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Simple parametrization of neutrino mixing matrix

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We propose simple forms of the neutrino mixing matrix in analogy with the Wolfenstein parametrization of the quark mixing matrix by adopting the smallest mixing angle θ_{13} as a measure of expansion parameters with the tribimaximal pattern as the base matrix. The triminimal parametrization technique is utilized to expand the mixing matrix under two schemes, i.e., the standard Chau-Keung scheme and the original Kobayashi-Maskawa scheme. The new parametrizations have their corresponding Wolfenstein-like parametrizations of the quark mixing matrix, and therefore they share the same intriguing features of the Wolfenstein parametrization. The newly introduced expansion parameters for neutrinos are connected to the Wolfenstein parameters for quarks via the quark-lepton complementarity.

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The recent establishment of a nonzero and relatively large value of the smallest neutrino mixing angle θ_{13} by a number of experiments [1–3] can be considered as a signal for the high-precision era of neutrino oscillations. Among the four parameters for the description of the neutrino mixing matrix, the three mixing angles have been measured to rather high precision, with only the charge-parity (CP)-violating phase δ still unknown. It is thus time to reconsider our understanding of the neutrino mixing pattern and seek a simple parametrization of the neutrino mixing matrix, in analogy with the Wolfenstein parametrization [4] of the quark mixing matrix.

In the standard model of particle physics, the mixing is well described by the Cabibbo-Kobayashi-Maskawa (CKM) matrix V_{CKM} [5,6] for quarks and the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix U_{PMNS} [7] for leptons. Among many options, the standard parametrization, i.e., the Chau-Keung (CK) scheme [8], expresses the mixing matrix by three angles θ_{12}, θ_{13}, θ_{23} and one CP-violating phase angle δ in a form

$$V/U = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}\mathrm{e}^{-\mathrm{i}\delta} \\ -c_{12}s_{23}s_{13}\mathrm{e}^{i\delta} - s_{12}c_{23} & -s_{12}s_{23}s_{13}\mathrm{e}^{\mathrm{i}\delta} + c_{12}c_{23} & s_{23}c_{13} \\ -c_{12}c_{23}s_{13}\mathrm{e}^{\mathrm{i}\delta} + s_{12}s_{23} & -s_{12}c_{23}s_{13}\mathrm{e}^{\mathrm{i}\delta} - c_{12}s_{23} & c_{23}c_{13} \end{pmatrix}, \tag{1}$$

where $s_{ij} = \sin\theta_{ij}$ and $c_{ij} = \cos\theta_{ij}$ ($i, j = 1, 2, 3$). For describing the quark mixing, another simple form of parametrization, i.e., the Wolfenstein parametrization [4], was proposed with the redefinition that $s_{12} = \lambda$, $s_{23} = A\lambda^2$ and $s_{13}\mathrm{e}^{\mathrm{i}\delta} = A\lambda^3(\rho + \mathrm{i}\eta)$. Its explicit form at the accuracy of $\mathcal{O}(\lambda^4)$ is

$$V_{\mathrm{CKM}} = \begin{pmatrix} 1-\frac{1}{2}\lambda^2 & \lambda & A\lambda^3(\rho - \mathrm{i}\eta) \\ -\lambda & 1-\frac{1}{2}\lambda^2 & A\lambda^2 \\ A\lambda^3(1-\rho-\mathrm{i}\eta) & -A\lambda^2 & 1 \end{pmatrix} + \mathcal{O}(\lambda^4). \tag{2}$$

Such a parametrization has a number of advantages, such as simplicity in form, convenience of use, and hierarchical feature in structure, thus it is widely adopted in theoretical analysis and phenomenological applications.

The Wolfenstein parametrization can be considered as expansions in orders of λ around the unit matrix. For the neutrino mixing, the situation becomes complicated as expansions around the unit matrix are inadequate due to the large values of neutrino mixing angles θ_{12} and θ_{23}. There have been attempts to parametrize the PMNS matrix based on some mixing patterns with finite mixing angles, such as the bimaximal (BM) pattern [9] with $\theta_{12} = \theta_{23} = 45°$ and the tribimaximal (TB) pattern [10] with $\theta_{12} = 35.26°$ and $\theta_{23} = 45°$, and the base matrices of the two patterns are

$$U_{\mathrm{BM}} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \end{pmatrix},$$
$$U_{\mathrm{TB}} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \tag{3}$$

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where an additional factor $P_\nu = \mathrm{Diag}\{e^{-i\alpha/2}, e^{-i\beta/2}, 1\}$ should be multiplied for the Majorana neutrino case. In both of the above mixing patterns, the mixing angle θ_{13} is chosen to be zero.

