examples for the formula (6.10) for the stress-energy tensor. Let us start with the simple theory

$$\mathcal{L} = R^2_{\mu\nu\rho\sigma} = R^{\mu\nu}_{\rho\sigma}R^{\rho\sigma}_{\mu\nu}. \tag{6.11}$$

The coefficients N, N_i are computed as described previously by taking derivatives with respect to the Riemann tensor and evaluating on the background AdS solution. We find that

$$N = P^{\eta\varepsilon}_{\lambda\kappa}\left(\frac{\partial \mathcal{L}}{\partial R^{\eta\varepsilon}_{\lambda\kappa}}\right)_{\bar{g}} = P^{\eta\varepsilon}_{\lambda\kappa}(2R^{\lambda\kappa}_{\eta\varepsilon})_{\bar{g}} = P^{\eta\varepsilon}_{\lambda\kappa}\left(\frac{4\Lambda}{(d-1)(d-2)}(\delta^\lambda_\eta\delta^\kappa_\varepsilon - \delta^\lambda_\varepsilon\delta^\kappa_\eta)\right) = \frac{8\Lambda}{(d-1)(d-2)} \tag{6.12}$$

and

$$N_i = P^{(i)\eta\varepsilon\alpha\beta}_{\lambda\kappa\gamma\delta}\left(\frac{\partial^2 \mathcal{L}}{\partial R^{\gamma\delta}_{\alpha\beta}\partial R^{\eta\varepsilon}_{\lambda\kappa}}\right)_{\bar{g}} = P^{(i)\eta\varepsilon\alpha\beta}_{\lambda\kappa\gamma\delta}\left(\frac{\partial}{\partial R^{\gamma\delta}_{\alpha\beta}}2R^{\lambda\kappa}_{\eta\varepsilon}\right)_{\bar{g}} = 2P^{(i)\eta\varepsilon,\alpha\beta}_{\lambda\kappa\gamma\delta}\delta^\lambda_\gamma\delta^\kappa_\delta\delta^\alpha_\eta\delta^\beta_\varepsilon, \tag{6.13}$$

so $N_1 = 2$, $N_2 = N_3 = 0$. Using Eq. (6.10), we obtain

$$T_{\mu\nu} = -\frac{8\Lambda}{d-1}\mathcal{G}^L_{\mu\nu} + 2H^{(1)}_{\mu\nu} + 4H^{(2)}_{\mu\nu}, \tag{6.14}$$

which matches the result of Ref. [13].

For a more complicated example, consider the six-derivative theory

$$\mathcal{L} = R + b_1R^2 + b_2(R^2_{\mu\nu\rho\sigma} - 4R^2_{\mu\nu} + R^2) + b_3R^2_{\mu\nu} + c_1R^{\mu\nu}{}_{\alpha\beta}R^{\alpha\beta}{}_{\lambda\rho}R^{\lambda\rho}{}_{\mu\nu} + c_2R^{\mu\nu}{}_{\rho\sigma}R^{\rho\tau}{}_{\lambda\mu}R^{\sigma}{}_{\tau}{}^{\lambda}{}_{\nu}, \tag{6.15}$$

whose stress tensor was computed explicitly in Ref. [24]. The results are summarized in the following table:

$\mathcal{L}$	N	N_1	N_2	N_3
R	1	0	0	0
R^2	$2d(d-1)k$	0	2	0
$R_{\mu\nu}^2$	$2(d-1)k$	0	0	2
$R_{\mu\nu\rho\sigma}^2$	$4k$	2	0	0
$R_{\mu\nu\rho\sigma}^2 - 4R_{\mu\nu}^2 + R^2$	$2(d-3)(d-2)k$	2	2	-8
$R^{\mu\nu}{}_{\alpha\beta}R^{\alpha\beta}{}_{\lambda\rho}R^{\lambda\rho}{}_{\mu\nu}$	$12k^2$	$12k$	0	0
$R^{\mu\nu}{}_{\rho\sigma}R^{\rho\tau}{}_{\lambda\mu}R^{\sigma\lambda}{}_{\tau\nu}$	$3(d-2)k^2$	$-3k$	0	$6k$

where

$$k \equiv \frac{2\Lambda}{(d-1)(d-2)}. \tag{6.16}$$

Substituting the above results into Eq. (6.10) gives

$$T_{\mu\nu} = \alpha_1 \mathcal{G}^L_{\mu\nu} + \alpha_2 H^{(1)}_{\mu\nu} + \alpha_3 H^{(2)}_{\mu\nu}, \tag{6.17}$$

where

$$\begin{aligned}
\alpha_1 &= 1 + \frac{4d\Lambda b_1}{d-2} + \frac{4(d-3)(d-4)\Lambda b_2}{(d-2)(d-1)} + \frac{4\Lambda b_3}{d-1} \\
&\quad - \frac{48(2d-3)\Lambda^2 c_1}{(d-2)^2(d-1)^2} + \frac{36\Lambda^2 c_2}{(d-2)(d-1)^2}, \\
\alpha_2 &= 2b_1 + b_3 + \frac{24\Lambda c_1}{(d-2)(d-1)}, \\
\alpha_3 &= b_3 + \frac{48\Lambda c_1}{(d-2)(d-1)} - \frac{6\Lambda c_2}{(d-2)(d-1)}.
\end{aligned} \tag{6.18}$$

This reproduces precisely the stress tensor obtained in Ref. [24].

