PHYSICAL REVIEW D **87,** 024014 (2013)

Axially symmetric static sources: A general framework and some analytical solutions

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(Received 19 November 2012; published 8 January 2013)

We provide all basic equations and concepts required to carry out a general study on axially symmetric static sources. The Einstein equations and the conservation equations are written down for a general anisotropic static fluid endowed with axial symmetry. The structure scalars are calculated and the inhomogeneity factors are identified. Finally some exact analytical solutions were found. One of these solutions describes an incompressible spheroid with isotropic pressure and becomes the well-known interior Schwarzschild solution in the spherically symmetric limit; however, it cannot be matched smoothly to any Weyl exterior metric. Another family of solutions was found that corresponds to an anisotropic fluid distribution and can in principle be matched to a Weyl exterior.

DOI: 10.1103/PhysRevD.87.024014 PACS numbers: 04.40.−b, 04.40.Nr, 04.40.Dg

I. INTRODUCTION

Observational evidence seems to suggest that deviations from spherical symmetry in compact self-gravitating objects (white dwarfs, neutron stars) are likely to be incidental rather than basic features of these systems. This explains why spherical symmetry is so commonly assumed in the study of self-gravitating compact objects.

However, the situation is not so simple. Indeed (putting aside the evident fact that astrophysical objects are generally endowed with angular momentum, and therefore excluding all stationary sources), it is well known that the only regular static and asymptotically flat vacuum spacetime possessing a regular horizon is the Schwarzchild solution [1], and all the other Weyl exterior solutions [2–5] exhibit singularities in the curvature invariants (as the boundary of the source approaches the horizon). This in turn implies that, for very compact objects, a bifurcation appears between any finite perturbation of Schwarzschild spacetime and any Weyl solution, even when the latter is characterized by parameters arbitrarily close to those corresponding to spherical symmetry (see Refs. [6–11] and references therein for a discussion on this point).

From the above comments it should be clear that a rigorous description of static axially symmetric sources, including finding exact analytical solutions, is a praiseworthy endeavor.

Accordingly, in this work we provide all ingredients and equations required for such a study. Of course, this issue has already been considered by several authors in the past. Without pretending to be exhaustive in the revision of the literature on this problem, let us mention the pioneering paper by Hernandez [12], where a general method for obtaining solutions describing axially symmetric sources is presented. Such a method, or some of its modifications, were used in Refs. [13–15] to find sources of different Weyl spacetimes.

This problem has also been considered in Refs. [16–19]. However, in all these last references the line element has been assumed to satisfy the so-called Weyl gauge, which, of course, severely restricts the family of possible sources (see the next section).

In this work we present a general description of axially symmetric sources by deploying all relevant equations without resorting to the Weyl gauge, and considering the most general matter content consistent with the symmetries of the problem.

With this purpose in mind it would be useful to introduce the so-called structure scalars. These form a set of scalar functions obtained from the orthogonal splitting of the Riemann tensor. They were originally defined in the discussion about the structure and evolution of spherically symmetric fluid distributions. Such scalars (five in the spherically symmetric case) were shown to be endowed with distinct physical meaning [20–23].

In particular, they control inhomogeneities in the energy density [20], and the evolution of the expansion scalar and the shear tensor [20–23]. Also in the static case all possible anisotropic solutions are determined by two structure scalars [20].

Furthermore, the role of electric charge and cosmological constant in structure scalars has also been recently investigated [24].

More recently, such scalars and their applications were discussed also in the context of cylindrical [25,26] and planar symmetry [27].

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A set of differential equations for some of these scalars allows us to identify the inhomogeneity factors.

Finally, we exhibit two families of solutions. One of them corresponds to an incompressible spheroid with isotropic pressure. It cannot be matched smoothly to any Weyl exterior spacetime. The second one corresponds to an anisotropic fluid and in principle is matchable to a Weyl exterior.

Our paper is organized as follows: In the next section we shall describe the line element corresponding to the most general nonvacuum, axially symmetric static spacetime. Next we provide a full description of the source that is represented by a general anisotropic matter. Einstein equations and conservation equations are explicitly written for such a system. We also calculate the electric part of the Weyl tensor (its magnetic part vanishes) as well as the electric and magnetic part of the Riemann tensor. With this information we are able to obtain all the nonvanishing structure scalars corresponding to our problem. Two differential equations for such scalars allow us to identify the inhomogeneity factors. The two families of solutions found are described in Secs. V and VI. A summary of the obtained results as well as a list of some unsolved issues are presented in Sec. VII. Finally, an Appendix with the expressions for the components of the electric Weyl tensor is included.

II. THE METRIC AND THE SOURCE

We shall consider bounded, static, and axially symmetric sources. For such a system the most general line element may be written in cylindrical coordinates as

$$ds^2 = -A^2dt^2 + B^2[(dx^1)^2 + (dx^2)^2] + D^2d\phi^2, \quad (1)$$

where A, B, D are positive functions of x^1 and x^2. We number the coordinates $x^0 = t$, $x^1 = \rho$, $x^2 = z$, $x^3 = \phi$.

We shall work in "Weyl spherical coordinates" (r, θ) defined by

$$\rho = r\sin\theta, \qquad z = r\cos\theta. \quad (2)$$

In these coordinates the line element reads

$$ds^2 = -A^2dt^2 + B^2(dr^2 + r^2d\theta^2) + D^2d\phi^2. \quad (3)$$

It is important not to confound these coordinates with Erez-Rosen coordinates $(\hat{r}, \hat{\theta})$ given by

$$\rho^2 = (\hat{r}^2 - 2m\hat{r})\sin^2\hat{\theta}, \qquad z = (\hat{r} - m)\cos\hat{\theta}, \quad (4)$$

where m is a constant to be identified with the monopole of the source.

It should be stressed that our line element is defined by three independent functions, unlike the vacuum case where it is always possible to reduce the line element so that only two independent metric functions appear. In the interior this is not possible in general, though obviously one may assume that as an additional restriction (the so-called Weyl gauge), which amounts to assume that $R^3_3 + R^0_0 = 0$.

In our notation the Weyl gauge is expressed by

$$D = \frac{r\sin\theta}{A}. \quad (5)$$

Let us now provide a full description of the source. To give physical significance to the components of the energy momentum tensor, we shall apply the Bondi approach [28].

