First law of mechanics for black hole binaries with spins

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We use the canonical Hamiltonian formalism to generalize to spinning point particles the first law of mechanics established for binary systems of nonspinning point masses moving on circular orbits [A. Le Tiec, L. Blanchet, and B. F. Whiting, Phys. Rev. D **85,** 064039 (2012)]. We find that the redshift observable of each particle is related in a very simple manner to the canonical Hamiltonian and, more generally, to a class of Fokker-type Hamiltonians. Our results are valid through linear order in the spin of each particle, but hold also for quadratic couplings between the spins of different particles. The knowledge of spin effects in the Hamiltonian allows us to compute spin-orbit terms in the redshift variable through 2.5PN order, for circular orbits and spins aligned or anti-aligned with the orbital angular momentum. To describe extended bodies such as black holes, we supplement the first law for spinning point-particle binaries with some "constitutive relations" that can be used for diagnosis of spin measurements in quasi-equilibrium initial data.

I. INTRODUCTION

The need to develop faithful template waveforms for the search of gravitational waves from compact-object binary systems (compact binaries, for short) with current interferometric detectors [1,2] has led, during the last years, to synergies and unexpected common ground among previously distinct research areas. The two main analytical frameworks used to study the relativistic dynamics of compact binaries are (i) the post-Newtonian (PN) approximation [3,4], which describes the inspiraling motion beyond the Newtonian limit, in the small velocity and weak-field regime ($v \ll c$), and (ii) black hole perturbation theory [5], which describes both the weak-field and strong-field dynamics in the small mass-ratio limit ($m_1 \ll m_2$). The other main research area in the study of the two-body problem is numerical relativity [6], which aims at describing the strong-field regime by solving numerically the exact Einstein field equations. Furthermore, in order to build templates for the search of gravitational waves, the effective-one-body (EOB) approach [7,8] combines the information from those different techniques in a flexible and effective way, and can provide accurate merger templates for advanced LIGO and Virgo. As said above, several studies [9–21] at the interface between those different frameworks have improved our knowledge of the two-body dynamics and gravitational-wave emission.

Recently, Le Tiec, Blanchet, and Whiting [22] (henceforth, Paper I) derived a "first law of mechanics" for binary systems of point particles moving on exact circular orbits (compatible with an helical Killing symmetry). Using the first law, the authors found a very simple relation between the PN binding energy of the binary and Detweiler's redshift observable [11]—which can also be

interpreted as the particle's Killing energy associated with the helical symmetry. This relation allowed the computation of previously unknown high-order PN terms in the circular-orbit binding energy by using the redshift observable computed numerically within the gravitational self-force (GSF) formalism [11–13,23]. Moreover, the results in Paper I led to the following applications:

(1) Reference [19] computed the *exact* binding energy and angular momentum of two nonspinning compact objects (modeled as point particles) moving on circular orbits, at leading order beyond the test-particle approximation, and recovered the exact frequency shift of the Schwarzschild innermost stable circular orbit induced by the conservative piece of the GSF [24].

(2) Reference [20] built upon the works [14,15,19,22] and derived the exact expressions of the EOB metric components through first order in the (symmetric) mass ratio. Quite interestingly, the results in Refs. [16,18,19] strongly suggest that the domain of validity of black hole perturbation theory calculations may extend well beyond the extreme mass-ratio limit.

Given the relevance of the first law of binary mechanics in enhancing our knowledge of the two-body dynamics in the comparable-mass case by using information from the perturbative GSF framework, we generalize it here to point particles carrying spins. We derive the first law of mechanics using the canonical Arnowitt-Deser-Misner (ADM) formalism [25] applied to spinning point particles [26,27]. In the PN context, both the ADM formalism and the harmonic-coordinate approach have been developed up to high orders in the approximation, in order to compute

the conservative part of the orbital dynamics of compact binaries. For nonspinning point masses, the Hamiltonian [28–30] and the harmonic-coordinates equations of motion [31–35] are known through 3PN order. More recently, partial results at 4PN order have been reported [36,37]. The logarithmic contributions to the conservative dynamics at 4PN and 5PN orders are also known [13,22]. High-order spin-orbit effects have been computed in harmonic coordinates [38–40], in the Hamiltonian [26,41,42], and within the effective field theory (EFT) approach [43]. See Ref. [44] for a review of the canonical ADM formalism for spinning point particles, and its application in PN theory. In this paper we shall not consider spin-spin couplings or higher-order effects in the spins. High-order spin-spin effects have been computed in the Hamiltonian [26,45–48] and using EFT techniques [49–54].

The paper is organized as follows. In Secs. II and III we review the Lagrangian and Hamiltonian formalisms for systems of point particles carrying spins. In particular, we derive a crucial relationship between the redshift observable and the variation of a general (Fokker-type) Hamiltonian, valid at linear order in the spins. In Sec. IV we derive the first law of mechanics for binary systems of spinning point particles using the canonical ADM formalism, and discuss some mathematical and physical consequences of this law. Then, in Sec. V, we employ the first law of mechanics and the ADM Hamiltonian to compute the spin-orbit contributions to the redshift observable through the (leading plus subleading) 2.5PN order. We find full agreement between our results and those recently obtained from a direct computation based on the near-zone PN metric [55]. In Sec. VI we discuss the first law in the particular case of binary black holes in corotation. Finally, Sec. VII summarizes our main results and discusses some prospects. Throughout this paper we use "geometrized units" where $G = c = 1$.

II. LAGRANGIAN AND HAMILTONIAN OF A SPINNING POINT PARTICLE

In this section, we review some necessary material for constructing a Lagrangian and then a Hamiltonian for a spinning point particle in curved spacetime. The formalism we shall use derives from the early works [56,57]. It has recently been developed in the context of the EFT framework [49]. Alternatives and variants to this formalism can be found in Refs. [26,44,58]. The formalism yields for the equations of motion of spinning particles and the precession of the spins the classic results valid in general relativity [59–66].

Let us consider a single spinning point particle moving in a given curved "background" metric $g_{\mu\nu}(x)$. The particle follows the worldline $r^\mu(s)$, with tangent four-velocity $u^\mu = \mathrm{d}r^\mu/\mathrm{d}s$, where s is a parameter along the representative worldline. In a first stage we do not require that the four-velocity be normalized, i.e., s need not be the proper

time elapsed along the worldline. To describe some internal degrees of freedom to be associated with the particle's spin, we introduce a moving tetrad $e_\alpha{}^\mu(s)$ along the trajectory. The tetrad is orthonormal, in the sense that $g_{\mu\nu}e_\alpha{}^\mu e_\beta{}^\nu = \eta_{\alpha\beta}$, and defines a "body-fixed" frame.[1] The rotation tensor $\Omega^{\mu\nu}$ associated with that tetrad is defined by

$$\frac{\mathrm{D}e_\alpha{}^\mu}{\mathrm{d}s} = -\Omega^{\mu\nu}e_{\alpha\nu}, \qquad (2.1)$$

where $\mathrm{D}/\mathrm{d}s \equiv u^\nu\nabla_\nu$ is the covariant derivative with respect to the parameter s along the worldline. Equivalently, we have

$$\Omega^{\mu\nu} = e^{\alpha\mu}\frac{\mathrm{D}e_\alpha{}^\nu}{\mathrm{d}s}. \qquad (2.2)$$

Because of the normalization of the tetrad the rotation tensor is antisymmetric: $\Omega^{\mu\nu} = -\Omega^{\nu\mu}$.

A. Lagrangian formulation

Following Refs. [49,56], we look for an action for the spinning particle that is at once (i) a covariant scalar, (ii) an internal Lorentz scalar, and (iii) reparametrization invariant; i.e., its form must be independent of the parameter used to follow the particle's worldline. We shall assume that the dynamical degrees of freedom are the particle's position r^μ and the tetrad $e_\alpha{}^\mu$. We restrict ourselves to a Lagrangian depending only on the four-velocity u^μ, the rotation tensor $\Omega^{\mu\nu}$, and the metric $g_{\mu\nu}$. In particular, this confines the formalism we are using to a pole-dipole model and to terms linear in the spins.[2] Thus, the postulated "particle" action is of the type

$$S_{\mathrm{part}}[r^\mu, e_\alpha{}^\mu] = \int \mathrm{d}s\,\hat{L}_{\mathrm{part}}(u^\mu, \Omega^{\mu\nu}, g_{\mu\nu}). \qquad (2.3)$$

We cover the Lagrangian $\hat{L}_{\mathrm{part}}$ with a hat in order to distinguish it from the usual definition of the Lagrangian, $L_{\mathrm{part}} = \hat{L}_{\mathrm{part}}\mathrm{d}s/\mathrm{d}t$ (with t the coordinate time), that we shall use later.

As it is written, depending only on Lorentz scalars, $\hat{L}_{\mathrm{part}}$ is automatically a Lorentz scalar. By performing an infinitesimal coordinate transformation, one easily sees that the requirement that the Lagrangian be a covariant scalar specifies its dependence on the metric as being (see, e.g., Ref. [57])

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[1]Here $\eta_{\alpha\beta} = \mathrm{diag}(-1, 1, 1, 1)$ denotes the Minkowski metric. The indices $\mu, \nu, \ldots$ are the usual spacetime covariant indices, while $\alpha, \beta, \ldots$ are the internal Lorentz indices. The inverse (or dual) tetrad $e^\alpha{}_\mu$, defined by $e_\alpha{}^\mu e^\alpha{}_\nu = \delta^\mu_\nu$, satisfies $\eta_{\alpha\beta}e^\alpha{}_\mu e^\beta{}_\nu = g_{\mu\nu}$. We also have the completeness relation $e_\alpha{}^\mu e^\beta{}_\mu = \delta^\beta_\alpha$.

[2]Such a model is "universal" in the sense that it can be used for black holes as well as neutrons stars. Indeed, the internal structure of the spinning body appears only at the next $\mathcal{O}(S^2)$, e.g., through the rotationally induced quadrupole moment.

$$2\frac{\partial \hat{L}_{\text{part}}}{\partial g_{\mu\nu}} = p^\mu u^\nu + S^\mu{}_\rho \Omega^{\nu\rho}, \qquad (2.4)$$

where we have *defined* the conjugate linear momentum p^μ and the antisymmetric spin tensor $S^{\mu\nu}$ as

$$p_\mu \equiv \frac{\partial \hat{L}_{\text{part}}}{\partial u^\mu}\bigg|_{\Omega,g}, \qquad (2.5a)$$

$$S_{\mu\nu} \equiv 2\frac{\partial \hat{L}_{\text{part}}}{\partial \Omega^{\mu\nu}}\bigg|_{u,g}. \qquad (2.5b)$$

Note that the right-hand side (RHS) of Eq. (2.4) is necessarily symmetric by exchange of the indices μ and ν. Finally, imposing the invariance of the action (2.3) by reparametrization of the worldline, we find that the Lagrangian must be a homogeneous function of degree one in the velocities u^μ and $\Omega^{\mu\nu}$. Applying Euler's theorem to the function $\hat{L}_{\text{part}}(u^\mu, \Omega^{\mu\nu})$ immediately gives

$$\hat{L}_{\text{part}} = p_\mu u^\mu + \frac{1}{2} S_{\mu\nu}\Omega^{\mu\nu}, \qquad (2.6)$$

where the functions $p_\mu(u^\rho, \Omega^{\rho\sigma})$ and $S_{\mu\nu}(u^\rho, \Omega^{\rho\sigma})$ must be reparametrization invariant. Note that, at this stage, their explicit expressions are not known. They will be specified only once a *spin supplementary condition* (SSC) is imposed, as discussed in Sec. II C below.