A large and nonzero θ_{13} poses a challenge to previous mixing patterns, such as the TB pattern which has been studied using basic symmetries. We need to make the upgrade from previous parametrizations which are mostly based on the speculation of a small θ_{13} [11]. A new mixing pattern has been suggested [12] based on a self-complementary relation [13] $\theta_{12}^\nu + \theta_{13}^\nu = \theta_{23}^\nu = 45°$ between neutrino mixing angles. The new base matrix and also the consequent Wolfenstein-like parametrization are complicated. Therefore, a new strategy for the parametrization of the PMNS matrix should be designed.

In this paper we take the smallest mixing angle θ_{13} as a Wolfenstein-like parameter for the PMNS matrix expansion. We utilize the triminimal parametrization technique [14,15] and adopt the three parameters $s \sim 0.1$, $a \sim 1$, and $b \sim 1$ from the following relations:

$$\begin{aligned}\sin\theta_{12} &= \frac{1}{\sqrt{3}}(1-2as^3),\\ \sin\theta_{23} &= \frac{1}{\sqrt{2}}(1-bs^2),\\ \sin\theta_{13} &= \sin 0° + \sqrt{2}s = \sqrt{2}s,\end{aligned} \tag{4}$$

where the trimaximal bases are taken to be the TB mixing pattern. From Eq. (4) we get the corresponding trigonometric functions,

$$\begin{aligned}\cos\theta_{12} &= \sqrt{1-\sin^2\theta_{12}} = \sqrt{\frac{2}{3}}(1+as^3) + \mathcal{O}(s^4 \to s^6),\\ \cos\theta_{23} &= \sqrt{1-\sin^2\theta_{23}} = \frac{1}{\sqrt{2}}(1+bs^2) + \mathcal{O}(s^4),\\ \cos\theta_{13} &= \sqrt{1-\sin^2\theta_{13}} = 1 - s^2 + \mathcal{O}(s^4).\end{aligned} \tag{5}$$

Substituting the above expressions into Eq. (1), we obtain

$$\begin{aligned}U_{\mathrm{PMNS}} = &\begin{pmatrix} \sqrt{\frac{2}{3}}(1-s^2+as^3) & \frac{1}{\sqrt{3}}(1-s^2-2as^3) & 0 \\ -\frac{1}{\sqrt{6}}(1+bs^2-2as^3) & \frac{1}{\sqrt{3}}(1+bs^2+as^3) & \frac{1}{\sqrt{2}}(1-s^2-bs^2) \\ \frac{1}{\sqrt{6}}(1-bs^2-2as^3) & -\frac{1}{\sqrt{3}}(1-bs^2+as^3) & \frac{1}{\sqrt{2}}(1-s^2+bs^2) \end{pmatrix}\\ &+ \begin{pmatrix} 0 & 0 & \sqrt{2}se^{-i\delta} \\ -\sqrt{\frac{2}{3}}se^{i\delta}(1-bs^2) & -\frac{1}{\sqrt{3}}se^{i\delta}(1-bs^2) & 0 \\ -\sqrt{\frac{2}{3}}se^{i\delta}(1+bs^2) & -\frac{1}{\sqrt{3}}se^{i\delta}(1+bs^2) & 0 \end{pmatrix} + \mathcal{O}(s^4),\end{aligned} \tag{6}$$

which can be rewritten in a form

$$\begin{aligned}U_{\mathrm{PMNS}} = U_{\mathrm{TB}} \circledast &\begin{pmatrix} 1-s^2+as^3 & 1-s^2-2as^3 & 0 \\ 1+bs^2-2as^3 & 1+bs^2+as^3 & 1-s^2-bs^2 \\ 1-bs^2-2as^3 & 1-bs^2+as^3 & 1-s^2+bs^2 \end{pmatrix}\\ &- \frac{1}{\sqrt{3}}se^{i\delta}\begin{pmatrix} 0 & 0 & -\sqrt{6}e^{-2i\delta} \\ \sqrt{2}(1-bs^2) & (1-bs^2) & 0 \\ \sqrt{2}(1+bs^2) & (1+bs^2) & 0 \end{pmatrix} + \mathcal{O}(s^4),\end{aligned} \tag{7}$$

where the sign $\circledast$ means direct multiplications between the corresponding elements of the two matrices. For $a = b = 0$, the parametrization reduces to the latest parametrization by King [16] with a similar form