Thus, following Bondi, let us introduce purely locally Minkowski coordinates (l.M.f.) (τ, x, y, z) (or equivalently, consider a tetrad field attached to such I.M.f.) by

$$\begin{aligned} d\tau &= Adt, \qquad dx = Bdr, \\ dy &= Brd\theta, \qquad dz = Dd\phi. \end{aligned} \quad (6)$$

Denoting by a hat the components of the energy-momentum tensor in such a locally defined coordinate system, we have that the matter content is given by

$$\hat{T}_{\alpha\beta} = \begin{pmatrix} \mu & 0 & 0 & 0 \\ 0 & P_{xx} & P_{xy} & 0 \\ 0 & P_{yx} & P_{yy} & 0 \\ 0 & 0 & 0 & P_{zz} \end{pmatrix}, \quad (7)$$

where μ, P_{xy}, P_{xx}, P_{yy}, P_{zz} denote the energy density and different stresses, respectively, as measured by our locally defined Minkowskian observer.

Also observe that $P_{xy} = P_{yx}$ and, in general, $P_{xx} \neq P_{yy} \neq P_{zz}$.

Introducing

$$\begin{aligned} \hat{V}_\alpha &= (-1, 0, 0, 0), \qquad \hat{K}_\alpha = (0, 1, 0, 0), \\ \hat{L}_\alpha &= (0, 0, 1, 0), \end{aligned} \quad (8)$$

we have

$$\begin{aligned} \hat{T}_{\alpha\beta} = {} & (\mu + P_{zz})\hat{V}_\alpha\hat{V}_\beta + P_{zz}\eta_{\alpha\beta} + (P_{xx} - P_{zz})\hat{K}_\alpha\hat{K}_\beta \\ & + (P_{yy} - P_{zz})\hat{L}_\alpha\hat{L}_\beta + 2P_{xy}\hat{K}_{(\alpha}\hat{L}_{\beta)}, \end{aligned} \quad (9)$$

where $\eta_{\alpha\beta}$ denotes the Minkowski metric.

Then transforming back to our coordinates, we obtain the components of the energy momentum tensor in terms of the physical variables as defined in the l.M.f.

$$\begin{aligned} T_{\alpha\beta} = {} & (\mu + P_{zz})V_\alpha V_\beta + P_{zz}g_{\alpha\beta} + (P_{xx} - P_{zz})K_\alpha K_\beta \\ & + (P_{yy} - P_{zz})L_\alpha L_\beta + 2P_{xy}K_{(\alpha}L_{\beta)}, \end{aligned} \quad (10)$$

where

$$\begin{aligned} V_\alpha &= (-A, 0, 0, 0), \qquad K_\alpha = (0, B, 0, 0), \\ L_\alpha &= (0, 0, Br, 0). \end{aligned} \quad (11)$$

Alternatively we may write the energy momentum tensor in the "canonical" form:

$$T_{\alpha\beta} = (\mu + P)V_\alpha V_\beta + Pg_{\alpha\beta} + \Pi_{\alpha\beta}, \quad (12)$$

with

$$\Pi_{\alpha\beta} = (P_{xx} - P_{zz})\left(K_\alpha K_\beta - \frac{h_{\alpha\beta}}{3}\right) + (P_{yy} - P_{zz})\left(L_\alpha L_\beta - \frac{h_{\alpha\beta}}{3}\right) + 2P_{xy}K_{(\alpha}L_{\beta)}, \quad (13)$$

and

$$P = \frac{P_{xx} + P_{yy} + P_{zz}}{3}, \qquad h_{\mu\nu} = g_{\mu\nu} + V_\nu V_\mu. \quad (14)$$

With the above information we can write the Einstein equations, which read

$$8\pi\mu = -\frac{1}{B^2}\left\{\frac{B''}{B} + \frac{D''}{D} + \frac{1}{r}\left(\frac{B'}{B} + \frac{D'}{D}\right) - \left(\frac{B'}{B}\right)^2 + \frac{1}{r^2}\left[\frac{B_{\theta\theta}}{B} + \frac{D_{\theta\theta}}{D} - \left(\frac{B_\theta}{B}\right)^2\right]\right\}, \quad (15)$$

$$8\pi P_{xx} = \frac{1}{B^2}\left[\frac{A'B'}{AB} + \frac{A'D'}{AD} + \frac{B'D'}{BD} + \frac{1}{r}\left(\frac{A'}{A} + \frac{D'}{D}\right) + \frac{1}{r^2}\left(\frac{A_{\theta\theta}}{A} + \frac{D_{\theta\theta}}{D} - \frac{A_\theta B_\theta}{AB} + \frac{A_\theta D_\theta}{AD} - \frac{B_\theta D_\theta}{BD}\right)\right], \quad (16)$$

$$8\pi P_{yy} = \frac{1}{B^2}\left[\frac{A''}{A} + \frac{D''}{D} - \frac{A'B'}{AB} + \frac{A'D'}{AD} - \frac{B'D'}{BD} + \frac{1}{r^2}\left(\frac{A_\theta B_\theta}{AB} + \frac{A_\theta D_\theta}{AD} + \frac{B_\theta D_\theta}{BD}\right)\right], \quad (17)$$

$$8\pi P_{zz} = \frac{1}{B^2}\left\{\frac{A''}{A} + \frac{B''}{B} - \left(\frac{B'}{B}\right)^2 + \frac{1}{r}\left(\frac{A'}{A} + \frac{B'}{B}\right) + \frac{1}{r^2}\left[\frac{A_{\theta\theta}}{A} + \frac{B_{\theta\theta}}{B} - \left(\frac{B_\theta}{B}\right)^2\right]\right\}, \quad (18)$$

$$8\pi P_{xy} = \frac{1}{B^2}\left\{\frac{1}{r}\left[-\frac{A'_\theta}{A} - \frac{D'_\theta}{D} + \frac{B_\theta}{B}\left(\frac{A'}{A} + \frac{D'}{D}\right) + \frac{B'}{B}\frac{A_\theta}{A} + \frac{B'}{B}\frac{D_\theta}{D}\right] + \frac{1}{r^2}\left(\frac{A_\theta}{A} + \frac{D_\theta}{D}\right)\right\}, \quad (19)$$

where prime and subscript θ denote derivatives with respect to r and θ, respectively.

