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Dust thin shell limit of a thick wall

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We consider the vanishing thickness limit of a wall separating Minkowski spacetime from Schwarzschild or Friedmann spacetime with the purpose of obtaining the thin shell described by the Israel matching formalism. We show that the thin shell cannot be a limit of the wall consisting of Lemaitre-Tolman-Bondi dust. To successfully implement the required limit, it is necessary to add anisotropy to the stress tensor of the wall matter. For such anisotropic matter, we derive boundary conditions that allow carrying out the limit. Finally, we provide an example of a solution satisfying these conditions.

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I. INTRODUCTION

Thin shells in general relativity have been receiving considerable attention ever since the classic paper of Israel [1]. Different aspects of thin shell dynamics are quite interesting in themselves [2–5], and there is a decent amount of application in wormhole theory [6], and especially in cosmology. The latter is mainly inspired by the presence of such structures as voids and walls in the Universe.

The study of thin shells in cosmology started from the works of Maeda and Sato [7], where the equations of motion for a shell surrounding a void were derived by using the Israel formalism to match Friedmann-Robertson-Walker spacetimes with different densities. A similar model was applied to the dynamics of bubbles [8,9], with the only difference being that the bubbles can be large enough for just one of them to enclose the visible part of Universe.

A new wave of interest was raised after the discovery of the accelerated expansion of the Universe and the realization that it can be affected by matter inhomogenities. One of the possible ways of simulating these inhomogenities is using thin shells surrounding empty or underdense regions (voids). The basic model of a single void is usually called a "compensated void" model, because the absence of matter in the void is compensated for by the mass of the surrounding shell. The relation between the angular diameter distance and redshift in such models was studied in Refs. [10,11]. In Ref. [12], the same was done for the Universe model consisting only of multiple spherically symmetric thin shells with a common center. Later, the anisotropy of the cosmic microwave background in compensated void models was investigated in Refs. [13–15]. And recently, Maeda [16] also considered how a compensated void is seen in redshift space by a distant observer.

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However, infinitely thin objects in cosmology can only be an idealization, as real galaxy walls have nonzero thickness. Therefore, the transition from a thick wall to a thin shell deserves thorough study. The investigation of this thin shell limit has already been undertaken by Khosravi, Khakshournia and Mansouri in Refs. [17,18], where they discussed the dynamics of thick shells. We cannot agree, however, with their conclusion that after taking the zero thickness limit, the equations of motion for the dust thick shell reduce to the ones for the thin shell. In this paper, we show why this is not true for pure dust, and then find conditions that make the limit possible after adding anisotropy to the stress-energy tensor of matter constituting the shell.

Units with $G = c = 1$ are used throughout.

II. THE LEMAITRE-TOLMAN-BONDI THICK WALL

To obtain a dust thin shell as a limit of some regular matter distribution, the most natural choice for the matter would be dust. This is described by the Lemaitre-Tolman-Bondi (LTB) solution,

$$ds^2 = -d\tau^2 + \frac{r'^2(\tau, R)}{1 + E(R)} dR^2 + r^2(\tau, R) d\Omega^2, \quad (1)$$

where the function $r(\tau, R)$ should satisfy

$$\ddot{r} = -\frac{m(R)}{r^2}, \quad (2)$$

with $m(R)$ the Misner-Sharp mass.

Let $R = R_1$ and $R = R_2$ be the interior and exterior boundaries of the dust layer under consideration. Because of the comoving frame, both R_1 and R_2 are constant. Our wall is joined to Minkowski space on R_1 and to Schwarzschild or Friedmann space on R_2. Following the Israel-Darmois-Lichnerowich matching formalism, we demand the continuity of the first fundamental forms on both boundary surfaces. From the LTB sides of the boundaries, these are

$$d\sigma_{1,2}^2 = -d\tau^2 + r^2(\tau, R_{1,2})d\Omega^2. \tag{3}$$

Another Israel-Darmois-Lichnerowich requirement is the continuity of the extrinsic curvature on the matching surfaces. In spherically symmetric spacetimes, this gives two independent junction conditions for every matching surface. However, due to the absence of pressure in all parts of the model, one of the conditions is satisfied identically. The other yields $m(R_1) = 0$ and $m(R_2) = M$, where $M \neq 0$ is the total mass of the layer.