$$U_{\mathrm{PMNS}} = \begin{pmatrix} \sqrt{\frac{2}{3}}(1-s^2) & \frac{1}{\sqrt{3}}(1-s^2) & \sqrt{2}se^{-i\delta} \\ -\frac{1}{\sqrt{6}} - \sqrt{\frac{2}{3}}se^{i\delta} & \frac{1}{\sqrt{3}} - \sqrt{\frac{1}{3}}se^{i\delta} & \frac{1}{\sqrt{2}}(1-s^2) \\ \frac{1}{\sqrt{6}} - \sqrt{\frac{2}{3}}se^{i\delta} & -\frac{1}{\sqrt{3}} - \sqrt{\frac{1}{3}}se^{i\delta} & \frac{1}{\sqrt{2}}(1-s^2) \end{pmatrix} + \mathcal{O}(s^2), \tag{8}$$

from which we see that the deviation could be of the order $\mathcal{O}(s^2)$ instead of $\mathcal{O}(s^3)$ if we adopt $\theta_{23} \approx 40°$ as our input for $b \sim 1$.

Equation (7) can be considered as our proposal for a simple Wolfenstein-like parametrization for neutrino mixing with respect to the standard parametrization Eq. (1). It is a full parametrization with four independent parameters s, a, b and δ, corresponding to the three Euler angles θ_{13}, θ_{12}, θ_{23} and the CP-violating phase δ in the CK scheme. One advantage of

Eq. (7) is that the leading order of the matrix corresponds to the tribimaximal pattern $U_{\rm TB}$. Similar to the Wolfenstein parametrization in which the largest mixing angle $\theta_{\rm C}$ serves as the main expansion parameter $\lambda = \sin(\theta_{\rm C})$, the main expansion parameter s for neutrinos corresponds to the smallest mixing angle θ_{13} by $\sqrt{2}s = \sin(\theta_{13})$. Another advantage of Eq. (7) is that the CP-violating terms are expressed in the matrix with the factor $e^{i\delta}$; therefore, this parametrization is also convenient for the analysis of CP violation in neutrino oscillations. Taking the latest results from phenomenological analysis [17] as our input,

$$\theta_{12} = (33.96^{+1.03}_{-0.99}(^{+3.22}_{-2.91}))^\circ, \qquad \theta_{23} = (40.40^{+4.64}_{-1.74}(^{+12.77}_{-4.64}))^\circ, \qquad \theta_{13} = (9.07 \pm 0.63(\pm 1.89))^\circ, \tag{9}$$

we obtain $s \approx 0.111$, $a \approx 11.7$, and $b \approx 6.71$.

In an alternative to the CK scheme, the Kobayashi-Maskawa (KM) scheme [6] has also been shown to own some intriguing features that are convenient for applications [18–23]. One example is a simple Wolfenstein-like parametrization of the form [22],

$$V = \begin{pmatrix} 1 - \frac{\lambda^2}{2} & \lambda & e^{-i\tilde{\delta}} h\lambda^3 \\ -\lambda & 1 - \frac{\lambda^2}{2} & (f + e^{-i\tilde{\delta}} h)\lambda^2 \\ f\lambda^3 & -(f + e^{i\tilde{\delta}} h)\lambda^2 & 1 \end{pmatrix} + \mathcal{O}(\lambda^4), \tag{10}$$

with respect to the rephased form [22] of mixing matrix in the KM scheme,

$$V_{\rm KM} = \begin{pmatrix} c_1 & s_1 c_3 & s_1 s_3 e^{-i\tilde{\delta}} \\ -s_1 c_2 & c_1 c_2 c_3 - s_2 s_3 e^{i\tilde{\delta}} & s_2 c_3 + c_1 c_2 s_3 e^{-i\tilde{\delta}} \\ s_1 s_2 & -c_1 s_2 c_3 - c_2 s_3 e^{i\tilde{\delta}} & c_2 c_3 - c_1 s_2 s_3 e^{-i\tilde{\delta}} \end{pmatrix}, \tag{11}$$

in which $s_i = \sin\theta_i$ and $c_i = \cos\theta_i$ correspond to Euler angles $\theta_i (i = 1, 2, 3)$, and $\tilde{\delta}$ is the CP-violating phase in the KM parametrization. One advantage of the KM scheme is that it allows for an almost perfect maximal CP violation of the quark mixing [18–23], i.e., $\tilde{\delta}^{\rm quark} = 90^\circ$. With an ansatz of maximal CP violation $\tilde{\delta}^{\rm lepton} = 90^\circ$ also for leptons [24], we get the mixing angles in the KM parametrization [17],

$$\begin{aligned} \theta_1 &= (35.01^{+1.01}_{-0.97})^\circ, \\ \theta_2 &= (39.86^{+5.14}_{-1.97})^\circ, \\ \theta_3 &= (15.96^{+1.11}_{-1.18})^\circ. \end{aligned} \tag{12}$$