Also, the nonvanishing components of the conservation equations $T^{\alpha\beta}_{\ ;\beta} = 0$ yield

$$\dot{\mu} = 0, \quad (20)$$

where the overdot denotes the derivative with respect to t, and

$$P'_{xx} + \frac{A'}{A}(\mu + P_{xx}) + \frac{B'}{B}(P_{xx} - P_{yy}) + \frac{D'}{D}(P_{xx} - P_{zz}) + \frac{1}{r}\left[\left(\frac{A_\theta}{A} + 2\frac{B_\theta}{B} + \frac{D_\theta}{D}\right)P_{xy} + P_{xy,\theta} + P_{xx} - P_{yy}\right] = 0, \quad (21)$$

$$P_{yy,\theta} + \frac{A_\theta}{A}(\mu + P_{yy}) + \frac{B_\theta}{B}(P_{yy} - P_{xx}) + \frac{D_\theta}{D}(P_{yy} - P_{zz}) + r\left[\left(\frac{A'}{A} + 2\frac{B'}{B} + \frac{D'}{D}\right)P_{xy} + P'_{xy}\right] + 2P_{xy} = 0. \quad (22)$$

Equation (20) is a trivial consequence of the staticity, whereas (21) and (22) are the hydrostatic equilibrium equations.

III. THE STRUCTURE SCALARS

We calculate here the structure scalars for the static axially symmetric case. For that purpose, let us first obtain the electric part of the Weyl tensor (the magnetic part vanishes identically).

The components of the electric Weyl tensor can be obtained directly from its definition,

$$E_{\mu\nu} = C_{\mu\alpha\nu\beta}V^\alpha V^\beta, \quad (23)$$

where $C_{\mu\alpha\nu\beta}$ denotes the Weyl tensor. These are exhibited in the Appendix.

Equivalently, the electric part of the Weyl tensor may also be written as

$$E_{\alpha\beta} = \mathcal{E}_1(K_\alpha L_\beta + L_\alpha K_\beta) + \mathcal{E}_2\left(K_\alpha K_\beta - \frac{1}{3}h_{\alpha\beta}\right) + \mathcal{E}_3\left(L_\alpha L_\beta - \frac{1}{3}h_{\alpha\beta}\right), \quad (24)$$

where explicit expressions for the three scalars $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_3$ are given in the Appendix.

Next, let us calculate the electric part of the Riemann tensor (the magnetic part vanishes identically), which is defined by

$$Y^\rho_\beta = V^\alpha V^\mu R^\rho_{\alpha\beta\mu}. \quad (25)$$

After some lengthy calculations we find

$$Y_{\alpha\beta} = Y_{TF_1}(K_\alpha L_\beta + K_\beta L_\alpha) + Y_{TF_2}\left(K_\alpha K_\beta - \frac{1}{3}h_{\alpha\beta}\right) + Y_{TF_3}\left(L_\alpha L_\beta - \frac{1}{3}h_{\alpha\beta}\right) + \frac{1}{3}Y_T h_{\alpha\beta}, \quad (26)$$

where

$$Y_T = 4\pi(\mu + P_{xx} + P_{yy} + P_{zz}), \quad (27)$$

$$Y_{TF_1} = \mathcal{E}_1 - 4\pi P_{xy}, \quad (28)$$

$$Y_{TF_2} = \mathcal{E}_2 - 4\pi(P_{xx} - P_{zz}), \quad (29)$$

$$Y_{TF_3} = \mathcal{E}_3 - 4\pi(P_{yy} - P_{zz}). \tag{30}$$

Finally, we shall find the tensor associated with the double dual of Riemann tensor, defined as

$$X_{\alpha\beta} = {}^*R^*_{\alpha\gamma\beta\delta}V^\gamma V^\delta = \frac{1}{2}\eta_{\alpha\gamma}{}^{\epsilon\rho}R^*_{\epsilon\rho\beta\delta}V^\gamma V^\delta, \tag{31}$$

with $R^*_{\alpha\beta\gamma\delta} = \frac{1}{2}\eta_{\epsilon\rho\gamma\delta}R_{\alpha\beta}{}^{\epsilon\rho}$. Thus, we find

$$X_{\alpha\beta} = X_{TF_1}(K_\alpha L_\beta + K_\beta L_\alpha) + X_{TF_2}\left(K_\alpha K_\beta - \frac{1}{3}h_{\alpha\beta}\right) + X_{TF_3}\left(L_\alpha L_\beta - \frac{1}{3}h_{\alpha\beta}\right) + \frac{1}{3}X_T h_{\alpha\beta}, \tag{32}$$

where

$$X_T = 8\pi\mu, \tag{33}$$

$$X_{TF_1} = -(\mathcal{E}_1 + 4\pi P_{xy}), \tag{34}$$

$$X_{TF_2} = -[\mathcal{E}_2 + 4\pi(P_{xx} - P_{zz})], \tag{35}$$

$$X_{TF_3} = -[\mathcal{E}_3 + 4\pi(P_{yy} - P_{zz})]. \tag{36}$$

The scalars Y_T, Y_{TF1}, Y_{TF2}, Y_{TF3}, X_T, X_{TF1}, X_{TF2}, X_{TF3} are the structure scalars for our problem.