We now investigate the unconstrained variations of the action (2.3) with respect to $e_\alpha{}^\mu$, r^μ, and $g^{\mu\nu}$. First, we vary it with respect to the tetrad $e_\alpha{}^\mu$ while keeping the position r^μ fixed. We must have a way to distinguish intrinsic variations of the tetrad from those which are induced by a change of the metric $g_{\mu\nu}$. This is done by decomposing the variation $\delta e_\alpha{}^\nu$ according to

$$\delta e_\alpha{}^\nu = e_{\alpha\mu}\left(\delta\theta^{\mu\nu} + \frac{1}{2}\delta g^{\mu\nu}\right), \qquad (2.7)$$

in which we have introduced the antisymmetric tensor $\delta\theta^{\mu\nu} \equiv e^{\alpha[\mu}\delta e_\alpha{}^{\nu]}$, and where the corresponding symmetric part is simply given by the variation of the metric, i.e., $e^{\alpha(\mu}\delta e_\alpha{}^{\nu)} = \frac{1}{2}\delta g^{\mu\nu}$. Then we can consider the independent variations $\delta\theta^{\mu\nu}$ and $\delta g^{\mu\nu}$. Varying with respect to $\delta\theta^{\mu\nu}$, but holding the metric fixed, gives the equation of spin precession which is found to be

$$\frac{DS_{\mu\nu}}{ds} = \Omega_\mu{}^\rho S_{\nu\rho} - \Omega_\nu{}^\rho S_{\mu\rho}, \qquad (2.8)$$

or, alternatively, using the fact that the RHS of Eq. (2.4) is symmetric,

$$\frac{DS_{\mu\nu}}{ds} = p_\mu u_\nu - p_\nu u_\mu. \qquad (2.9)$$

We next vary with respect to the particle's position r^μ while holding the tetrad $e_\alpha{}^\mu$ fixed. Operationally, this means that we have to parallel-transport the tetrad along the displacement vector; i.e., we have to impose

$$\delta r^\nu \nabla_\nu e_\alpha{}^\mu = 0. \qquad (2.10)$$

A simple way to derive the result is to use locally inertial coordinates, such that $\Gamma^\mu_{\nu\rho} = 0$ along the particle's worldline $r^\mu(s)$. Then, Eq. (2.10) yields $\delta e_\alpha{}^\mu = \delta r^\nu \partial_\nu e_\alpha{}^\mu = -\delta r^\nu \Gamma^\mu_{\nu\rho} e_\alpha{}^\rho = 0$, and the variation gives the well-known Mathisson-Papapetrou [61–63] equation of motion

$$\frac{Dp_\mu}{ds} = -\frac{1}{2} u^\nu R_{\mu\nu\rho\sigma} S^{\rho\sigma}. \qquad (2.11)$$

With more work, Eq. (2.11) can also be derived using an arbitrary coordinate system.

Finally, varying with respect to the metric (while keeping $\delta\theta^{\mu\nu} = 0$) gives the stress-energy tensor of the spinning particle. We must take into account the scalarity of the action, as imposed by Eq. (2.4), and we obtain the standard result [59–66]

$$T^{\mu\nu}_{\text{part}} = \int ds\, p^{(\mu}u^{\nu)}\frac{\delta^{(4)}(x-r)}{\sqrt{-g}} - \nabla_\rho \int ds\, S^{\rho(\mu}u^{\nu)}\frac{\delta^{(4)}(x-r)}{\sqrt{-g}}, \qquad (2.12)$$

where $\delta^{(4)}(x - r)$ denotes the four-dimensional Dirac function, such that $\int d^4 x\, \delta^{(4)}(x) = 1$. It can easily be checked that $\nabla_\nu T^{\mu\nu}_{\text{part}} = 0$ as a consequence of the equation of motion (2.11) and the equation of spin precession (2.9).

B. Hamiltonian formulation

We now want to define a Hamiltonian associated with the Lagrangian (2.6). Because of the reparametrization-invariance condition, performing a Legendre transformation with respect to the variables u^μ and $\Omega^{\mu\nu}$ yields a vanishing result. Different routes are possible. One may add in the action a mass-shell constraint with a Lagrange multiplier and vary the action, keeping the momentum p_μ as an independent variable. Through various changes of variables and gauge fixings, the action is transformed into an action which possesses Euler-Lagrange equations of the form of Hamilton's equations, such that the canonical (ADM) Hamiltonian can be read off from these equations (see Ref. [44] for a review).

A different strategy, that we shall follow here, is closer to the usual procedure of classical mechanics. It consists of using as internal dynamical variables the 6 rotational degrees of freedom of Lorentz matrices, and defining the Hamiltonian by means of a $3 + 1$ split, with a preferred time coordinate, with the usual Legendre transformation [58]. The tetrad $e_\alpha{}^\mu$ is decomposed into the product of an internal Lorentz transformation $\Lambda^\beta{}_\alpha$ and a reference orthonormal tetrad field $\epsilon_\alpha{}^\mu(x)$, which is associated with the background metric $g_{\mu\nu}(x)$ and is evaluated at the particle's position, $x^\mu = r^\mu(s)$.[3] The point is that using as dynamical variables the six internal angles ϕ^a of Lorentz matrices

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[3]Such decomposition was already implicit in the variation of the tetrad we performed to derive the spin precession equation (2.9).

$\Lambda^\beta{}_\alpha$ allows us to have usual-looking Euler-Lagrange equations. We thus write[4]

$$e_\alpha{}^\mu(\phi, r) = \Lambda^\beta{}_\alpha(\phi)\epsilon_\beta{}^\mu(r). \qquad (2.13)$$

Next, we perform a $3 + 1$ decomposition of all the fields. In particular, we split the particle's position and coordinate velocity according to $r^\mu = (t, \mathbf{r})$ and $v^\mu = (1, \mathbf{v})$, where $\mathbf{v} \equiv d\mathbf{r}/dt$ and boldface letters denote ordinary spatial vectors (often also denoted with spatial indices $i, j, \ldots = 1, 2, 3$). We choose for the parameter s along the worldline the proper time τ and define the so-called redshift variable z by [11,22]

$$z \equiv \frac{d\tau}{dt} = [-g_{\mu\nu}(t, \mathbf{r})v^\mu v^\nu]^{1/2}. \qquad (2.14)$$

Then the ordinary Lagrangian $L_{\text{part}} \equiv z\hat{L}_{\text{part}}$, such that $S_{\text{part}} = \int dt L_{\text{part}}$, becomes a function of the variables t, $\mathbf{r}$, $\mathbf{v}$, ϕ^a, and the ordinary derivative $\dot{\phi}^a \equiv d\phi^a/dt$, with the dependence in t and $\mathbf{r}$ entering only through the background metric or, rather, through the reference tetrad evaluated at the particle's location:

$$L_{\text{part}} = L_{\text{part}}[\mathbf{v}, \phi^a, \dot{\phi}^a; \epsilon_\alpha{}^\mu(t, \mathbf{r})]. \qquad (2.15)$$

We have ordinary Euler-Lagrange equations for the generalized coordinates $(\mathbf{r}, \phi^a)$ of the spinning particle (spatial vectors ranging over three components $i = 1, 2, 3$ while the rotation label runs over $a = 1, \ldots, 6$):

$$\frac{dP_i}{dt} = \frac{\partial L_{\text{part}}}{\partial r^i}, \qquad (2.16a)$$

$$\frac{dP_{\phi_a}}{dt} = \frac{\partial L_{\text{part}}}{\partial \phi^a}, \qquad (2.16b)$$

where the conjugate momenta are defined in the usual way by $P_i \equiv \partial L_{\text{part}}/\partial v^i$ and $P_{\phi_a} \equiv \partial L_{\text{part}}/\partial \dot{\phi}^a$. They explicitly read

$$P_i = p_i + \frac{1}{2} S_{\mu\nu} \epsilon^{\alpha\mu} \nabla_i \epsilon_\alpha{}^\nu, \qquad (2.17a)$$

$$P_{\phi_a} = \frac{1}{2} S_{\alpha\beta} \Lambda^{\gamma\beta} \frac{\partial \Lambda_\gamma{}^\alpha}{\partial \phi^a}, \qquad (2.17b)$$

where $S_{\alpha\beta} = e_\alpha{}^\mu e_\beta{}^\nu S_{\mu\nu}$ denote the tetrad components of the spin tensor in the "body-fixed" frame. The Euler-Lagrange equations (2.16a) and (2.16b) are equivalent to the equations of motion (2.11) and spin precession (2.9), respectively [58]. In particular, the spin precession equation is recovered in this formalism as the natural fact that the spin components remain constant in the frame attached to the body:

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[4]The indices $a, b = 1, \ldots, 6$ label the internal angles ϕ^a of Lorentz matrices satisfying $\eta_{\gamma\delta}\Lambda^\gamma{}_\alpha \Lambda^\delta{}_\beta = \eta_{\alpha\beta}$. Since $\epsilon_\alpha{}^\mu$ is a tetrad we have $g_{\mu\nu}\epsilon_\alpha{}^\mu \epsilon_\beta{}^\nu = \eta_{\alpha\beta}$ and, of course, still $g_{\mu\nu}e_\alpha{}^\mu e_\beta{}^\nu = \eta_{\alpha\beta}$.

$$\frac{dS_{\alpha\beta}}{d\tau} = 0. \qquad (2.18)$$

Finally, the Hamiltonian is simply given in this approach by the Legendre transformation as

$$H_{\text{part}} = P_i v^i + P_{\phi_a} \dot{\phi}^a - L_{\text{part}}, \qquad (2.19)$$

and Hamilton's equations yield Eqs. (2.11) and (2.9), or equivalently (2.18) [58]. The Hamiltonian is a function of conjugate variables and again depends on the position r^μ only through the background tetrad:

$$H_{\text{part}} = H_{\text{part}}[P_i, \phi^a, P_{\phi_a}; \epsilon_\alpha{}^\mu(t, \mathbf{r})]. \qquad (2.20)$$

An explicit computation of the Hamiltonian (2.19) with the help of Eqs. (2.6) and (2.17) yields [58]

$$H_{\text{part}} = -p_t - \frac{1}{2} S_{\mu\nu} \epsilon^{\alpha\mu} \nabla_t \epsilon_\alpha{}^\nu. \qquad (2.21)$$

Both terms should be understood as being functions of the conjugate variables, like in Eq. (2.20). As in ordinary classical mechanics, this is obtained by inverting Eqs. (2.17) to obtain v^i and $\dot{\phi}^a$ as functions of r^i, P_i, ϕ^a, and P_{ϕ_a}. However, here we need an additional relation to express p_t as a function of the conjugate variables; this is provided by Eq. (2.6) which, in the $3 + 1$ split, gives $p_t = L_{\text{part}} - v^i p_i - \frac{z}{2} S_{\mu\nu} \Omega^{\mu\nu}$.