VII. THE DERIVATION OF THE ENERGY FORMULA

Given the result (6.10), the final step in the derivation of the energy formula is to write $\bar{\xi}_\nu T^{\mu\nu}$ as a total derivative. For this purpose, we follow the steps in Ref. [12]. For the first term, $\bar{\xi}_\nu \mathcal{G}_L^{\mu\nu}$, the result has already been given in Eq. (3.6). It is straightforward to show that the second term can be written as

$$\bar{\xi}_\nu H^{(1)\mu\nu} = \bar{\nabla}_\alpha(\bar{\xi}^\mu\bar{\nabla}^\alpha R_L + R_L\bar{\nabla}^\mu\bar{\xi}^\alpha - \bar{\xi}^\alpha\bar{\nabla}^\mu R_L). \tag{7.1}$$

The third term, $\bar{\xi}_\nu H^{(2)\mu\nu}$, is more complicated and turns out to give an additional contribution of the form $\bar{\xi}_\nu \mathcal{G}_L^{\mu\nu}$. To see this, we can rewrite

$$\begin{aligned}
\bar{\xi}_\nu \bar{\Box}\mathcal{G}_L^{\mu\nu} &= \bar{\nabla}_\alpha(\bar{\xi}_\nu\bar{\nabla}^\alpha \mathcal{G}_L^{\mu\nu} - \bar{\xi}_\nu\bar{\nabla}^\mu \mathcal{G}_L^{\alpha\nu} - \mathcal{G}_L^{\mu\nu}\bar{\nabla}^\alpha\bar{\xi}_\nu \\
&\quad + \mathcal{G}_L^{\alpha\nu}\bar{\nabla}^\mu\bar{\xi}_\nu) + \mathcal{G}_L^{\mu\nu}\bar{\Box}\bar{\xi}_\nu + \bar{\xi}_\nu\bar{\nabla}_\alpha\bar{\nabla}^\mu\mathcal{G}_L^{\alpha\nu} \\
&\quad - \mathcal{G}_L^{\alpha\nu}\bar{\nabla}_\alpha\bar{\nabla}^\mu\bar{\xi}_\nu.
\end{aligned} \tag{7.2}$$

Since $\bar{\xi}^\nu$ is a Killing vector, it satisfies

$$\begin{aligned}
\bar{\nabla}_\alpha\bar{\nabla}_\mu\bar{\xi}_\nu &= \bar{R}^\rho{}_{\nu\mu\alpha}\bar{\xi}_\rho \\
&= \frac{2\Lambda}{(d-2)(d-1)}(\bar{g}_{\nu\alpha}\bar{\xi}_\mu - \bar{g}_{\alpha\mu}\bar{\xi}_\nu),
\end{aligned} \tag{7.3}$$

$$\bar{\Box}\bar{\xi}_\nu = -\frac{2\Lambda}{d-2}\bar{\xi}_\nu. \tag{7.4}$$

Then the last terms of Eq. (7.2) simplify to

$$\bar{\xi}_\nu\bar{\nabla}_\alpha\bar{\nabla}^\mu\mathcal{G}_L^{\alpha\nu} = \frac{2\Lambda d}{(d-2)(d-1)}\bar{\xi}_\nu\mathcal{G}_L^{\mu\nu} + \frac{\Lambda}{d-1}\bar{\nabla}^\mu R_L, \tag{7.5}$$

$$\begin{aligned}
&\mathcal{G}_L^{\mu\nu}\bar{\Box}\bar{\xi}_\nu + \bar{\xi}_\nu\bar{\nabla}_\alpha\bar{\nabla}^\mu\mathcal{G}_L^{\alpha\nu} - \mathcal{G}_L^{\alpha\nu}\bar{\nabla}_\alpha\bar{\nabla}^\mu\bar{\xi}_\nu \\
&\quad = \frac{\Lambda}{d-1}\left(\frac{4}{d-2}\mathcal{G}_L^{\mu\nu}\bar{\xi}_\nu + \bar{\xi}^\mu R_L - \frac{2}{d-2}\mathcal{G}_L\bar{\xi}^\mu\right).
\end{aligned} \tag{7.6}$$