IV. DIFFERENTIAL EQUATIONS FOR THE STRUCTURE SCALARS AND THE INHOMOGENEITY FACTORS

Two differential equations for the Weyl tensor may be obtained using Bianchi identities [29,30]; they have been found before for the spherically symmetric and the cylindrically symmetric cases (see Refs. [25,31], and references therein). Here we calculate them for our case. We obtain

$$\frac{\mathcal{E}_{1\theta}}{r} + \frac{1}{3}(2\mathcal{E}_2 - \mathcal{E}_3)' + \frac{\mathcal{E}_1}{r}\left(\frac{2B_\theta}{B} + \frac{D_\theta}{D}\right) + \mathcal{E}_2\left(\frac{B'}{B} + \frac{D'}{D} + \frac{1}{r}\right) - \mathcal{E}_3\left(\frac{B'}{B} + \frac{1}{r}\right) = \frac{4\pi}{3}(2\mu + P_{xx} + P_{yy} + P_{zz})' + 4\pi(\mu + P_{xx})\frac{A'}{A} + 4\pi P_{xy}\frac{A_\theta}{Ar}, \tag{37}$$

$$\mathcal{E}_1' + \frac{1}{3r}(2\mathcal{E}_3 - \mathcal{E}_2)_\theta + \mathcal{E}_1\left(\frac{2B'}{B} + \frac{D'}{D} + \frac{2}{r}\right) - \frac{\mathcal{E}_2 B_\theta}{Br} + \frac{\mathcal{E}_3}{r}\left(\frac{B_\theta}{B} + \frac{D_\theta}{D}\right) = \frac{4\pi}{3r}(2\mu + P_{xx} + P_{yy} + P_{zz})_\theta + 4\pi(\mu + P_{yy})\frac{A_\theta}{Ar} + 4\pi P_{xy}\frac{A'}{A}, \tag{38}$$

which, using (27)–(30) and (33)–(36), may be written in terms of structure scalars:

$$\frac{8\pi\mu'}{3} = -\frac{1}{r}[X_{TF1\theta} + X_{TF1}(\ln B^2 D)_\theta] - \left[\frac{2}{3}X'_{TF2} + X_{TF2}(\ln BDr)'\right] + \left[\frac{1}{3}X'_{TF3} + X_{TF3}(\ln Br)'\right], \tag{39}$$

$$\frac{8\pi\mu_\theta}{3r} = \frac{1}{r}\left[\frac{1}{3}X_{TF2\theta} + X_{TF2}(\ln B)_\theta\right] - \frac{1}{r}\left[\frac{2}{3}X_{TF3\theta} + X_{TF3}(\ln BD)_\theta\right] - [X'_{TF1} + X_{TF1}(\ln B^2 D r^2)'] . \tag{40}$$

Let us now turn to the inhomogeneity factors. The inhomogeneity factors (say, Ψ_i) are the specific combinations of physical and geometric variables, such that their vanishing is a necessary and sufficient condition for the homogeneity of energy density (i.e., for the vanishing of all spatial derivatives of the energy density).

In the spherically symmetric case it has been shown that in the absence of dissipation the necessary and sufficient condition for the vanishing of the (invariantly defined) spatial derivative of the energy density is the vanishing of the scalar associated with the trace-free part of $X_{\alpha\beta}$ (see Refs. [20,31]).

We shall now identify the inhomogeneity factors in our case.

First, observe that from (39) and (40) it follows at once that $X_{TF1} = X_{TF2} = X_{TF3} = 0 \Rightarrow \mu' = \mu_\theta = 0$. To identify the above scalars as the inhomogeneity factors, we need to prove that the inverse is also true (i.e., $\mu' = \mu_\theta = 0 \Rightarrow X_{TF1} = X_{TF2} = X_{TF3} = 0$).

For that purpose we shall first establish the behavior of different variables in the neighborhood of $r \approx 0$. We shall demand that both A and B are regular functions, and

$$D(r, \theta) \approx r\sin\theta, \tag{41}$$

at $r = 0$. Then from (21) it follows that in the neighborhood of $r = 0$

$$P_{xy} \approx r, \qquad P_{xx} - P_{yy} \approx r, \qquad P_{xx} - P_{zz} \approx r, \tag{42}$$

and from (19)

$$A_\theta(0, \theta) \approx r^3, \quad B_\theta(0, \theta) \approx r^3, \quad A(0, \theta)'_\theta \approx r^2. \tag{43}$$

Using the above in (A9) we have that

$$\mathcal{E}_1(0, \theta) \approx r, \tag{44}$$

implying because of (34)

$$X_{TF1}(0, \theta) \approx r. \tag{45}$$

Finally we shall assume that the three structure scalars X_{TF1}, X_{TF2}, X_{TF3} are analytical functions (class C^∞) in the neighborhood of $r = 0$.

Assuming $\mu' = \mu_\theta = 0$, then evaluating (40) in the neighborhood of $r \approx 0$, since $X'_{TF1}(r = 0)$ is regular, it follows that

$$X'_{TF1}(r = 0) = 0, \tag{46}$$

and

$$X_{TF2\theta} - 2X_{TF3\theta} - X_{TF3}\cot\theta = 0, \tag{47}$$

where (41), (43), and (46) have been used (in what follows it is understood that all expressions are evaluated at $r \approx 0$).

Next, from (39) we obtain

$$2X_{TF2} = X_{TF3}, \tag{48}$$

where (46) and the regularity of first derivatives of structure scalars have been used.

Then, feeding back (48) into (47) we obtain

$$X_{TF2} = \frac{\alpha}{\sin^{2/3}\theta}, \tag{49}$$

where α is a constant, implying $X_{TF2} = X_{TF3} = 0$.

Thus in the neighborhood of $r \approx 0$ we have $X_{TF1} = X'_{TF1} = X_{TF2} = X_{TF3} = 0$.

Next, taking the r derivative of (40) and evaluating at $r \approx 0$, we get $X''_{TF1} = 0$. Continuing this process it follows that $X^{(n)}_{TF1} \equiv \frac{\partial^n X_{TF1}}{\partial r^n} = 0$ for any $n \geq 0$.

Also, from the r derivative of (39) evaluated in the neighborhood of $r \approx 0$ it follows that

$$2X'_{XTF2} = X'_{TF3}. \tag{50}$$

Feeding this equation back into the r derivative of (40) produces

$$X'_{TF2} = \frac{\beta}{\sin^{2/3}\theta}, \tag{51}$$

where β is a constant, implying $X'_{TF2} = X'_{TF3} = 0$. It is not difficult to see that this procedure can be continued to obtain, in the neighborhood of $r \approx 0$, $X^{(n)}_{TF1} = X^{(n)}_{TF2} = X^{(n)}_{TF3} = 0$, for any $n \geq 0$. Therefore we can continue analytically their value at the center, from which we infer

$$X_{TF1} = X_{TF2} = X_{TF3} = 0 \Leftrightarrow \mu' = \mu_\theta = 0. \tag{52}$$

This last result allows us to identify the three structure scalars X_{TF1}, X_{TF2}, X_{TF3} as the inhomogeneity factors.