C. Relating the Hamiltonian to the redshift

In this section we shall relate the Lagrangian L_{part} and the Hamiltonian H_{part} to the redshift variable z. To do so we must introduce a realistic physical model for the spin of the point particle in the pole-dipole approximation.

Up to now we have considered unconstrained variations of the action (2.3), which describes the particle's internal degrees of freedom by the six independent components of the tetrad $e_\alpha{}^\mu$, a 4×4 matrix subject to the ten constraints $g_{\mu\nu}e_\alpha{}^\mu e_\beta{}^\nu = \eta_{\alpha\beta}$, or equivalently by the six internal angles ϕ^a. To correctly account for the number of degrees of freedom associated with the spin, we must impose three SSC. In this section, we adopt the Tulczyjew covariant conditions [59,60]

$$S^{\mu\nu} p_\nu = 0. \qquad (2.22)$$

It would be possible to specify the Lagrangian in (2.3) so that the constraints (2.22) are directly the consequence of the equations derived from that Lagrangian [56]. Alternatively, one could also introduce Lagrange multipliers into the action to enforce these constraints [44]. Here, for the sake of simplicity, we shall rather impose the constraints (2.22) directly in the space of solutions. Furthermore, by contracting Eq. (2.9) with p^ν and using the equation of motion (2.11), one obtains the relation

linking the four-momentum p_μ to the four-velocity u_μ as[5]

$$p_\mu (pu) + m^2 u_\mu = \frac{1}{2} u^\lambda R^\nu{}_{\lambda\rho\sigma} S^{\rho\sigma} S_{\mu\nu}, \quad (2.23)$$

where we have defined $m^2 \equiv -g^{\mu\nu} p_\mu p_\nu$. The parameter m is the mass of the particle, and it can be checked using Eqs. (2.22) and (2.23) that it is constant along the trajectory, that is, $dm/d\tau = 0$.

Henceforth, we restrict our attention to spin-orbit (SO) interactions, which are *linear* in the spins. Neglecting quadratic spin-spin (SS) and higher-order interactions, the linear momentum is simply proportional to the four-velocity: $p_\mu = m u_\mu + \mathcal{O}(S^2)$. To linear order, Eq. (2.9) reduces to the equation of parallel transport for the spin tensor, $DS_{\mu\nu}/d\tau = \mathcal{O}(S^2)$, and the corresponding stress-energy tensor reads

$$T_{\text{part}}^{\mu\nu} = m \frac{v^\mu v^\nu}{z} \frac{\delta^{(3)}(\mathbf{x} - \mathbf{r})}{\sqrt{-g}}$$
$$- \nabla_\rho \left(S^{\rho(\mu} v^{\nu)} \frac{\delta^{(3)}(\mathbf{x} - \mathbf{r})}{\sqrt{-g}} \right) + \mathcal{O}(S^2), \quad (2.24)$$

where $\delta^{(3)}(\mathbf{x} - \mathbf{r})$ is the three-dimensional Dirac function, such that $\int d^3x \delta^{(3)}(\mathbf{x}) = 1$, and we use the parametrization by the proper time τ and the notation (2.14), with $v^\mu = z u^\mu$. Furthermore, thanks to (2.6) the Lagrangian at linear order in the spin reads

$$L_{\text{part}} = z \left(-m + \frac{1}{2} S_{\mu\nu} \Omega^{\mu\nu} \right) + \mathcal{O}(S^2). \quad (2.25)$$

We now want to compute from Eq. (2.25) the partial derivative of the Lagrangian with respect to the mass m of the particle, while holding the dynamical degrees of freedom $\mathbf{r}$, $\mathbf{v}$, ϕ^a, and $\dot{\phi}^a$ fixed. The first term in (2.25) will contribute in an obvious way; however, we must also control the partial derivative of the spin tensor $S_{\mu\nu}$, which is the conjugate of the rotation tensor $\Omega^{\mu\nu}$, with respect to the mass. Let us show that, in fact, the second term in Eq. (2.25) will not contribute to the result at the required $\mathcal{O}(S)$.

To show this we choose the timelike tetrad vector to agree with the four-momentum rescaled by the mass, i.e., $e_0{}^\mu = p^\mu/m$. Then, using Eq. (2.1) and the equation of motion (2.11) we get $\Omega^{\mu\nu} p_\nu = \frac{1}{2} u^\nu R^\mu{}_{\nu\rho\sigma} S^{\rho\sigma}$. Comparing with the SSC (2.22) and using $p_\mu = m u_\mu + \mathcal{O}(S^2)$, we infer that $S_{\mu\nu}$ and $\Omega_{\mu\nu}$ must be proportional, up to a small curvature coupling to the spin and some higher-order spin terms: $S_{\mu\nu} \propto \Omega_{\mu\nu} - \frac{1}{2m} R_{\mu\nu\rho\sigma} S^{\rho\sigma} + \mathcal{O}(S^3)$. For instance, such a proportionality relation is verified in models for the

relativistic spherical top [49,56], and the constant of proportionality is associated with the moment of inertia, say I, of the spherical top. The previous relation can be solved iteratively, yielding an expression for $S_{\mu\nu}$ as a function of m, I, $R_{\mu\nu\rho\sigma}$, and $\Omega^{\mu\nu}$, which schematically reads $S \sim I[\Omega - \frac{I}{2m} R\Omega + (\frac{I}{2m})^2 RR\Omega + \cdots]$. The functional dependence on m is explicit in this expression. Taking a partial derivative with respect to the mass, and replacing back $\Omega^{\mu\nu}$ in terms of the spin, we deduce that $\partial S_{\mu\nu}/\partial m = \mathcal{O}(S)$. Since the rotation tensor is also $\mathcal{O}(S)$, we finally get $\Omega^{\mu\nu} \partial S_{\mu\nu}/\partial m = \mathcal{O}(S^2)$. We thus conclude that the second term in Eq. (2.25) will not contribute at linear order in the spin. Thus, we have proven within the covariant SSC (2.22) that

$$\frac{\partial L_{\text{part}}}{\partial m} = -z + \mathcal{O}(S^2). \quad (2.26)$$

Using the properties of the Legendre transformation (2.19), we readily find that the partial derivative of the Hamiltonian with respect to the mass, holding the conjugate variables $\mathbf{r}$, P_i, ϕ^a, and P_{ϕ_a} fixed, is

$$\frac{\partial H_{\text{part}}}{\partial m} = z + \mathcal{O}(S^2). \quad (2.27)$$

The latter result will be central to the derivation of the first law of mechanics for binary systems of spinning particles in Sec. IV. For the moment, we note that it has been proven only for a single spinning particle moving in a given curved "background" spacetime.

III. LAGRANGIAN AND HAMILTONIAN OF INTERACTING SPINNING PARTICLES

In this section, we consider a self-gravitating matter system consisting of N spinning point particles, which we shall label by $A, B = 1, \ldots, N$. The Lagrangian for each particle is of the form (2.15) in a given metric, so the system of N particles (interacting only through gravitation) is described by the matter Lagrangian

$$L_{\text{mat}} = \sum_{A=1}^N L_{\text{part}}[\mathbf{v}_A, \phi_A^a, \dot{\phi}_A^a, m_A; \epsilon_\alpha{}^\mu(r_A)]. \quad (3.1)$$

Here we have added, for later convenience the dependence on the mass $m_A = (-p_\mu^A p_A^\mu)^{1/2}$. Neglecting terms quadratic or higher order in the spins, this Lagrangian reduces to

$$L_{\text{mat}} = \sum_{A=1}^N z_A \left[-m_A + \frac{1}{2} S_{\mu\nu}^A \Omega_A^{\mu\nu} + \mathcal{O}(S_A^2) \right]. \quad (3.2)$$

In the above equation, the spins $S_{\mu\nu}^A$ should be seen as functions of r_A, $\mathbf{v}_A$, ϕ_A^a, and $\dot{\phi}_A^a$ through their relations to the rotation tensors $\Omega_A^{\mu\nu}$, as discussed in the previous section. Next, we add the usual Einstein-Hilbert term for

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[5]We denote $(pu) \equiv p_\nu u^\nu$. By further contracting Eq. (2.23) with u^μ we obtain an explicit expression for (pu), which can then be substituted back into (2.23).

gravitation, $L_{\text{EH}} = \frac{1}{16\pi} \int d^3x \sqrt{-g} R$, and obtain the total Lagrangian describing the "matter + gravitation" system as

$$L = L_{\text{mat}}[\mathbf{v}_A, \phi_A^a, \dot{\phi}_A^a, m_A; \epsilon_\alpha{}^\mu(r_A)] + L_{\text{EH}}[g_{\mu\nu}], \quad (3.3)$$

in which the spacetime metric reads $g_{\mu\nu} = \epsilon_{\alpha\mu} \epsilon^\alpha{}_\nu$, and the brackets refer, with obvious notations, to a functional dependence. Varying the Lagrangian (3.3) with respect to the tetrad $\epsilon_\alpha{}^\mu$ we get the Einstein field equations $G^{\mu\nu} = 8\pi T_{\text{mat}}^{\mu\nu}$, where the matter tensor is a sum of terms of the type (2.24), one for each particle.