We introduce the three parameters,

$$\begin{aligned} \sin\theta_1 &= \frac{1}{\sqrt{3}}(1 - 2\tilde{a}\tilde{s}^4), \\ \sin\theta_2 &= \frac{1}{\sqrt{2}}(1 - \tilde{b}\tilde{s}^2), \\ \sin\theta_3 &= \sqrt{2}\tilde{s}, \end{aligned} \tag{13}$$

from which we get

$$\begin{aligned} \cos\theta_1 &= \sqrt{1 - \sin^2\theta_1} = \sqrt{\frac{2}{3}}(1 + \tilde{a}\tilde{s}^4) + \mathcal{O}(\tilde{s}^8), \\ \cos\theta_2 &= \sqrt{1 - \sin^2\theta_2} = \frac{1}{\sqrt{2}}(1 + \tilde{b}\tilde{s}^2) + \mathcal{O}(\tilde{s}^4), \\ \cos\theta_3 &= \sqrt{1 - \sin^2\theta_3} = 1 - \tilde{s}^2 + \mathcal{O}(\tilde{s}^4), \end{aligned} \tag{14}$$

where $\tilde{s} \approx 0.194$, $\tilde{a} \approx 2.20$, and $\tilde{b} \approx 2.48$. Substituting the above trigonometric functions into Eq. (11), we obtain the simple parametrization of the PMNS matrix with respect to the KM scheme,

$$\begin{aligned} U_{\rm PMNS} = U_{\rm TB} \circledast &\begin{pmatrix} 1 & 1 - \tilde{s}^2 & 0 \\ 1 + \tilde{b}\tilde{s}^2 & 1 - \tilde{s}^2 + \tilde{b}\tilde{s}^2 & 1 - \tilde{s}^2 - \tilde{b}\tilde{s}^2 \\ 1 - \tilde{b}\tilde{s}^2 & 1 - \tilde{s}^2 - \tilde{b}\tilde{s}^2 & 1 - \tilde{s}^2 + \tilde{b}\tilde{s}^2 \end{pmatrix} \\ &+ \tilde{s} e^{i\tilde{\delta}} \begin{pmatrix} 0 & 0 & \sqrt{2/3} e^{-2i\tilde{\delta}} \\ 0 & -1 & \sqrt{2/3} e^{-2i\tilde{\delta}} \\ 0 & -1 & -\sqrt{2/3} e^{-2i\tilde{\delta}} \end{pmatrix} + \mathcal{O}(\tilde{s}^3). \end{aligned} \tag{15}$$

In comparison with Eq. (7), this form of the PMNS matrix looks more simple.

We now provide connections between the newly introduced parameters of neutrinos and the Wolfenstein parameters of quarks via the quark-lepton complementarity (QLC) [25–27]. The QLC in forms of mixing matrices can lead to a simple relation $U_{e3} \approx \lambda/\sqrt{2}$ [12,28,29], thus we have

$$s \approx \lambda/2, \qquad \tilde{s} \approx \sqrt{3}\lambda/2, \tag{16}$$

where the first relation can also be obtained from alternative arguments [16,30]. From the QLC in the form of mixing angles, i.e., $\theta^q_{23} + \theta^l_{23} = 45^\circ$, we arrive at the relation $bs^2 \sim \tilde{b}\tilde{s}^2 \sim \kappa A\lambda^2$, with $\kappa_{23} = (45^\circ - \theta^l_{23})/\theta^q_{23} = 1.96$ and $\kappa_2 = (45^\circ - \theta^l_2)/\theta^q_2 = 2.18$ being the adjusting factors of the data. Then we get

$$b \sim 4\kappa_{23} A, \qquad \tilde{b} \sim 4\kappa_2 A/3. \tag{17}$$

Thus, our newly proposed simple parametrizations, Eqs. (7) and (15), can be also considered as expansions in terms of the Wolfenstein parameters λ and A of quarks. Adopting $\lambda = 0.2253$ and $A = 0.808$, we obtain

$$s \approx 0.113, \quad b \sim 6.32; \quad \tilde{s} \approx 0.195, \quad \tilde{b} \sim 2.37, \tag{18}$$

which are compatible with the above-estimated values from neutrino oscillation data. Further accurate measurements can test and improve the above suggested connections between quarks and leptons.