We shall next find some explicit analytical solutions.

V. THE INCOMPRESSIBLE, ISOTROPIC SPHEROID

We shall now find an analytical solution corresponding to a bounded spheroid with isotropic pressures and homogeneous energy density. From the results of the previous section, (28)–(30) and (34)–(36), it is evident that such a solution is also conformally flat.

Thus let us assume $P_{xx} = P_{yy} = P_{zz} = P$, $P_{xy} = 0$, and $\mu = \mu_0 = \text{const}$.

For simplicity we shall assume the boundary surface Σ to be defined by the equation

$$r = r_1 = \text{const}. \tag{53}$$

Then to satisfy Darmois conditions (continuity of the first and second fundamental forms) we demand that all metric functions, as well as r derivatives, be continuous across Σ (see Ref. [12]). Obviously θ derivatives of A, B, D and A', B', D' are continuous too across Σ.

From the above and (16) and (19) it follows that

$$P \overset{\Sigma}{=} 0. \tag{54}$$

Under the conditions above (21) and (22) can be integrated to obtain

$$P + \mu_0 = \frac{\zeta}{A}, \tag{55}$$

and

$$P + \mu_0 = \frac{\xi(r)}{A}, \tag{56}$$

where ξ is an arbitrary function of its argument. Using boundary conditions (54) in (55) and (56) it follows that

$$A(r_1, \theta) = \text{const} = \frac{\alpha}{\mu_0}, \qquad \zeta = \text{const}. \tag{57}$$

Since, as mentioned before, our solution is conformally flat, then using $\mathcal{E}_1 = \mathcal{E}_2 = \mathcal{E}_3 = 0$ and $P_{xy} = 0$, in (19) and (A6) we obtain

$$\frac{A'_\theta}{A} - \frac{A'}{A}\frac{B_\theta}{B} - \frac{A_\theta}{A}\left(\frac{B'}{B} + \frac{1}{r}\right) = 0, \tag{58}$$

$$\frac{D'_\theta}{D} - \frac{D'}{D}\frac{B_\theta}{B} - \frac{D_\theta}{D}\left(\frac{B'}{B} + \frac{1}{r}\right) = 0. \tag{59}$$

Introducing the auxiliary function $\bar{A}(r, \theta)$ defined by

$$A(r, \theta) = \bar{A}(r, \theta)B(r, \theta), \tag{60}$$

and assuming

$$D(r, \theta) = B(r, \theta)r\sin\theta, \tag{61}$$

Eqs. (58) and (59) can be integrated to obtain

$$\bar{A}(r, \theta) = \tilde{A}(r) + r\chi(\theta), \qquad B(r, \theta) = \frac{1}{R(r) + r\omega(\theta)}, \tag{62}$$

where $\tilde{A}$, χ, R, and ω are arbitrary functions of their argument.

Next, from (A7) and (A8), taking into account (61) and (62) we get

$$\chi = a\cos\theta, \qquad \tilde{A}(r) = \alpha r^2 + \beta, \tag{63}$$

where a, α, and β are constants of integration.

From the above it follows that conformally flat solutions are described by the line element

$$ds^2 = \frac{1}{[R(r) + r\omega(\theta)]^2}[-(\alpha r^2 + \beta + ar\cos\theta)^2 dt^2 + dr^2 + r^2 d\theta^2 + r^2\sin^2(\theta) d\phi^2]. \tag{64}$$

Next, from the condition $P_{yy} - P_{zz} = 0$ and Eqs. (17) and (18) we get

$$\omega(\theta) = b\cos\theta, \tag{65}$$

and from the condition $P_{xx} - P_{yy} = 0$ and Eqs. (16)–(19) we obtain

$$R(r) = \gamma r^2 + \delta, \tag{66}$$

where b, γ, and δ are constants of integration.

Finally, the metric of incompressible conformally flat isotropic fluids can be written as follows:

$$ds^2 = \frac{1}{(\gamma r^2 + \delta + br\cos\theta)^2}[-(\alpha r^2 + \beta + ar\cos\theta)^2 dt^2 + dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2]. \tag{67}$$

Next, the physical variables can be easily calculated. Thus, using (67) into (15) the energy density reads

$$8\pi\mu = 12\gamma\delta - 3b^2. \tag{68}$$

To obtain the pressure, we shall use (55) and (57), which produce

$$8\pi P = (3b^2 - 12\gamma\delta) \times \left[1 - \frac{\alpha r_1^2 + \beta}{\gamma r_1^2 + \delta}\frac{\gamma r^2 + \delta + br\cos\theta}{\alpha r^2 + \beta + ar\cos\theta}\right], \tag{69}$$

where

$$\zeta = \mu_0\frac{\alpha r_1^2 + \beta}{\gamma r_1^2 + \delta}, \qquad a = \frac{\alpha r_1^2 + \beta}{\gamma r_1^2 + \delta}b, \tag{70}$$

in order to satisfy the junction condition (54).

It may be instructive to recover the spherically symmetric case (the interior Schwarzschild solution). In this case we have $a = b = 0$.

To see how this comes about, let us perform the transformation

$$\bar{r} = \frac{r}{\gamma r^2 + \delta}, \qquad \bar{\theta} = \theta, \qquad \bar{t} = t, \qquad \bar{\phi} = \phi, \tag{71}$$

where the overbar denotes the usual Schwarzschild coordinates. Then, it is a simple matter to check that (67) and (69) are identical to the well-known expressions characterizing the interior Schwarzschild solution:

$$g_{\bar{t}\bar{t}} = \frac{1}{4}\left[3\left(1 - \frac{2M}{\bar{r}_1}\right)^{(1/2)} - \left(1 - \frac{2m(\bar{r})}{\bar{r}}\right)^{(1/2)}\right], \tag{72}$$

$$g_{\bar{r}\bar{r}} = \left(1 - \frac{2m(\bar{r})}{\bar{r}}\right)^{-1}, \tag{73}$$