A. Fokker Lagrangian

We shall define the Fokker Lagrangian [67] by formally solving the Einstein field equations and eliminating the gravitational field degrees of freedom in Eq. (3.3). Writing the metric as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, the field equations are solved perturbatively in powers of $h_{\mu\nu}$ by using the graviton propagator. As usual we discard in the Einstein-Hilbert action a total derivative and consider instead the Landau-Lifshitz Lagrangian, which involves only first-order derivatives of the metric; symbolically $L_{\text{LL}} = \frac{1}{16\pi} \times \int d^3x \sqrt{-g} \Gamma\Gamma$. Furthermore, we need to specify a coordinate system to solve the field equations; this is done by adding some *gauge-fixing* terms to the action. Choosing the harmonic coordinate system, this yields the "harmonic-gauge-relaxed" Lagrangian $L_{\text{LL}}^{\text{harm}}$ which contains at quadratic order the graviton propagator $\mathcal{P}_{\mu\nu\rho\sigma}$, say $L_{\text{LL}}^{\text{harm}} = -\frac{1}{2} \int d^3x [h\mathcal{P}^{-1}h + \mathcal{O}(h^3)]$ (see, e.g., Appendix A of Ref. [68]). We thus consider

$$L^{\text{harm}} = L_{\text{mat}}[\mathbf{v}_A, \phi_A^a, \dot{\phi}_A^a, m_A; \epsilon_\alpha{}^\mu(r_A)] + L_{\text{LL}}^{\text{harm}}[g_{\mu\nu}]. \quad (3.4)$$

Next, we perturbatively solve the harmonic-gauge Einstein field equations, and we formally re-expand the solution in a PN approximation. Expanding the retardations of retarded integrals in a PN scheme yields accelerations and time derivatives of accelerations. It will also generate higher time derivatives of the spin variables. The solution, valid at any point x^μ in the near zone, is a function of all the source parameters,

$$\bar{\epsilon}_\alpha{}^\mu(x) \equiv \bar{\epsilon}_\alpha{}^\mu[x; \mathbf{r}_B, \mathbf{v}_B, \mathbf{a}_B, \phi_B^a, \dot{\phi}_B^a, \ddot{\phi}_B^a, m_B], \quad (3.5)$$

where we collectively denote the higher time derivatives by $\mathbf{a}_B = (d\mathbf{v}_B/dt, d^2\mathbf{v}_B/dt^2, \cdots)$ and symbolically by $\ddot{\phi}_B^a$ for the spin variables. Then the near-zone metric solution is $\bar{g}_{\mu\nu} = \bar{\epsilon}_{\alpha\mu} \bar{\epsilon}^\alpha{}_\nu$. Thus, by construction the solution (3.5) satisfies

$$\frac{\delta L^{\text{harm}}}{\delta \bar{\epsilon}_\alpha{}^\mu} \equiv \frac{\delta L_{\text{mat}}}{\delta \epsilon_\alpha{}^\mu}[\mathbf{v}_A, \phi_A^a, \dot{\phi}_A^a, m_A; \bar{\epsilon}_\alpha{}^\mu(r_A)]$$
$$+ \frac{\delta L_{\text{LL}}^{\text{harm}}}{\delta \epsilon_\alpha{}^\mu}[\bar{g}_{\mu\nu}] = 0. \quad (3.6)$$

The Fokker Lagrangian [67] is now obtained by inserting the formal solution (3.5) into (3.4); hence

$$L_{\text{F}}^{\text{harm}} = L_{\text{mat}}[\mathbf{v}_A, \phi_A^a, \dot{\phi}_A^a, m_A; \bar{\epsilon}_\alpha{}^\mu(r_A)] + L_{\text{LL}}^{\text{harm}}[\bar{g}_{\mu\nu}]. \quad (3.7)$$

This Lagrangian gives the correct equations of motion (and spin precession) for the matter variables. Its variational derivative with respect to either the position $\mathbf{r}_A$ or spin parameters ϕ_A^a [which we collectively denote as $\boldsymbol{\sigma}_A \equiv (\mathbf{r}_A, \phi_A^a)$] reads

$$\frac{\delta L_{\text{F}}^{\text{harm}}}{\delta \boldsymbol{\sigma}_A} = \frac{\delta L_{\text{mat}}}{\delta \boldsymbol{\sigma}_A}[\mathbf{v}_B, \phi_B^a, \dot{\phi}_B^a, m_B; \bar{\epsilon}_\alpha{}^\mu(r_B)] + \frac{\delta \bar{\epsilon}_\alpha{}^\mu}{\delta \boldsymbol{\sigma}_A} \frac{\delta L^{\text{harm}}}{\delta \bar{\epsilon}_\alpha{}^\mu}, \quad (3.8)$$

where the last term vanishes thanks to Eq. (3.6).[6] The equations of motion and precession are precisely those we expect for spinning particles within the background $\bar{\epsilon}_\alpha{}^\mu$ generated by all the particles themselves, i.e.,

$$\frac{\delta L_{\text{mat}}}{\delta \boldsymbol{\sigma}_A}[\mathbf{v}_B, \phi_B^a, \dot{\phi}_B^a, m_B; \bar{\epsilon}_\alpha{}^\mu(r_B)] = 0. \quad (3.9)$$

To obtain this result, the Fokker Lagrangian (3.7) must involve not only the matter part evaluated in the background generated by the particles, but also, crucially, the Einstein-Hilbert (or Landau-Lifshitz) part for the gravitational field.

We can now use the properties of the Fokker Lagrangian to compute its partial derivative with respect to one of the masses, say m_A, holding all dynamical variables (and the other masses) fixed. As before we obtain

$$\frac{\partial L_{\text{F}}^{\text{harm}}}{\partial m_A} = \frac{\partial L_{\text{mat}}}{\partial m_A}[\mathbf{v}_B, \phi_B^a, \dot{\phi}_B^a, m_B; \bar{\epsilon}_\alpha{}^\mu(r_B)] + \frac{\partial \bar{\epsilon}_\alpha{}^\mu}{\partial m_A} \frac{\delta L^{\text{harm}}}{\delta \bar{\epsilon}_\alpha{}^\mu}, \quad (3.10)$$

where again the last term vanishes thanks to Eq. (3.6), such that it remains only

$$\frac{\partial L_{\text{F}}^{\text{harm}}}{\partial m_A} = \frac{\partial L_{\text{part}}}{\partial m_A}[\mathbf{v}_A, \phi_A^a, \dot{\phi}_A^a, m_A; \bar{\epsilon}_\alpha{}^\mu(r_A)], \quad (3.11)$$

where we used Eq. (3.1) to express the result in terms of the single-particle Lagrangian for particle A. From (3.2) and the proof given above Eq. (2.26), we have, at linear order in the spin S_A,

$$\frac{\partial L_{\text{F}}^{\text{harm}}}{\partial m_A} = -z_A + \mathcal{O}(S_A^2). \quad (3.12)$$

We emphasize that this result is valid in the background field generated by the system of N particles; the redshift variable therein reads, in full glory,

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[6]Notice that in the last term of Eq. (3.8) the functional derivative $\delta \bar{\epsilon}_\alpha{}^\mu / \delta \boldsymbol{\sigma}_A$ is to be taken in a generalized sense because of the presence of accelerations $\mathbf{a}_B$ and $\dot{\phi}_B^a$.

$$z_A = (-\bar{g}_{\mu\nu}[r_A; \mathbf{r}_B, \mathbf{v}_B, \mathbf{a}_B, \phi_B^a, \dot{\phi}_B^a, \ddot{\phi}_B^a, m_B] v_A^\mu v_A^\nu)^{1/2}. \tag{3.13}$$

In particular, it depends on the masses m_B of the other particles in the system.

B. Fokker Hamiltonian

That a relationship analogous to Eq. (3.12) holds also for the Hamiltonian is not completely straightforward because of the presence of higher-order derivatives (notably accelerations) in the harmonic-coordinate Fokker Lagrangian (3.7). The higher-order derivatives come from the replacement of the metric by the perturbative solution (3.5), and from explicit time derivatives in the self-interaction terms (such as $\partial h \partial h$) of the graviton action. However, as shown in Ref. [69] (see also Refs. [33,70]), one can remove from a PN expanded Lagrangian the higher-order derivatives of $\mathbf{v}_A$ through a suitable redefinition of the position variables, $\mathbf{r}_A \rightarrow \mathbf{r}_A^{\text{new}}$, and the addition of a total time derivative to the Lagrangian.[7] This process can be implemented to *all* orders in PN theory [33,69,70]. The perturbative method of Ref. [69] is very general, and there should be no obstacle in applying it to the spin variables, i.e., in removing from a Lagrangian the higher-order derivatives of $\dot{\phi}_A^a$ through a redefinition $\phi_A^a \rightarrow (\phi_A^a)^{\text{new}}$ and the addition of a total time derivative. Thus, we can eliminate all the higher-order time derivatives in the harmonic-coordinate Fokker Lagrangian (3.7) by introducing the new (nonharmonic) Lagrangian

$$L_{\text{F}} = L_{\text{F}}^{\text{harm}} + \frac{\delta L_{\text{F}}^{\text{harm}}}{\delta \mathbf{r}_A} \delta \mathbf{r}_A + \frac{\delta L_{\text{F}}^{\text{harm}}}{\delta \phi_A^a} \delta \phi_A^a + \frac{\mathrm{d}G}{\mathrm{d}t}, \tag{3.14}$$

where we denote $\delta \mathbf{r}_A \equiv \mathbf{r}_A^{\text{new}} - \mathbf{r}_A$ and $\delta \phi_A^a \equiv (\phi_A^a)^{\text{new}} - \phi_A^a$, and G is some function of time that does not affect the dynamics. The Lagrangian (3.14) now depends on $\mathbf{r}_A^{\text{new}}$ and $\mathbf{v}_A^{\text{new}}$, but no longer on accelerations $\mathbf{a}_A^{\text{new}}$; and it depends on the spins only through $(\phi_A^a)^{\text{new}}$ and $(\dot{\phi}_A^a)^{\text{new}}$. Also, the nonharmonic metric (or tetrad) solution of the field equations in the new variables takes the same form as in Eq. (3.5), but without the contributions from higher time derivatives. This is because the redefinition of the positions $\mathbf{r}_A \rightarrow \mathbf{r}_A^{\text{new}}$ can be seen as being induced by a coordinate transformation of the "bulk" near-zone metric, and that metric, when evaluated at the location of the particles, is necessarily free of higher derivatives (see Sec. 3 in Ref. [70]). Thus, we have the same result as in Eq. (3.12). We shall, from now on, remove the superscript "new" on all variables, but we shall keep in mind that they correspond to nonharmonic coordinates.

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[7]As usual, nonlinear contributions in accelerations can be removed by the addition of so-called "multiple-zero" terms [71,72].

Finally, we have an ordinary Lagrangian L_{F} depending on the dynamical variables $(\mathbf{r}_A, \mathbf{v}_A, \phi_A^a, \dot{\phi}_A^a)$; like in Eq. (2.19) we apply a standard Legendre transformation to obtain the corresponding Hamiltonian, say H_{F}. Defining $P_i^A \equiv \partial L_{\text{F}} / \partial v_A^i$ and $P_{\phi_a}^A \equiv \partial L_{\text{F}} / \partial \dot{\phi}_A^a$, we have

$$H_{\text{F}} = \sum_{A=1}^{N} (P_i^A v_A^i + P_{\phi_a}^A \dot{\phi}_A^a) - L_{\text{F}}, \tag{3.15}$$

where $\mathbf{v}_A$ and $\dot{\phi}_A^a$ are now viewed as functions of the conjugate variables $\mathbf{r}_B$, P_i^B, ϕ_B^a, and $P_{\phi_a}^B$, obtained by inverting the definitions of the momenta P_i^A and $P_{\phi_a}^A$. And like in Eq. (2.27), a consequence of the Legendre transformation (3.15) is that the partial derivative of the Hamiltonian with respect to the mass m_A, holding the conjugate variables $\mathbf{r}_B$, P_i^B, ϕ_B^a, and $P_{\phi_a}^B$ (and the masses $m_B \neq m_A$) fixed, is

$$\frac{\partial H_{\text{F}}}{\partial m_A} = z_A + \mathcal{O}(S_A^2). \tag{3.16}$$

The result (3.16) is the main ingredient of the first law of mechanics for binary systems of spinning point particles, which is the topic of Sec. IV.