Now we decrease the difference $R_2 - R_1$ in such a way that the wall's total mass M remains the same and the matching with the interior and exterior regions is maintained. In the vanishing width limit, we expect to have a thin shell described by the Israel formalism. Therefore, the first fundamental forms calculated from opposite sides of the shell should coincide. This means that $r(\tau, R_1) \rightarrow r(\tau, R_2)$ when $R_1 \rightarrow R_2$. And due to the comoving framework, continuity holds also for all the time derivatives of $r(\tau, R)$, i.e.,

$$\ddot{r}(\tau, R_1) \overset{R_1 \rightarrow R_2}{=} \ddot{r}(\tau, R_2). \tag{4}$$

But that contradicts Eq. (2), because its right-hand side cannot be continuous on a shell with nonzero total mass M. So, we conclude that the thin shell separating Minkowski space from either Schwarzschild or Friedmann space cannot be a limit of a LTB thick wall.

III. A THICK WALL OF ANISOTROPIC MATTER

To finally get the desired dust shell limit, one has to choose another solution inside the thick wall. Joining with either Schwarzschild or Friedmann spacetime yields the vanishing of the radial pressure on the matching surface. Thus, it is natural to retain inside the wall matter without radial pressure, but to add an anisotropy to ensure it will differ from LTB dust. The stress-energy tensor then takes the form

$$T_{\alpha\beta} = \rho u_\alpha u_\beta + P(u_\alpha u_\beta - n_\alpha n_\beta + g_{\alpha\beta}), \tag{5}$$

where ρ and P are the density and tangential pressure, u_α is the four-velocity, and n_α is the unit normal to the dust layers. The general exact solution describing the dynamics of such matter was reduced to a quadrature by Magli [19]. That was possible by utilizing the analog of special coordinates first introduced in Ref. [20] for studying the gravitational collapse of charged dust. The idea is to use the area coordinate $r(\tau, R)$ instead of the comoving time coordinates τ. The line element in this coordinate system has the form

$$ds^2 = -\frac{1}{v^2(r,R)}dr^2 + \frac{2K(r,R)}{v(r,R)}\sqrt{1+E(r,R)}drdR - K^2(r,R)\left(1 - \frac{2m(R)}{r}\right)dR^2 + r^2 d\Omega^2. \tag{6}$$

Here, $m(R)$ and $E(r, R)$ are arbitrary functions, but the latter depends on both the time and the radial coordinates, unlike the LTB solution. The function $v(r, R)$ is just a notation for $\sqrt{E + 2m/r}$ and has the meaning of the velocity of matter, and $K(r, R)$ satisfies

$$\dot{K} = \frac{v'}{2v^{3/2}\sqrt{1+E}}, \tag{7}$$

with the dot now denoting the derivative with respect to r, while the prime still means differentiation by R.

The density ρ and tangential pressure P are given by

$$\rho = \frac{m'}{4\pi K v r^2 \sqrt{1+E}}, \tag{8}$$

$$P = \frac{m'\dot{E}}{16\pi K v r(1+E)^{3/2}}. \tag{9}$$

Now we put the matter described above into a spherical thick layer of matter with boundaries $R = R_1$ and $R = R_2$, as we did with the LTB dust. Again, matching with the interior Minkowski spacetime and the exterior Schwarzschild or Friedmann spacetime gives

$$m(R_1) = 0 \quad \text{and} \quad m(R_2) = M \neq 0. \tag{10}$$

In the limit $R_1 \rightarrow R_2$, we expect to obtain a thin shell. The shell's surface density σ and surface tangential pressure p are determined by the difference between the Lanczos tensor components taken at the exterior and the interior boundaries

$$\sigma = S_\tau^\tau|_2 - S_\tau^\tau|_1, \tag{11}$$

$$p = S_\theta^\theta|_2 - S_\theta^\theta|_1, \tag{12}$$

where the Lanczos tensor is calculated via the extrinsic curvature of the boundary K_{ab}:

$$S_{ab} = -\frac{1}{8\pi}(K_{ab} - K_c^c g_{ab}). \tag{13}$$

The inner times (different for the interior and exterior boundaries) on the matching surfaces can always be chosen in such a way that the first fundamental forms take the form

$$d\sigma^2 = -d\tau^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{14}$$

with $g_{\tau\tau} = -1$. With that choice, the nontrivial components of the Lanczos tensor become

$$S_\tau^\tau = \frac{\sqrt{1+E}}{4\pi r}, \tag{15}$$

$$S_\theta^\theta = S_\phi^\phi = \frac{1}{8\pi r\sqrt{1+E}}\left(1 + E + \frac{1}{2}r\dot{E}\right). \tag{16}$$