$$P = \mu\left[\frac{(1 - \frac{2m(\bar{r})}{\bar{r}})^{1/2} - (1 - \frac{2M}{\bar{r}_1})^{1/2}}{3(1 - \frac{2M}{\bar{r}_1})^{1/2} - (1 - \frac{2m(\bar{r})}{\bar{r}})^{1/2}}\right], \tag{74}$$

where $m(\bar{r})$, M, and $\bar{r}_1$ denote the mass function, the total mass, and the radius of the sphere, respectively, and the following relationships are satisfied:

$$m = \frac{4\pi}{3}\mu\bar{r}^3 = \frac{2\gamma\delta r^3}{(\gamma r^2 + \delta)^3}, \tag{75}$$

$$m(\bar{r}_1) = M, \tag{76}$$

and

$$r_1^2 = \frac{\beta + \delta}{\gamma - \alpha}. \tag{77}$$

At this point it is pertinent to ask the question: to what specific exterior spacetime can we match smoothly our solution? This is a relevant question, since there are as many different (physically distinguishable) Weyl solutions as there are different harmonic functions.

The answer to the above question is the following: our solution cannot be matched to any Weyl exterior, even though it has a surface of vanishing pressure. This is so because the first fundamental form is not continuous across the boundary surface.

Indeed, from the continuity of g_{tt} and $g_{\phi\phi}$ components at $r = r_1$, we have

$$A_W(r_1, \theta) = \frac{\alpha r_1^2 + \beta + ar_1\cos\theta}{\gamma r_1^2 + \delta + br_1\cos\theta} \tag{78}$$

and

$$A_W(r_1, \theta) = \gamma r_1^2 + \delta + br_1\cos\theta, \tag{79}$$

where $A_W(r_1, \theta)$ denotes the $\sqrt{g_{tt}}$ of any Weyl exterior solution (evaluated on the boundary surface). It is a simple matter to check that the two equations above cannot be satisfied unless $a = b = 0$, which corresponds to the spherically symmetric case. This result is in agreement with theorems indicating that static, perfect fluid (isotropic in pressure) sources are spherical (see Ref. [32], and references therein).

Observe that the above result is a consequence of (61). Therefore, to find matchable solutions we should relax this condition. In this later case, of course, neither isotropy of pressure nor homogeneity of the energy density is preserved.

The remaining possibility is trying to match on a boundary surface given by the equation $r = r_1(\theta)$. However, this does not seem to solve the problem since in our case $A(r_1, \theta) = \text{const}$.

VI. ANISOTROPIC INHOMOGENEOUS SPHEROIDS

To find solutions that could be matched to a Weyl exterior, we shall relax the condition of isotropy of pressure and energy density homogeneity.

The introduction of the pressure anisotropy is well motivated from purely physical considerations. Indeed, it is well known that compact objects (white dwarfs and neutron stars) are endowed with strong magnetic fields (see Refs. [33–39], and references therein). On the other hand, it has been shown that the effect of such a magnetic field on a degenerate Fermi gas manifests itself through the appearance of a strong anisotropy due to the magnetic stresses (see Refs. [40–43], and references therein). This in turn may severely affect some important characteristics of compact objects (see Refs. [44–46], and references therein). Besides the magnetic field, local anisotropy of pressure may be produced by a variety of physical phenomena (see Ref. [47], and references therein).

Thus, let us assume $P_{xy} = \mathcal{E}_1 = \mathcal{E}_3 = P_{yy} - P_{zz} = 0$, although $\mathcal{E}_2 \neq 0$ and $P_{xx} \neq P_{yy}$. Then from Eqs. (19) and (A9) we obtain Eqs. (58) and (59).

Next, introducing the auxiliary functions $\tilde{A}(r, \theta)$ and $R(r, \theta)$ defined by

$$A(r, \theta) = \tilde{A}(r, \theta)B(r, \theta)r, \qquad D(r, \theta) = R(r, \theta)B(r, \theta)r, \tag{80}$$

Eqs. (58) and (59) can be rewritten as

$$\frac{\tilde{A}'_\theta}{\tilde{A}} = \frac{R'_\theta}{R} = Br\left(\frac{1}{Br}\right)'_\theta. \tag{81}$$

From $P_{yy} - P_{zz} = \mathcal{E}_3 = 0$ and Eqs. (A8) and (A11) we get

$$\frac{R''}{R} + \frac{R'}{R}\left(\frac{1}{r} - \frac{\tilde{A}'}{\tilde{A}}\right) + \frac{1}{r^2}\left(\frac{\tilde{A}_{\theta\theta}}{\tilde{A}} - \frac{\tilde{A}_\theta}{\tilde{A}}\frac{R_\theta}{R}\right) = 0, \tag{82}$$

$$\frac{R'}{R}\left(\frac{\tilde{A}'}{\tilde{A}} + \frac{B'}{B} + \frac{1}{r}\right) + \frac{1}{r^2}\left(\frac{\tilde{A}_\theta}{\tilde{A}}\frac{R_\theta}{R} + \frac{B_\theta}{B}\frac{R_\theta}{R} + 2\frac{B_\theta^2}{B^2} - \frac{\tilde{A}_{\theta\theta}}{\tilde{A}} - \frac{B_{\theta\theta}}{B}\right) = 0. \tag{83}$$

To find a simple solution that satisfies the boundary condition, we choose

$$R(r, \theta) = \sin\theta[b(r_1) + \gamma\cos\theta], \tag{84}$$

then from Eqs. (81)–(83) we find

$$\tilde{A}(r, \theta) = \alpha\left[\frac{\gamma}{2}\sin^2\theta - b(r_1)\right]\cos\theta + a(r), \tag{85}$$

$$B(r, \theta) = \frac{1}{\beta r[\frac{\gamma}{2}\sin^2\theta - b(r_1)\cos\theta] + b(r)}, \tag{86}$$

where α, γ, β and $a(r)$, $b(r)$ are arbitrary constants and functions of integration, respectively. $a(r)$, α, and β have the dimension of an inverse of length, whereas the others are dimensionless.