C. Constructing the canonical Hamiltonian

Working within the ADM Hamiltonian formalism, the authors of Refs. [26,27] showed that, starting from the unconstrained Hamiltonian depending on the spin variables $S^{\mu\nu}$, it is possible to build the constrained canonical Hamiltonian for gravitationally interacting particles by imposing suitable constraints and gauge conditions, and making a change of variables for the positions, momenta, and spins of the particles. On the other hand, starting from the Hamiltonian (2.19) which depends on the spin variables ϕ^a and P_{ϕ_a}, the construction of the constrained canonical Hamiltonian for a single spinning particle in a given curved spacetime was explicitly done in Ref. [58], thereby extending to curved spacetime the classic work by Hanson and Regge [56] for a relativistic top in flat spacetime. Being guaranteed that a constrained, canonical ADM Hamiltonian exists for gravitationally interacting spinning point particles [26,27,44], henceforth we assume that the construction of Ref. [58] can be extended to interacting particles. Let us then briefly review how the constrained Hamiltonian for a single spinning particle was built in Ref. [58].

We start from the unconstrained Hamiltonian for a single spinning particle $H_{\text{part}}(\mathbf{r}, P_i, \phi^a, P_{\phi_a})$, together with the Poisson bracket operation $\{\cdot, \cdot\}$ (see Sec. III B in Ref. [58] for the explicit expressions of the Poisson brackets between the relevant variables). We consider a set of constraints $\xi_k = 0$ with $k = 1, \ldots, 2M$ (with $M < 9$) such that the matrix $C_{kl} = \{\xi_k, \xi_l\}$ is not singular. Following Ref. [56], these constraints can be imposed by

(i) replacing the original Poisson brackets with Dirac brackets,[8] and (ii) inserting the constraints directly in the original, unconstrained Hamiltonian. More specifically, Ref. [58] first imposed the Newton-Wigner SSC [56], suitably generalized to curved spacetime, to eliminate three out of the six variables P_{ϕ_a} ($a = 1, \ldots, 6$), and an additional constraint to eliminate three out of the six variables ϕ^a, such that the constraint hypersurface contains the same number of configuration coordinates and conjugate momenta. Then, the constrained Hamiltonian $H_C(\mathbf{r}, \mathbf{p}, \mathbf{S})$ and the phase-space algebra of the constrained system were computed, showing that they are canonical at linear order in the particle's spin. Thus, the variables $\mathbf{r}$, $\mathbf{p}$, and $\mathbf{S}$—with $S^i = \frac{1}{2} \varepsilon^{ijk} S_{jk}$ where $S_{jk} = \epsilon_j{}^\mu \epsilon_k{}^\nu S_{\mu\nu}$ (i, j, $k = 1, 2, 3$)—obey the standard commutation relations $\{r^i, p_j\} = \delta^i_j$ and $\{S^i, S^j\} = \varepsilon^{ijk} S^k$, all other brackets vanishing. As usual, this algebra can be used to compute the dynamical evolution of any (time-independent) function f of the canonical variables according to $\dot{f} \equiv \mathrm{d}f/\mathrm{d}t = \{f, H\}$. Finally, the constrained canonical Hamiltonian constructed in Ref. [58] coincides with the ADM Hamiltonian for a spinning particle moving in a given background.

As said, we shall assume that the construction above can be extended to gravitationally interacting spinning point particles. The resulting constrained Hamiltonian $H_C(\mathbf{r}_A, \mathbf{p}_A, \mathbf{S}_A)$ with $A = 1, \ldots, N$ with usual canonical variables will coincide with the ADM canonical Hamiltonian. Furthermore, being constructed from the class of Fokker Hamiltonians (3.15), it necessarily satisfies Eq. (3.16). Thereafter, we restrict ourselves to a binary system of spinning point masses, i.e., $A = 1, 2$, and we denote the canonical ADM Hamiltonian by H.

IV. FIRST LAW OF MECHANICS FOR BINARY POINT PARTICLES WITH SPINS

A. Derivation of the first law

In Sec. III B we proved the remarkable relationship (3.16), valid when neglecting nonlinear terms in the spins of the type $\mathcal{O}(S_1^2)$ and $\mathcal{O}(S_2^2)$, i.e.,

$$\frac{\partial H}{\partial m_A} = z_A. \qquad (4.1)$$

Note that this relation is not only valid for linear contributions $\mathcal{O}(S_1)$ and $\mathcal{O}(S_2)$, but also for nonlinear spin contributions of the type $\mathcal{O}(S_1 S_2)$. Here the redshift quantity z_A is defined in terms of the coordinate four-velocity $v_A^\mu = (1, \mathbf{v}_A)$ of particle A, in ADM-type coordinates, by

$$z_A = \frac{\mathrm{d}\tau_A}{\mathrm{d}t} = (-\bar{g}_{\mu\nu}[r_A; \mathbf{r}_B, \mathbf{v}_B, \mathbf{S}_B, m_B] v_A^\mu v_A^\nu)^{1/2}, \qquad (4.2)$$

with $\bar{g}_{\mu\nu}$ being the background metric generated by the two particles, and evaluated at the location of particle A. Note that the particle velocity $\mathbf{v}_A$ in the RHS of Eq. (4.2) should be viewed as a function of the conjugate variables $\mathbf{r}_A$, $\mathbf{p}_A$, $\mathbf{S}_A$, obtained by inverting the definition of the momentum $\mathbf{p}_A$. We emphasize that Eqs. (4.1) and (4.2) are valid for *generic* orbits and spin configurations. For circular orbits and spins aligned or anti-aligned with respect to the orbital angular momentum, z_A would coincide with the (gauge-invariant) redshift observable introduced by Detweiler [11]. This is the redshift of a photon emitted from the particle and observed at a large distance, along the symmetry axis $\hat{\mathbf{z}}$ perpendicular to the orbital plane.

In order to derive the first law, we reduce the ADM Hamiltonian $H(\mathbf{r}_A, \mathbf{p}_A, \mathbf{S}_A, m_A)$, given in a generic frame of reference, to the center-of-mass Hamiltonian, say $H(\mathbf{r}, \mathbf{p}, \mathbf{S}_A, m_A)$. This is achieved by first imposing that the ADM linear momentum $\mathbf{P}_{\mathrm{ADM}} = \mathbf{p}_1 + \mathbf{p}_2$ vanishes, then substituting the individual momenta $\mathbf{p}_A$ for the relative linear momentum $\mathbf{p} \equiv \mathbf{p}_1 = -\mathbf{p}_2$, and denoting the coordinate separation by $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$.

We shall also limit our study to spins aligned or anti-aligned with the orbital angular momentum $\mathbf{L}$, and to circular orbits. Defining $\mathbf{L} = L\hat{\mathbf{z}}$, where $\hat{\mathbf{z}}$ is the unit vector orthogonal to the orbital plane, we have $\mathbf{S}_A = S_A \hat{\mathbf{z}}$, with $|S_A| < m_A^2$. Using polar coordinates (r, φ) in the orbital plane, we can express the Hamiltonian as a function of the separation $r = |\mathbf{r}|$, of the radial momentum p_r, of the azimuthal momentum $p_\varphi = L$, and of the masses m_A and amplitudes S_A of the spins of the two point particles. An unconstrained variation of the Hamiltonian $H(r, p_r, p_\varphi, S_A, m_A)$ thus gives

$$\delta H = \frac{\partial H}{\partial r} \delta r + \frac{\partial H}{\partial p_r} \delta p_r + \frac{\partial H}{\partial L} \delta L$$
$$+ \sum_A \left(\frac{\partial H}{\partial m_A} \delta m_A + \frac{\partial H}{\partial S_A} \delta S_A \right), \qquad (4.3)$$

where to ease the notation we have omitted to indicate that the partial derivatives with respect to r, p_r, L, S_A, and m_A are computed keeping all the other variables fixed. If the variation compares one solution of the Hamiltonian dynamics to a neighboring solution, then Hamilton's equations of motion must be satisfied. For circular orbits, this yields $\partial H/\partial r = -\dot{p}_r = 0$, as well as $\partial H/\partial p_r = \dot{r} = 0$. The constant circular-orbit frequency is given by $\partial H/\partial L = \dot{\varphi} = \Omega$, and the Hamiltonian is numerically equal to the ADM mass: $H = M$. Therefore, "on shell" we have

$$\delta M = \Omega \delta L + \sum_A \left(\frac{\partial H}{\partial m_A} \delta m_A + \frac{\partial H}{\partial S_A} \delta S_A \right). \qquad (4.4)$$

At linear order in the spins, the Hamiltonian can be written as the sum of an orbital part H_O, which does not

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[8]For two phase-space functions A and B, the Dirac brackets are given by $\{A, B\}_D = \{A, B\} + \{A, \xi_k\}\{B, \xi_l\}[C^{-1}]_{kl}$, and represent the projection of the original symplectic structure onto the phase-space surface defined by the constraints.

depend on the spins S_A, and a spin part $H_S = \sum_A \Omega_A S_A$. The partial derivatives of the Hamiltonian with respect to the spins,

$$\Omega_A = \frac{\partial H}{\partial S_A}, \qquad (4.5)$$

are the so-called *precession* frequencies of the spins. Indeed, using the algebra satisfied by the canonical variables, the spins $\mathbf{S}_A$ can easily be seen to satisfy—in the most general, precessing case—the Newtonian-looking (but exact) precession equations $\dot{\mathbf{S}}_A = \mathbf{\Omega}_A \times \mathbf{S}_A$. Thus, the usual Euclidean norms of the canonical spins are conserved, $\mathbf{S}_A \cdot \mathbf{S}_A = \text{const}$, and $\mathbf{S}_A$ are also referred to as the constant-in-magnitude spins, or *constant* spins.[9]

Finally, combining Eqs. (4.1), (4.4), and (4.5), the first law of mechanics takes on the simple form[10]

$$\delta M - \Omega \delta L = \sum_A (z_A \delta m_A + \Omega_A \delta S_A). \qquad (4.6)$$

This differential relation gives the changes in the ADM mass M and orbital angular momentum L of the *binary* system under small changes in the *individual* masses m_A and spins S_A of the two point particles. We emphasize that Eq. (4.6) is valid for any spin magnitude $|S_A| < m_A^2$.

It is convenient to replace the orbital angular momentum L in favor of the total (ADM-like) angular momentum J. For aligned or anti-aligned spins, we simply have

$$J = L + \sum_A S_A. \qquad (4.7)$$

In terms of J, the first law (4.6) reads

$$\delta M - \Omega \delta J = \sum_A [z_A \delta m_A + (\Omega_A - \Omega) \delta S_A]. \qquad (4.8)$$

We recall that the expressions (4.6) and (4.8) of the first law are, in principle, valid only at linear order in the amplitudes S_A of the constant (or canonical) spin variables $\mathbf{S}_A$. It is true that the definitions (4.5) of the precession frequencies do not require that the Hamiltonian H be linear in the spins; if contributions quadratic (or higher) in the spins were included in the Hamiltonian, then the precession frequencies Ω_A would become functions of the spins S_A. Nevertheless, our proof in Sec. III of the result (3.16) is only valid at linear order in the spins.