The surface density of the thin shell obtained in the considered zero-thickness limit should be positive, so

$$\lim_{R_1\to R_2}(\sqrt{1+E_2}-\sqrt{1+E_1})>0. \qquad (17)$$

In order to obtain not a general shell but a dust shell in particular, the surface tangential pressure p should vanish. This yields the second boundary condition for E,

$$\lim_{R_1\to R_2}\left[\frac{1}{\sqrt{1+E_2}}\left(1+E_2+\frac{1}{2}r\dot{E}_2\right) - \frac{1}{\sqrt{1+E_1}}\left(1+E_1+\frac{1}{2}r\dot{E}_1\right)\right]=0. \qquad (18)$$

Any pair of functions $m(R)$ and $E(r,R)$ satisfying the conditions of Eqs. (10), (17), and (18) produces a solution of the Einstein equations that allows a dust thin shell limit.

Using Eqs. (8) and (9), the expression of Eq. (16) for S^θ_θ can be rewritten as

$$S^\theta_\theta=\frac{1}{2}\left(1+2\frac{P}{\rho}\right)S^\tau_\tau. \qquad (19)$$

Therefore, the boundary condition [Eq. (18)] for the vanishing of the surface tangential pressure [Eq. (12)] becomes

$$p=\frac{1}{2}\left(1+2\frac{P_2}{\rho_2}\right)S^\tau_\tau|_2-\frac{1}{2}\left(1+2\frac{P_1}{\rho_1}\right)S^\tau_\tau|_1=0. \qquad (20)$$

This promptly suggests the example of the solution possessing a dust shell limit. Obviously, with a linear equation of state $P=k\rho$, the required conditions are satisfied if and only if $k=-1/2$.

The solution of the Einstein equations for $E(r,R)$ when $P=-\rho/2$ can be easily found from Eqs. (8) and (9), and has the form

$$1+E(r,R)=\frac{A^2(R)}{r^2}, \qquad (21)$$

where $A(R)$ is another arbitrary function. $K(r,R)$ is then obtained from Eq. (7), and the corresponding integral can even be expressed in terms of elementary functions, although the result is cumbersome.

IV. DISCUSSION

We showed that a thin shell cannot be obtained as a limit of a regular LTB dust matter distribution. On the other hand, Israel or Maeda dust thin shells can be obtained as the limit of an anisotropic matter layer. That is a rather nonintuitive result, because in order to get a thin shell with vanishing pressure, i.e., a shell composed of noninteracting particles (galaxies), we have to reconsider the source dust stress-energy tensor by adding an anisotropy, which looks like introducing an interaction of some kind. Moreover, with the conventional simulation of galaxies constituting the Universe as pure LTB dust, not only can we not obtain the dust shell limit, but we cannot obtain any thin shell limit at all. Does this mean that all cosmology models using shells are invalid? Or does the problem lie within the LTB solution?

In our opinion, the above difficulties are related to the applicability of the dust stress-energy tensor and the LTB solution. Usually it is reckoned that with the growth of the dust density, the particles of dust come very close to each other and eventually begin to interact. That gives rise to an isotropic pressure. However, in cosmology, this is not quite true. Even if we imagine the passing of one galaxy wall through another, the only interaction between them would be gravity. The applicability violation of the LTB solution appears in the other way: It is due to the fact that the stress-energy tensor of dust is a macroscopic quantity resulting from some tacit averaging of a discrete galaxy distribution. That averaging works well in a normal situation, but close to a shell crossing, the correlations of individual galaxy fields can be considerable. The problem of averaging in general relativity is not yet definitively solved, due to the nonlinearity of the theory. The latest review was given by van den Hoogen [21]. In general, an averaging procedure leads to additional terms in the field equations, which can be viewed as a modification of the stress-energy tensor. This can give a possible interpretation of the anisotropy needed to obtain the dust shell limit.

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