For the above metric, Einstein equations yield the following expressions for physical variables:

$$8\pi\mu = (\beta r\Gamma + b)^2\left\{\frac{2b''}{\beta r\Gamma + b} - \frac{3(\beta\Gamma + b')^2}{(\beta r\Gamma + b)^2} - \frac{3(\beta\Lambda\sin\theta)^2}{(\beta r\Gamma + b)^2} + \frac{1}{r}\left[\frac{4(\beta\Gamma + b')}{\beta r\Gamma + b} + \frac{4\beta\Sigma}{\beta r\Gamma + b}\right] - \frac{1}{r^2}\left[1 - \frac{b_1 + 4\gamma\cos\theta}{\Lambda}\right]\right\}, \tag{87}$$

$$8\pi P_{xx} = (\beta r\Gamma + b)^2\left\{\frac{-2a'(\beta\Gamma + b')}{(\beta r\Gamma + b)(\alpha\Gamma + a)} + \frac{3(\beta\Gamma + b')^2}{(\beta r\Gamma + b)^2} + \frac{3(\beta\Lambda\sin\theta)^2}{(\beta r\Gamma + b)^2} + \frac{1}{r}\left[\frac{2a'}{\alpha\Gamma + a} - \frac{6(\beta\Gamma + b')}{\beta r\Gamma + b} - \frac{4\beta\Sigma}{\beta r\Gamma + b} - \frac{2\alpha\beta\Lambda^2\sin^2\theta}{(\alpha\Gamma + a)(\beta r\Gamma + b)}\right] - \frac{1}{r^2}\left[3 + \frac{2\alpha\Sigma}{\alpha\Gamma + a} - \frac{b_1 + 4\gamma\cos\theta}{\Lambda}\right]\right\}, \tag{88}$$

$$8\pi P_{yy} = 8\pi P_{zz} = (\beta r\Gamma + b)^2\left\{\frac{a''}{\alpha\Gamma + a} - \frac{2b''}{\beta r\Gamma + b} - \frac{2a'(\beta\Gamma + b')}{(\beta r\Gamma + b)(\alpha\Gamma + a)} + \frac{3(\beta\Gamma + b')^2}{(\beta r\Gamma + b)^2} + \frac{3(\beta\Lambda\sin\theta)^2}{(\beta r\Gamma + b)^2} + \frac{1}{r}\left[\frac{3a'}{\alpha\Gamma + a} - \frac{4(\beta\Gamma + b')}{\beta r\Gamma + b} - \frac{2\beta\Sigma}{\beta r\Gamma + b} - \frac{2\alpha\beta\Lambda^2\sin^2\theta}{(\alpha\Gamma + a)(\beta r\Gamma + b)}\right] + \frac{1}{r^2}\left(1 + \frac{\alpha\Sigma}{\alpha\Gamma + a}\right)\right\}, \tag{89}$$

where

$$\Lambda = b_1 + \gamma\cos\theta, \qquad \Gamma = \frac{\gamma\sin^2\theta}{2} - b_1\cos\theta, \qquad b(r_1) = b_1,$$
$$\Sigma = \Lambda\cos\theta - \gamma\sin^2\theta. \tag{90}$$

The equations above describe a wide class of solutions that can be matched to any specific Weyl metric by an appropriate choice of functions and constants of integration. Furthermore, physically reasonable models can be

obtained, at least from slight deviations from spherical symmetry.

VII. CONCLUSIONS

We have established the general framework to carry out a systematic analysis of general static axially symmetric sources. By ''general'' we mean that the Weyl gauge was not assumed and the matter description is the most general compatible with axial symmetry and staticity.

Thus, we started from the most general line element and considered a general anisotropic fluid as a source of the exterior Weyl spacetime. Relevant equations were then written down and structure scalars were calculated.

We have seen that the three structure scalars associated with the trace-free part of the tensor $X_{\alpha\beta}$ define the inhomogeneity factors.

We have found an exact analytical solution representing a spheroid of isotropic pressure with homogeneous energy density. In the spherically symmetric limit our solution becomes the well-known Schwarzschild interior solution. Such an interior cannot be matched (except in the spherically symmetric case) to any Weyl exterior.

Matchable solutions can be found by relaxing the conditions of isotropy of pressure and density inhomogeneity. The anisotropy of pressure was also justified on physical grounds. An example was given in the last section. To study the physical relevance of nonsphericity in the structure of the source, it is necessary to match the above-mentioned solution to a specific Weyl exterior (so that the arbitrary parameters of the source could be related to the parameters of the exterior metric), but such a task is beyond the scope of this paper.

In general, solutions as the one presented here and many others found by either analytical or numerical procedures could provide answers to important questions related to stellar structure, namely,

(i) What is the limit of compactness of a static axially symmetric source?
(ii) How is the above limit related to (influenced by) some exterior parameters such as the quadrupole moment of the source?
(iii) How are intrinsically nonspherical physical variables (e.g., P_{xy}) related to multipole moments (higher than monopole)?
(iv) Are specific Weyl exteriors related to specific sources?

ACKNOWLEDGMENTS

L. H. thanks Departamento de Física Teórica e Historia de la Ciencia, Universidad del País Vasco, for financial support and hospitality. A. D. P. and J. O. acknowledge hospitality of the Departamento de Física Teórica e Historia de la Ciencia, Universidad del País Vasco. This work was partially supported by the Spanish Ministry of Science and Innovation (Grant No. FIS2010-15492) and UFI 11/55 program of the Universidad del País Vasco. J. O. acknowledges financial support from the Spanish Ministry of Science and Innovation (Grant No. FIS2009-07238).