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[9]Note that the constant spins are not uniquely defined, because a local Euclidean rotation leaves the magnitude $\mathbf{S}_A \cdot \mathbf{S}_A$ of the spin $\mathbf{S}_A$ unchanged. However, for spins aligned or anti-aligned with the orbital angular momentum, this remaining gauge freedom is irrelevant, as it does not affect the (algebraic) magnitude S_A.

[10]A similar result was previously established in Ref. [73]—see Eq. (24) there—although without the crucial mass variation terms $z_A \delta m_A$ in the right-hand side.

B. Consequences of the first law

We now explore some consequences of the first law for spinning point particles. In particular, we establish algebraic expressions which can be regarded as first integrals associated with the variational relations (4.6) and (4.8).

For two spinning point particles with masses m_A and spins S_A, on a circular orbit with azimuthal frequency Ω, the ADM mass M, orbital angular momentum L, redshifts z_A and precession frequencies Ω_A are all functions of the five independent variables $(\Omega, m_1, m_2, S_1, S_2)$. Therefore, the first law (4.6) is equivalent to the following set of partial differential equations:

$$\frac{\partial M}{\partial \Omega} - \Omega \frac{\partial L}{\partial \Omega} = 0, \qquad (4.9a)$$

$$\frac{\partial M}{\partial m_A} - \Omega \frac{\partial L}{\partial m_A} = z_A, \qquad (4.9b)$$

$$\frac{\partial M}{\partial S_A} - \Omega \frac{\partial L}{\partial S_A} = \Omega_A. \qquad (4.9c)$$

Note that the relations (4.9a) and (4.9b) are unchanged with respect to the results in the nonspinning case [see Eqs. (2.40) and (2.41) of Paper I], where they have been checked up to 3PN order plus the logarithmic contributions at 4PN and 5PN order. However, in the spinning case we have the additional equations (4.9c) relating the precession frequencies Ω_A to the partial derivatives of M and L with respect to the spins S_A.

By combining Eqs. (4.9a) and (4.9b) with numerical calculations of the GSF effect on the redshift z_1 of a nonspinning point particle on a circular orbit around a Schwarzschild black hole [11,23,74], the *exact* expressions of the total mass M and orbital angular momentum L could be determined, at leading order beyond the test-particle approximation [19]. This result was used in Ref. [20] (see also Ref. [21]) to improve the knowledge of the EOB model [7,8,75] for nonspinning binaries. We leave to future work the generalization of these findings to spinning point masses.

By commutation of partial derivatives, Eqs. (4.9b) and (4.9c) can be combined to give the interesting relation

$$\frac{\partial z_A}{\partial S_B} = \frac{\partial \Omega_B}{\partial m_A}. \qquad (4.10)$$

This equation relates the variation in the redshift z_A of particle A under a small change in the spin S_B of particle B to the variation of the precession frequency Ω_B of particle B under a small change in the mass m_A of particle A. Therefore, for $A \neq B$, Eq. (4.10) reflects some equilibrium state of the spinning point particles under their mutual gravitational attraction. For $A = B$, we obtain a nontrivial relation between the redshift (coinciding, in adapted coordinates, with the helical Killing energy) and the precession frequency of each spinning point particle.

We now derive the first integral associated with the variational first law (4.6) by following the same steps as

in the proof given in the nonspinning case in Paper I. First, we introduce the convenient combination

$$\mathcal{M} \equiv M - \Omega L. \qquad (4.11)$$

Making the change of variables $(\Omega, m_1, m_2, S_1, S_2) \rightarrow (\Omega, m, \nu, \chi_1, \chi_2)$, where $m \equiv m_1 + m_2$ is the total mass, $\nu \equiv m_1 m_2/m^2$ the symmetric mass ratio, and $\chi_A \equiv S_A/m_A^2$ the dimensionless spins, Eqs. (4.9b) and (4.9c) can be combined to give

$$\sum_A (m_A z_A + 2\Omega_A S_A) = \sum_A \left(m_A \frac{\partial \mathcal{M}}{\partial m_A} + 2S_A \frac{\partial \mathcal{M}}{\partial S_A} \right)$$
$$= m \frac{\partial \mathcal{M}}{\partial m}. \qquad (4.12)$$

Next, we notice that the ratio $\mathcal{M}/m$ is dimensionless and symmetric by the exchange $m_1 \leftrightarrow m_2$ of the particles; it must thus be a function of $m\Omega$, ν, χ_1, and χ_2; see, e.g., Eqs. (5.7) below. This last observation implies the relationship $m\partial(\mathcal{M}/m)/\partial m = \Omega\partial(\mathcal{M}/m)/\partial\Omega$, which when combined with (4.9a) and (4.12) yields the first integral relation

$$M - 2\Omega L = \sum_A (m_A z_A + 2\Omega_A S_A). \qquad (4.13)$$

Alternatively, Euler's theorem for homogeneous functions provides a more straightforward proof of the result (4.13). Since the Einstein field equations do not contain any privileged mass scale, the ADM mass must be a homogeneous function of degree one in $L^{1/2}$, m_A, and $S_A^{1/2}$. Hence, Euler's theorem implies

$$M(L, m_A, S_A) = L^{1/2} \frac{\partial M}{\partial L^{1/2}} + \sum_A \left(m_A \frac{\partial M}{\partial m_A} + S_A^{1/2} \frac{\partial M}{\partial S_A^{1/2}} \right), \qquad (4.14)$$

which when combined with the first law (4.6) immediately yields the first integral relation (4.13). This result generalizes to spinning point particles the first integral relation derived in the nonspinning case [see Eq. (1.2) of Paper I], as it involves appropriate additional spin terms.

Furthermore, we can use the previous results to obtain a relationship between the spin contributions to the ADM mass M and the sum $\sum_A m_A z_A$ of the redshifted masses. Indeed, Eqs. (4.10) and (4.9c) successively imply

$$\frac{\partial}{\partial S_B} \sum_A m_A z_A = \sum_A m_A \frac{\partial \Omega_B}{\partial m_A} = \frac{\partial}{\partial S_B} \sum_A m_A \frac{\partial \mathcal{M}}{\partial m_A}. \qquad (4.15)$$

By combining Eqs. (4.12) and (4.13), we can easily replace the sum $\sum_A m_A \partial\mathcal{M}/\partial m_A$ in the RHS by the expression $-M + 2\mathcal{M} - 2\sum_A S_A \partial\mathcal{M}/\partial S_A$. At linear order in the spins, $\partial^2\mathcal{M}/(\partial S_A \partial S_B) = 0$, and we find

$$\frac{\partial}{\partial S_B}(M + m_1 z_1 + m_1 z_2) = 0. \qquad (4.16)$$

This simple relation immediately gives the SO contributions to the ADM mass M once those in the redshift observables z_1 and z_2 are known, and vice versa.

Until now, all results were given in terms of the orbital angular momentum L. We can rewrite them using the total angular momentum (4.7) instead. The partial differential equations (4.9) then become

$$\frac{\partial M}{\partial \Omega} - \Omega \frac{\partial J}{\partial \Omega} = 0, \qquad (4.17a)$$
$$\frac{\partial M}{\partial m_A} - \Omega \frac{\partial J}{\partial m_A} = z_A, \qquad (4.17b)$$
$$\frac{\partial M}{\partial S_A} - \Omega \frac{\partial J}{\partial S_A} = \Omega_A - \Omega. \qquad (4.17c)$$

Equation (4.17a) is the "thermodynamical" relation commonly used in PN theory for quasicircular orbits (see, e.g., Refs. [29,76]), or in the construction of sequences of quasi-equilibrium initial data for binary black holes and binary neutrons stars [77–80]. In terms of the total angular momentum, Eq. (4.13) becomes

$$M - 2\Omega J = \sum_A [m_A z_A - 2(\Omega - \Omega_A)S_A], \qquad (4.18)$$

which is, as expected, the first integral associated with the variational first law (4.8). The existence of such a simple, linear, algebraic relation between the local quantities z_A and Ω_A on one hand, and the global quantities M and J on the other hand, is noticeable.

V. SPIN-ORBIT EFFECTS IN THE PARTICLE'S REDSHIFT OBSERVABLE

We employ the ADM Hamiltonian to derive the spin-orbit terms at next-to-leading 2.5PN order in the particle's redshift observable (or helical Killing energy) using the relation (4.1). We keep neglecting all terms $\mathcal{O}(S_A^2)$ or higher. We do not include nonlinear spin interactions of the type $\mathcal{O}(S_1 S_2)$ either, even though they must satisfy our first law of mechanics. The center-of-mass ADM Hamiltonian can then be written as

$$H = H(\mathbf{r}, \mathbf{p}, \mathbf{S}_A, m_A) = H_{\mathrm{O}} + H_{\mathrm{SO}} + H_{\mathrm{NLSO}}, \qquad (5.1)$$

where H_{O} is the orbital (or nonspinning) Hamiltonian, which is known through 3PN order, while the leading-order 1.5PN spin-orbit term reads [81]

$$H_{\mathrm{SO}} = \frac{2}{r^3}\mathbf{S}_{\mathrm{eff}} \cdot \mathbf{L}, \qquad (5.2)$$

with $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ the orbital angular momentum, $\mathbf{n} = \mathbf{r}/r$ the unit vector pointing from m_2 to m_1, and

$$\mathbf{S}_{\mathrm{eff}} \equiv \left(1 + \frac{3m_2}{4m_1} \right)\mathbf{S}_1 + \left(1 + \frac{3m_1}{4m_2} \right)\mathbf{S}_2. \qquad (5.3)$$

The next-to-leading 2.5PN SO terms, computed first in the harmonic-coordinates approach [38,39], and then in the ADM Hamiltonian [41], read[11]

$$H_{\mathrm{NLSO}} = \frac{(\mathbf{L} \cdot \mathbf{S}_1)}{r^3}\left\{\left(-6m_1 - 13m_2 - \frac{5m_2^2}{m_1}\right)\frac{1}{r}\right.$$
$$+ \left[\left(\frac{3}{4}\frac{1}{m_1^2} + \frac{3}{2}\frac{1}{m_1 m_2}\right)(\mathbf{n} \cdot \mathbf{p})^2\right.$$
$$\left.\left.+ \left(\frac{3}{4}\frac{1}{m_1^2} + \frac{7}{4}\frac{1}{m_1 m_2} - \frac{5}{8}\frac{m_2}{m_1^3}\right)\mathbf{p}^2\right]\right\} + 1 \leftrightarrow 2.$$
(5.4)