Using these results, we find that the final form of the conserved energy is

$$\begin{aligned}
E &= \left(N - \frac{4\Lambda(d-3)}{(d-1)(d-2)}N_1\right)2\kappa E_0 + \left(N_1 + N_2 + \frac{N_3}{2}\right)\int_{\partial\Sigma} d^{d-2}x\sqrt{\bar{g}_{\partial\Sigma}}\, n_\mu r_\nu(\bar{\xi}^\mu\bar{\nabla}^\nu R_L + R_L\bar{\nabla}^\mu\bar{\xi}^\nu - \bar{\xi}^\nu\bar{\nabla}^\mu R_L) \\
&\quad + \left(2N_1 + \frac{N_3}{2}\right)\int_{\partial\Sigma} d^{d-2}x\sqrt{\bar{g}_{\partial\Sigma}}\, n_\mu r_\nu(\bar{\xi}_\alpha\bar{\nabla}^\nu\mathcal{G}_L^{\mu\alpha} - \bar{\xi}_\alpha\bar{\nabla}^\mu\mathcal{G}_L^{\nu\alpha} - \mathcal{G}_L^{\mu\alpha}\bar{\nabla}^\nu\bar{\xi}_\alpha + \mathcal{G}_L^{\nu\alpha}\bar{\nabla}^\mu\bar{\xi}_\alpha).
\end{aligned} \tag{7.7}$$

In asymptotically SdS spacetimes [see Eq. (2.10)], the last two terms in Eq. (7.7) fall off too fast at large r to contribute and the total energy is given by

$$E = \left(N - \frac{4\Lambda(d-3)}{(d-1)(d-2)}N_1\right)\frac{(d-2)\mathrm{Vol}(S^{d-2})}{2}r_0^{d-3}, \tag{7.8}$$

or in the full explicit form as in Eq. (2.3).

VIII. DISCUSSION

In this paper, we have derived a simple formula (2.3) for the ADT energy of any gravitational theory of the form $\mathcal{L} = \mathcal{L}(R^{\mu\nu}_{\rho\sigma}, g_{\mu\nu})$. We gave a detailed argument that the energy of such a theory takes the same basic form as in quadratic curvature gravity, but with coefficients modified by the higher-curvature terms. The coefficients are given by taking derivatives of the Lagrangian with respect to the

Riemann tensor, and in this sense our energy formula is reminiscent of Wald's entropy formula. We have demonstrated in a number of examples that our formula correctly reproduces previous results, but with significantly less computations. For more complicated theories in which following the full ADT procedure would be unmanageable in practice, it seems that our formula could still be applied relatively easily.

We note from the final formula for energy (2.3) that only N and N_1 appear, and it would be interesting to understand why this is the case. We also see that in $d = 3$, the contribution of the second derivative of the Lagrangian completely drops out. In the case of three-dimensional topologically massive gravity (TMG) [32–34], the action contains a gravitational Chern-Simons term so it is not of the form $\mathcal{L} = \mathcal{L}(R^{\mu\nu}_{\rho\sigma}, g_{\mu\nu})$. Indeed, the ADT energy for TMG has a different structure than Eq. (2.3) [35]. It is also known that Wald's entropy formula has to be modified in TMG, since the Chern-Simons term does not satisfy the diffeomorphism-covariance requirement in the original construction (see, e.g., Refs. [36,37]).

Given the final expression for the energy (7.8), it seems natural to define the effective gravitational coupling as

$$\frac{1}{2\kappa_{\text{eff}}} = N - \frac{4\Lambda(d-3)}{(d-1)(d-2)} N_1. \tag{8.1}$$

Then the energy can be written succinctly in terms of the Einstein gravity result as

$$E = \frac{\kappa}{\kappa_{\text{eff}}} E_0. \tag{8.2}$$

This is analogous to the way the entropy was written in Ref. [8] as

$$S = \frac{A}{4G_{\text{eff}}}, \tag{8.3}$$

where A is the black hole area. However, the effective coupling constant also has an interpretation in the tree-level scattering amplitude via the exchange of a graviton. When one looks at a similar process on the background [10], the effective coupling turns out to be the coefficient of $\mathcal{G}^L_{\mu\nu}$ in the stress-energy tensor:

$$\frac{1}{2\kappa_{\text{eff}}} = N - \frac{4\Lambda}{d-2} N_1 - \frac{2\Lambda}{(d-1)(d-2)} N_3. \tag{8.4}$$

The two definitions for the effective coupling coincide for Lanczos-Lovelock gravity, since any Lagrangian of the Lanczos-Lovelock type can be reduced to a Gauss-Bonnet quadratic theory [11]. This coincidence might be related to the fact that higher-derivative theories which are not of the Lanczos-Lovelock type exhibit ghosts and other inconsistencies [38,39]. In future work, it would be interesting to further understand this ambiguity in the definition of the effective gravitational coupling.

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