There is uncertainty with the choice of the powers of s or $\tilde{s}$ with respect to a, b or $\tilde{a}$, $\tilde{b}$. For example, we can alternatively choose $\sin\theta_{12} = \frac{1}{\sqrt{3}}(1 - 2as^2)$ in the CK scheme or $\sin\theta_1 = \frac{1}{\sqrt{3}}(1 - 2\tilde{a}\tilde{s}^3)$ in the KM scheme, so that the new PMNS matrix is

$$U_{\mathrm{PMNS}} = U_{\mathrm{TB}} \circledast \begin{pmatrix} 1 - s^2 + as^2 & 1 - s^2 - 2as^2 & 0 \\ 1 + bs^2 - 2as^2 & 1 + bs^2 + as^2 & 1 - s^2 - bs^2 \\ 1 - bs^2 - 2as^2 & 1 - bs^2 + as^2 & 1 - s^2 + bs^2 \end{pmatrix} - \frac{1}{\sqrt{3}} s e^{i\delta} \begin{pmatrix} 0 & 0 & -\sqrt{6}e^{-2i\delta} \\ \sqrt{2}(1 - bs^2) & (1 - bs^2) & 0 \\ \sqrt{2}(1 + bs^2) & (1 + bs^2) & 0 \end{pmatrix} + \mathcal{O}(s^4), \tag{19}$$

with $s \approx 0.111$, $a \approx 1.31$, and $b \approx 6.71$, or

$$U_{\mathrm{PMNS}} = U_{\mathrm{TB}} \circledast \begin{pmatrix} 1 + \tilde{a}\tilde{s}^3 & 1 - \tilde{s}^2 - 2\tilde{a}\tilde{s}^3 & 0 \\ 1 + \tilde{b}\tilde{s}^2 - 2\tilde{a}\tilde{s}^3 & 1 - \tilde{s}^2 + \tilde{b}\tilde{s}^2 + \tilde{a}\tilde{s}^3 & 1 - \tilde{s}^2 - \tilde{b}\tilde{s}^2 \\ 1 - \tilde{b}\tilde{s}^2 - 2\tilde{a}\tilde{s}^3 & 1 - \tilde{s}^2 - \tilde{b}\tilde{s}^2 + \tilde{a}\tilde{s}^3 & 1 - \tilde{s}^2 + \tilde{b}\tilde{s}^2 \end{pmatrix} + \frac{1}{\sqrt{3}} \tilde{s} e^{i\tilde{\delta}} \begin{pmatrix} 0 & 0 & \sqrt{2}e^{-2i\tilde{\delta}} \\ 0 & -\sqrt{3}(1 - \tilde{b}\tilde{s}^2) & \sqrt{2}(1 + \tilde{b}\tilde{s}^2)e^{-2i\tilde{\delta}} \\ 0 & -\sqrt{3}(1 + \tilde{b}\tilde{s}^2) & -\sqrt{2}(1 - \tilde{b}\tilde{s}^2)e^{-2i\tilde{\delta}} \end{pmatrix} + \mathcal{O}(\tilde{s}^4), \tag{20}$$

with $\tilde{s} \approx 0.194$, $\tilde{a} \approx 0.428$, and $\tilde{b} \approx 2.48$. The corresponding powers of s or $\tilde{s}$ related to b or $\tilde{b}$ could be 3 or higher, instead of 2, if the experimental value for θ_{23} or θ_2 is closer to 45°. Further improved measurements of neutrino mixing data or theoretical progress on the underlying quark-lepton connections can help to determine the explicit powers.

In conclusion, we suggest simple forms of the PMNS matrix with some intriguing features possessed by the Wolfenstein parametrization of the quark mixing matrix, such as full form of mixing matrix with four independent parameters, simplicity in form for convenient applications especially for CP-violation study, and also hierarchical structure as expansions around the tribimaximal pattern. Though the explicit forms of parametrization, such as the powers of s or $\tilde{s}$ with respect to the parameters a, b or $\tilde{a}$, $\tilde{b}$, might change according to future improved measurements of neutrino mixing parameters, parametrizations in the same spirit as the Wolfenstein parametrization might have a better chance for wide application.

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