As was done in Sec. IV, we now restrict ourselves to circular orbits ($p_r = \mathbf{n}\cdot\mathbf{p} = 0$, $\Omega = \partial H/\partial L$, $\partial H/\partial r = 0$)

and spins aligned or anti-aligned with the orbital angular momentum, and use Eq. (4.1) to compute the spin-orbit coupling through 2.5PN order in the redshift observable z_A. Remember that the partial derivative with respect to m_A in Eq. (4.1) is to be taken at fixed r, L, S_A, and $m_B \neq m_A$. A straightforward calculation gives the following leading 1.5PN SO contributions to the redshift associated with particle 1[12]:

$$z_1^{\mathrm{SO}} = \left[\left(-\frac{1}{3} + \frac{\Delta}{3} + \frac{2}{3}\nu\right)\nu\chi_1\right.$$
$$\left.+ \left(1 + \Delta - \frac{17}{6}\nu - \frac{5}{6}\Delta\nu + \frac{2}{3}\nu^2\right)\chi_2\right]x^{5/2}, \quad (5.5)$$

and similarly for the next-to-leading 2.5PN SO terms

$$z_1^{\mathrm{NLSO}} = \left[\left(\frac{1}{2} - \frac{\Delta}{2} + \frac{19}{18}\nu - \frac{19}{18}\Delta\nu - \frac{\nu^2}{9}\right)\nu\chi_1 + \left(\frac{3}{2} + \frac{3}{2}\Delta - \frac{17}{3}\nu - \frac{8}{3}\Delta\nu + \frac{179}{36}\nu^2 + \frac{41}{36}\Delta\nu^2 - \frac{\nu^3}{9}\right)\chi_2\right]x^{7/2}. \quad (5.6)$$

We have introduced the usual frequency-related PN parameter $x \equiv (m\Omega)^{2/3}$ and we recall the notation $\chi_A = S_A/m_A^2$. The SO and NLSO terms found in Eqs. (5.5) and (5.6) agree with the results of Ref. [55], obtained from a direct calculation based on the near-zone PN metric; see Eq. (3.13). This provides an important, nontrivial test of the validity of the relation (3.16) derived in Sec. III.

For consistency, we verified that Eqs. (4.9), (4.10), and (4.13), hold in the spin-orbit sector. We also verified that Eq. (4.16) is satisfied by the spin-orbit terms through 2.5PN order. To check those relations we use the spin-orbit contributions to the binding energy $E \equiv M - m$ and orbital angular momentum:

$$E_{\mathrm{SO}} = -\frac{m\nu x}{2}\left[\left(\frac{4}{3} - \frac{4}{3}\Delta - \frac{2}{3}\nu\right)\chi_1 + \left(\frac{4}{3} + \frac{4}{3}\Delta - \frac{2}{3}\nu\right)\chi_2\right]x^{3/2}, \quad (5.7a)$$

$$E_{\mathrm{NLSO}} = -\frac{m\nu x}{2}\left[\left(4 - 4\Delta - \frac{121}{18}\nu + \frac{31}{18}\Delta\nu + \frac{\nu^2}{9}\right)\chi_1 + \left(4 + 4\Delta - \frac{121}{18}\nu - \frac{31}{18}\Delta\nu + \frac{\nu^2}{9}\right)\chi_2\right]x^{5/2}, \quad (5.7b)$$

$$L_{\mathrm{SO}} = \frac{m\nu x}{\Omega}\left[\left(-\frac{5}{3} + \frac{5}{3}\Delta + \frac{5}{6}\nu\right)\chi_1 + \left(-\frac{5}{3} - \frac{5}{3}\Delta + \frac{5}{6}\nu\right)\chi_2\right]x^{3/2}, \quad (5.7c)$$

$$L_{\mathrm{NLSO}} = \frac{m\nu x}{\Omega}\left[\left(-\frac{7}{2} + \frac{7}{2}\Delta + \frac{847}{144}\nu - \frac{217}{144}\Delta\nu - \frac{7}{72}\nu^2\right)\chi_1 + \left(-\frac{7}{2} - \frac{7}{2}\Delta + \frac{847}{144}\nu + \frac{217}{144}\Delta\nu - \frac{7}{72}\nu^2\right)\chi_2\right]x^{5/2}. \quad (5.7d)$$

We need also the orbital (i.e., spin-independent) part of the precession frequencies at next-to-leading order. We compute it from the definition (4.5) together with the SO Hamiltonian (5.2), (5.3), and (5.4), and find

$$\Omega_1 = \Omega\left\{\left(\frac{3}{4} + \frac{3}{4}\Delta + \frac{\nu}{2}\right)x\right.$$
$$\left.+ \left(\frac{9}{16} + \frac{9}{16}\Delta + \frac{5}{4}\nu - \frac{5}{8}\Delta\nu - \frac{\nu^2}{24}\right)x^2\right\}. \quad (5.8)$$

Note that when computing the partial derivatives to prove Eqs. (4.9), (4.10), and (4.16), we need to hold fixed the spin variables S_A and not the reduced spins χ_A.

We conclude that the redshift observables z_A satisfy the first law of mechanics in the nonspinning sector, up to 3PN order included, as well as for the leading-order 4PN and next-to-leading-order 5PN logarithmic terms (Sec. II D of Paper I), and, in the spin-orbit sector, up to the next-to-leading 2.5PN order.

VI. BINARY BLACK HOLES IN COROTATION

In this section, we consider the particular case of corotating point particles on a circular orbit. By modeling each spinning point particle by an isolated rotating black hole, and comparing with the first law of binary black hole mechanics [82], we define and compute the proper rotation frequencies of the corotating point particles.

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[11]Note that spin-orbit terms in the ADM Hamiltonian are also known at 3.5PN order [42,48], but we do not include them here.

[12]Hereafter we assume, without any loss of generality, $m_1 \leq m_2$. We denote the reduced mass difference by $\Delta \equiv (m_2 - m_1)/m = \sqrt{1 - 4\nu}$. The redshift observable z_2 of particle 2 can immediately be deduced from z_1 by setting $\Delta \to -\Delta$ and $\chi_1 \leftrightarrow \chi_2$.

A. Spinning point particles as Kerr black holes

In Sec. IV we derived the first law of mechanics for spinning point particles [see Eq. (4.8)]. We now derive an alternative version of this result that can heuristically be applied to binary black holes carrying spins. Clearly, in order to derive a first law for *extended* bodies such as rotating black holes, we must supplement the pole-dipole model we have used so far with some "constitutive relations" $m_A(\mu_A, S_A, \ldots)$ specifying the energy content of the bodies, i.e., the relations of their masses m_A to the spins S_A and to some "irreducible" masses μ_A. In principle, these constitutive relations should also involve other parameters such as some external (tidal) multipole moments due to the environment; in the two-body case, they would thus depend on the orbital separation and the mass of the other body.

By analogy with the first law of mechanics for a single black hole [83], we *define* for each spinning point particle the analogue of an irreducible mass $\mu_A \equiv m_A^{\mathrm{irr}}$ via the variational relation $\delta m_A = c_A \delta \mu_A + \omega_A \delta S_A$, in which the "response coefficients" c_A and ω_A are associated with the internal structure:

$$c_A \equiv \frac{\partial m_A}{\partial \mu_A}\bigg|_{S_A}, \qquad \omega_A \equiv \frac{\partial m_A}{\partial S_A}\bigg|_{\mu_A}. \qquad (6.1)$$

In particular, ω_A can be interpreted as the proper rotation frequency of body A. Assuming the validity of the Christodoulou mass formula $m_A^2 = \mu_A^2 + S_A^2/(4\mu_A^2)$ for Kerr black holes [84,85], i.e., neglecting the influence of the other body, we have

$$c_A = \frac{\mu_A}{m_A}\left(1 - \frac{S_A^2}{4\mu_A^4}\right), \qquad (6.2a)$$

$$\omega_A = \frac{S_A}{4m_A\mu_A^2}. \qquad (6.2b)$$

For an isolated black hole, the coefficient c_A is related to the constant surface gravity κ_A by $c_A = 4\mu_A\kappa_A$. Note that, at linear order in the spins, we have $c_A = 1 + \mathcal{O}(S_A^2)$. While the first law (4.8) does not account for spin effects $\mathcal{O}(S_A^2)$ or higher, we shall not make the substitution $c_A \to 1$ in the equations and discussion below, in order to ease the comparison with the binary black hole case.

Identifying each spinning point mass with an "extended body" (or black hole) characterized by coefficients c_A and ω_A, we use the definitions (6.1) and find that the first law (4.8) can be rewritten as

$$\delta M - \Omega \delta J = \sum_A [c_A z_A \delta \mu_A + (z_A \omega_A + \Omega_A - \Omega)\delta S_A].$$
$$(6.3)$$

This variational relationship is reminiscent of the first law of binary black hole mechanics of Friedman, Uryū, and Shibata [82]:

$$\delta M - \Omega \delta J = \sum_A 4\mu_A \kappa_A \delta \mu_A, \qquad (6.4)$$

which holds for two actual black holes with irreducible masses μ_A and constant surface gravities κ_A (the surface areas of the black holes being $\mathcal{A}_A = 16\pi\mu_A^2$).

B. First law for corotating binaries

Both first laws (6.3) and (6.4) have been derived for circular orbits and spins aligned with the orbital angular momentum. However, whereas Eq. (6.3) is valid for arbitrary spin magnitudes, Eq. (6.4) can only describe *corotating* black holes. This key difference is intimately related to the fact that black holes are finite-sized objects, whereas point particles have (by definition) no spatial extension. The assumption of a circular orbit implies the existence of a helical symmetry. In the binary black hole case, the helical Killing field must be tangent to the null geodesic generators of the horizons, which entirely constrains the rotational state of each black hole. If corotation were not realized, the resulting nonvanishing shear would lead to the growth of the horizon's surface areas, in contradiction with the hypothesis of helical symmetry [82]. In the binary point-particle case, however, no such restriction occurs because of the pointlike nature of such idealized objects.

By analogy with Eq. (6.4), we see from Eq. (6.3) that the two point particles will be "corotating" if and only if

$$z_A \omega_A + \Omega_A = \Omega. \qquad (6.5)$$

This equation determines the values of the proper frequencies ω_A to be used to compute the corotating point particle spins S_A through the relation $S_A = 4m_A\mu_A^2\omega_A$ (and the Christodoulou mass formula). Physically, the condition (6.5) means that the redshifted proper rotation frequency of each particle, $z_A\omega_A$, must be equal to the circular-orbit frequency Ω, as seen in a frame rotating at the angular rate Ω_A with respect to an inertial frame of reference. Alternatively, we can associate with the proper rotation frequency ω_A a proper rotation angle ϕ_A by the definition $\omega_A \equiv d\phi_A/d\tau_A$, such that Eq. (6.5) becomes $d\phi_A = (\Omega - \Omega_A)dt = d\varphi - d\Phi_A$. This relationship gives the change $d\phi_A$ in the proper rotation angle in terms of the changes $d\varphi$ and $d\Phi_A$ in the orbital phase and precession angle during a coordinate time interval dt.