APPENDIX: EXPRESSION FOR THE COMPONENTS OF THE ELECTRIC WEYL TENSOR

The nonvanishing components as calculated from (23) are

$$E_{11} = \frac{1}{6}\Big[\frac{2A''}{A} - \frac{B''}{B} - \frac{D''}{D} - \frac{3A'B'}{AB} - \frac{A'D'}{AD} + \Big(\frac{B'}{B}\Big)^2 + \frac{3B'D'}{BD} + \frac{1}{r}\Big(2\frac{D'}{D} - \frac{B'}{B} - \frac{A'}{A}\Big)\Big] + \frac{1}{6r^2}\Big[-\frac{A_{\theta\theta}}{A} - \frac{B_{\theta\theta}}{B} + \frac{2D_{\theta\theta}}{D} + \frac{3A_\theta B_\theta}{AB} - \frac{A_\theta D_\theta}{AD} + \Big(\frac{B_\theta}{B}\Big)^2 - \frac{3B_\theta D_\theta}{BD}\Big], \tag{A1}$$

$$E_{22} = -\frac{r^2}{6}\Big[\frac{A''}{A} + \frac{B''}{B} - \frac{2D''}{D} - \frac{3A'B'}{AB} + \frac{A'D'}{AD} - \Big(\frac{B'}{B}\Big)^2 + \frac{3B'D'}{BD} + \frac{1}{r}\Big(\frac{D'}{D} + \frac{B'}{B} - \frac{2A'}{A}\Big)\Big] - \frac{1}{6}\Big[-\frac{2A_{\theta\theta}}{A} + \frac{B_{\theta\theta}}{B} + \frac{D_{\theta\theta}}{D} + \frac{3A_\theta B_\theta}{AB} + \frac{A_\theta D_\theta}{AD} - \Big(\frac{B_\theta}{B}\Big)^2 - \frac{3B_\theta D_\theta}{BD}\Big], \tag{A2}$$

$$E_{33} = -\frac{D^2}{6B^2}\Big[\frac{A''}{A} - \frac{2B''}{B} + \frac{D''}{D} - \frac{2A'D'}{AD} + 2\Big(\frac{B'}{B}\Big)^2 + \frac{1}{r}\Big(\frac{D'}{D} - \frac{2B'}{B} + \frac{A'}{A}\Big)\Big] - \frac{D^2}{6B^2r^2}\Big[\frac{A_{\theta\theta}}{A} - \frac{2B_{\theta\theta}}{B} + \frac{D_{\theta\theta}}{D} - \frac{2A_\theta D_\theta}{AD} + 2\Big(\frac{B_\theta}{B}\Big)^2\Big], \tag{A3}$$

$$E_{12} = \frac{1}{2}\Big[\frac{A'_\theta}{A} - \frac{D'_\theta}{D} + \frac{B_\theta}{B}\frac{D'}{D} - \frac{A'B_\theta}{AB} - \frac{B'A_\theta}{AB} + \frac{D_\theta}{D}\frac{B'}{B} - \frac{1}{r}\Big(\frac{A_\theta}{A} - \frac{D_\theta}{D}\Big)\Big]. \tag{A4}$$

These components are not independent since they satisfy the relationship

$$E_{11} + \frac{1}{r^2}E_{22} + \frac{B^2}{D^2}E_{33} = 0. \tag{A5}$$

For the three scalars $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_3$ we obtain

$$\mathcal{E}_1 = \frac{1}{2B^2}\left[\frac{1}{r}\left(\frac{A'_\theta}{A} - \frac{D'_\theta}{D} - \frac{B_\theta}{B}\frac{A'}{A} + \frac{D'}{D}\frac{B_\theta}{B} - \frac{B'}{B}\frac{A_\theta}{A} + \frac{D_\theta}{D}\frac{B'}{B}\right) + \frac{1}{r^2}\left(\frac{D_\theta}{D} - \frac{A_\theta}{A}\right)\right], \tag{A6}$$

$$\mathcal{E}_2 = -\frac{1}{2B^2}\left[-\frac{A''}{A} + \frac{B''}{B} + \frac{A'B'}{AB} + \frac{A'D'}{AD} - \left(\frac{B'}{B}\right)^2 - \frac{B'D'}{BD} + \frac{1}{r}\left(\frac{B'}{B} - \frac{D'}{D}\right)\right] - \frac{1}{2B^2r^2}\left[\frac{B_{\theta\theta}}{B} - \frac{D_{\theta\theta}}{D} - \frac{A_\theta B_\theta}{AB} + \frac{A_\theta D_\theta}{AD} - \left(\frac{B_\theta}{B}\right)^2 + \frac{B_\theta D_\theta}{BD}\right], \tag{A7}$$

$$\mathcal{E}_3 = -\frac{1}{2B^2}\left[\frac{B''}{B} - \frac{D''}{D} - \frac{A'B'}{AB} + \frac{A'D'}{AD} - \left(\frac{B'}{B}\right)^2 + \frac{B'D'}{BD} + \frac{1}{r}\left(\frac{B'}{B} - \frac{A'}{A}\right)\right] - \frac{1}{2B^2r^2}\left[\frac{B_{\theta\theta}}{B} - \frac{A_{\theta\theta}}{A} + \frac{A_\theta B_\theta}{AB} + \frac{A_\theta D_\theta}{AD} - \left(\frac{B_\theta}{B}\right)^2 - \frac{B_\theta D_\theta}{BD}\right]. \tag{A8}$$

Or, using Einstein equations we may also write

$$\mathcal{E}_1 = \frac{E_{12}}{B^2r} = 4\pi P_{xy} + \frac{1}{B^2r}\left[\frac{A'_\theta}{A} - \frac{A'B_\theta}{AB} - \frac{A_\theta}{A}\left(\frac{B'}{B} + \frac{1}{r}\right)\right], \tag{A9}$$

$$\mathcal{E}_2 = -\frac{2E_{33}}{D^2} - \frac{E_{22}}{B^2r^2} = 4\pi(\mu + 2P_{xx} + P_{yy}) - \frac{A'}{B^2A}\left(\frac{2D'}{D} + \frac{B'}{B} + \frac{1}{r}\right) + \frac{A_\theta}{AB^2r^2}\left(\frac{B_\theta}{B} - \frac{2D_\theta}{D}\right) - \frac{1}{B^2r^2}\frac{A_{\theta\theta}}{A}, \tag{A10}$$

$$\mathcal{E}_3 = -\frac{E_{33}}{D^2} + \frac{E_{22}}{B^2r^2} = 4\pi(P_{yy} - P_{zz}) - \frac{A'}{B^2A}\left(\frac{D'}{D} - \frac{B'}{B} - \frac{1}{r}\right) - \frac{A_\theta}{AB^2r^2}\left(\frac{D_\theta}{D} + \frac{B_\theta}{B}\right) + \frac{1}{B^2r^2}\frac{A_{\theta\theta}}{A}. \tag{A11}$$

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