For such corotating point particles, the first law (6.3) simplifies considerably to

$$\delta M - \Omega \delta J = \sum_A c_A z_A \delta \mu_A. \qquad (6.6)$$

It is almost *identical* to that derived for nonspinning binaries in Paper I, which is Eq. (6.6) with the substitutions $\delta\mu_A \to \delta m_A$ and $c_A \to 1$. Since the irreducible mass μ_A of a rotating black hole is the spin-independent part of its total mass m_A, this observation suggests that corotating binaries are very similar to nonspinning binaries, at least from the

perspective of the first law of mechanics. This helps to explain why the first law of binary black hole mechanics (6.4) has been observed—in the context of sequences of quasi-equilibrium initial data—to be remarkably well satisfied by *nonspinning* binary black holes [80], even though it should, in principle, hold only for corotating black hole binaries [82].

We also note that the variational first law (6.6) admits the first integral

$$M - 2\Omega J = \sum_A \mu_A c_A z_A, \qquad (6.7)$$

which again is very similar to the first integral relation derived in the nonspinning case, i.e., Eq. (6.7) with $\mu_A \to m_A$ and $c_A \to 1$. Equation (6.7) can also be derived from the first integral (4.18) valid for arbitrary spins, in which we replace the masses m_A by the irreducible masses μ_A and spins S_A using $m_A = \mu_A c_A + 2\omega_A S_A$ (the analogue of Smarr's formula [86] for spinning point particles treated as Kerr black holes), and impose the condition (6.5) for corotation.

Now, comparing the first law (6.4) for corotating black holes with the first law (6.6) for corotating point particles, we notice the formal analogy

$$c_A z_A \leftrightarrow 4\mu_A \kappa_A. \qquad (6.8)$$

This point-particle/black hole analogy was pointed out in Paper I (with $c_A \to 1$), in the case of nonspinning point masses. Up to the factors of $c_A = 1 + \mathcal{O}(S_A^2)$, the first law for corotating point particles expressed in terms of the irreducible masses μ_A is identical to the first law for nonspinning point particles expressed in terms of the total masses $m_A = \mu_A + \mathcal{O}(S_A^2)$. As derived in Paper I, the analogy (6.8), with $c_A \to 1$, was not entirely physically motivated, because it required the identification of the irreducible masses of the corotating black holes with the total masses of the nonspinning point particles. Having been established here in the case of corotating point particles, and requiring only the identification of irreducible masses, the relation (6.8) is much more compelling.

In the limit of large separation, the black holes or point particles can be viewed, in first approximation, as isolated. In that limit, we know that $4\mu_A \kappa_A \to c_A$ for each black hole, and $z_A \to 1$ for each point particle, such that the analogy (6.8) is consistent. Going beyond the large separation limit, Eq. (6.8) suggests that the deviations of the redshifts of the point particles from 1 provide a measure of the interaction between the black holes. In particular, since the redshifts $z_A = d\tau_A/dt$ must be less than 1, we expect

$$\kappa_A < \frac{c_A}{4\mu_A} \qquad (6.9)$$

in binary systems of corotating black holes on circular orbits. In other words, the tidal interaction between the holes should *decrease* the surface gravities with respect to

their values in isolation. This prediction could be checked by computing numerically κ_A in sequences of quasicircular initial data relying on the existence of a global helical Killing vector [77,78]. One could also compare quantitatively PN results for z_A [11–13,22] with numerical results for $4\mu_A \kappa_A$, as functions of the circular-orbit frequency Ω.

C. Proper rotation frequencies through 2PN order

The condition (6.5) for corotation can be solved for the proper rotation frequencies ω_A of the particles:

$$\omega_A = u_A^t(\Omega - \Omega_A), \qquad (6.10)$$

where $u_A^t = 1/z_A$ has been computed up to very high PN orders, for *nonspinning* binaries [11–13,22]. Here we only need the 1PN-accurate results. For circular orbits, and in the center-of-mass frame, the spin-independent contributions to the redshift associated with particle 1 read

$$u_1^t = 1 + \left(\frac{3}{4} + \frac{3}{4}\Delta - \frac{\nu}{2}\right)x$$
$$+ \left(\frac{27}{16} + \frac{27}{16}\Delta - \frac{5}{2}\nu - \frac{5}{8}\Delta\nu + \frac{\nu^2}{24}\right)x^2 + \mathcal{O}(x^3).$$
$$(6.11)$$

However, for corotating binaries we should also include spin contributions. The lowest order spin-related effect comes from the leading-order 1.5PN SO terms, as given by Eq. (5.5). Since $\chi_A \propto \Omega$ at leading order, this term will yield a 3PN contribution, which can be neglected here. Combining Eqs. (5.8) and (6.11), the condition (6.10) for corotation readily fixes at 2PN order

$$\omega_A = \Omega\left\{1 - \nu x + \nu\left(-\frac{3}{2} + \frac{\nu}{3}\right)x^2 + \mathcal{O}(x^3)\right\}. \qquad (6.12)$$

Noticeably, even though $u_1^t \neq u_2^t$ and $\Omega_1 \neq \Omega_2$, we find $\omega_1 = \omega_2$ up to 2PN order included. Future work should investigate whether this symmetry property still holds at the next 3PN order. Note that in the Newtonian limit $x \to 0$ or the test-particle limit $\nu \to 0$ we simply have $\omega_A = \Omega$, in agreement with physical intuition.

The observation that $\omega_1 = \omega_2$ up to high order suggests that the proper rotation frequency ω_A might physically correspond to the rotation rate of the tidal field of the companion $B \neq A$, as measured in the local asymptotic rest frame (LARF) of body A [87]. By matching the near-zone PN metric of a binary system of point particles to the metric of two tidally distorted Schwarzschild black holes, Alvi [88] (see also Ref. [89]) computed the 1PN-accurate expression of the rotation rate of the tidal field of body B in the LARF of body A. Interestingly, his result agrees with the restriction at 1PN order of our 2PN-accurate expression (6.12).

Furthermore, Caudill *et al.* [80] previously made that same hypothesis, in the context of quasi-equilibrium initial data for equal-mass black hole binaries. They found that

using the 1PN-accurate result $\omega_A = \Omega(1 - \nu x)$ instead of the leading-order result $\omega_A = \Omega$ considerably improved the agreement between two different quasilocal measures of the individual spins of the black holes; see Eqs. (54)–(56) and Figs. 4 and 5 of Ref. [80]. It would be interesting to revisit their work using our improved, 2PN-accurate expression (6.12) for the proper rotation frequencies.

In Ref. [76], the binding energy $E = M - m$ and total angular momentum J of a binary system of corotating point particles were computed by replacing the masses m_A and spins S_A by the irreducible masses μ_A and proper rotation frequencies ω_A, using the PN-expanded form of the relations between (m_A, S_A) and (μ_A, ω_A) obtained from the Christodoulou formula, and assuming the leading-order result $\omega_A = \Omega$. Using the corrected formula (6.12) instead, the results of Ref. [76] are modified. Redoing the calculation for the *additional* spin-related contributions E^{cor} and J^{cor} to the binding energy and total angular momentum in the corotating case, we find

$$E^{\text{cor}} = \mu(2 - 6\eta)x_\mu^3 + \mu\,\eta(-10 + 25\eta)x_\mu^4, \quad (6.13a)$$

$$\Omega J^{\text{cor}} = \mu(4 - 12\eta)x_\mu^3 + \mu\,\eta(-16 + 40\eta)x_\mu^4, \quad (6.13b)$$

where the total mass $\mu = \mu_1 + \mu_2$, the symmetric mass ratio $\eta = \mu_1\mu_2/\mu^2$, and the dimensionless invariant PN parameter $x_\mu = (\mu\Omega)^{2/3}$ are now expressed in terms of the irreducible masses μ_A, rather than the masses m_A. The 2PN and 3PN spin-related contributions (6.13) must be added to the known 3PN-accurate expressions of E and J for spinless binaries. As expected, the 1PN correction in Eq. (6.13) modifies the 3PN terms in (6.13) with respect to the results of Ref. [76]. It can be checked that the additional contributions (6.13) for corotating binaries now satisfy the relation

$$\frac{\partial E^{\text{cor}}}{\partial \Omega}\bigg|_{\mu_A} = \Omega\,\frac{\partial J^{\text{cor}}}{\partial \Omega}\bigg|_{\mu_A}. \quad (6.14)$$

Since the spin-independent contributions to E and J are known to satisfy that same relation (see Paper I), we conclude that the thermodynamical law is verified, as it should be according to the first law (6.6), for corotating point particles. It would be interesting to revisit the PN/NR comparison of Ref. [76] using the corrected formulas (6.13).

VII. SUMMARY AND PROSPECTS

In this paper we generalized to spinning point particles the first law of binary mechanics established in Paper I for nonspinning point masses. We derived a simple relation between the Hamiltonian and the redshift observable using a general formalism for the Fokker Lagrangian and Hamiltonian of a system of spinning particles. We then derived the first law within the canonical ADM Hamiltonian formalism, for binaries on circular orbits and with spins aligned or anti-aligned with the orbital angular momentum. We also obtained several useful relations linking the main quantities describing the two-body dynamics for circular orbits, mainly the ADM mass, angular momentum, and precession frequencies.

Similarly to previous work [19,20], a calculation of the gravitational self-force effect on the redshift observable of a particle on a circular, equatorial orbit around a Kerr black hole [55,90] could be used to compute the exact spin-orbit contributions to the binding energy and orbital angular momentum, for aligned or anti-aligned spins, beyond the test-particle approximation. This would allow computing the frequency shift of the Kerr innermost stable circular orbit under the effect of the conservative self-force (at linear order in the spin), an important strong-field benchmark.

New comparisons to numerical relativity simulations of spinning black hole binaries could then explore further the promise of using perturbation theory to model comparable-mass compact binaries [16,19]. This information could also be used to improve the EOB model for spinning binaries [91–96]. Furthermore, once second-order gravitational self-force calculations [97–100] become mature enough, these same ideas could be applied to compute the fully relativistic second-order contributions in the (symmetric) mass ratio to the binding energy and angular momentum, for circular orbits. All of these results would be valuable to improve template waveforms for inspiraling and coalescing binaries of compact objects.

Moreover, our simple relation between the redshift observable and the (ADM, or more generally Fokker-type) Hamiltonian allowed us to compute spin-orbit effects in the redshift through 2.5PN order, for circular orbits and spins aligned or anti-aligned. These results agree with a recent direct calculation based on the PN metric [55]. Future work can extend the knowledge of the redshift function to higher PN order using the recently computed spin-orbit effects at 3.5PN order [40,42]. It would also be interesting and useful to extend the first law to second order in the spins, in order to account for spin-spin contributions and quadrupolar deformations.

Finally, we specified our first law for binary systems of spinning point particles to the corotating case, allowing a comparison with the binary black hole case. Extending previous results [88,89], we computed the proper rotation frequencies of the particles through 2PN order. This finding could be used to improve the quasilocal measure of the individual spins of black holes in the context of quasi-equilibrium initial data [80].

ACKNOWLEDGMENTS

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