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Exact solutions of Bethe-Salpeter equations with instantaneous interactions

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The Bethe-Salpeter approach allows for quantum-field theoretic descriptions of relativistic bound states; its inherent complexity, however, usually prevents the finding of exact solutions. Under suitable simplifying assumptions about the systems discussed, we derive *analytically* examples of rigorous solutions to the instantaneous homogeneous Bethe-Salpeter equation by relating tentative solutions to the interactions responsible for the formation of bound states.

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I. INTRODUCTION

One of the great remaining challenges in theoretical elementary particle physics is the description of bound states in a way fully consistent with all requirements imposed by special relativity and quantum mechanics—that is, within the framework of relativistic quantum field theory. In principle, the appropriate tool to achieve this goal is the *Bethe-Salpeter formalism* [1–3]: for bound-state constituents with fixed features, the *homogeneous Bethe-Salpeter equation* governs all the bound states. However, for various practical reasons, some of which we recall below, in its applications to both quantum electrodynamics and quantum chromodynamics, some simplified equations, situated along a path of nonrelativistic reduction, are frequently used. Its cornerstones are the Salpeter equation [4] and the reduced Salpeter equation [5,6].

The nature of such bound-state equations renders it difficult to find their exact analytic solutions for given interactions of the bound-state constituents. However, a knowledge of at least *some* exact solutions facilitates the judging of the significance of the outcomes of reduction steps or numerical solution techniques. In view of this, we remember a straightforward idea used in *quantum physics*, e.g., by Neumann and Wigner [7], for studies of the following kind.

Within the formalism of *nonrelativistic quantum mechanics*, bound states are described by the time-independent Schrödinger equation, which is the eigenvalue equation of the Hamiltonian operator controlling the dynamics of the system under consideration for energy eigenvalues E and associated eigenfunctions ψ. The *bound states* correspond to the *discrete* eigenvalues in the spectrum of this operator. For a particle of mass m experiencing interactions induced by some potential V, this Schrödinger equation in configuration-space representation reads

$$\left[-\frac{\Delta}{2m}+V(\boldsymbol{x})\right]\psi(\boldsymbol{x})=E\psi(\boldsymbol{x}),\qquad \Delta\equiv\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}.$$

Typically, the interaction potential V is inferred from physical considerations or principles, and, for this potential, the solutions $\{(E,\psi)\}$ of the Schrödinger equation are then derived. However, for the purpose of just constructing examples for exact solutions (E,ψ) related to potentials V entering into the Schrödinger equation, one may also follow the opposite route: on an equal footing, one may postulate, for a chosen value of E, any preferred or convenient shape of the solution ψ and see whether one is able to find analytically the related potential V. This is easily done for vanishing binding energy E, i.e., for $E=0$, with the general result

$$V(\boldsymbol{x})=\frac{\Delta\psi(\boldsymbol{x})}{2m\psi(\boldsymbol{x})}.$$

Assuming the potential V to be *spherically symmetric*, $V(\boldsymbol{x})=V(r)$, $r\equiv|\boldsymbol{x}|$, the choice [7]

$$\psi(\boldsymbol{x})=\psi(r)=\frac{\sin(a^3r^3)}{r^2},\qquad a\equiv 2m,$$

for a likewise spherically symmetric stationary solution thus fixes [7] the central potential[1]:

$$V(r)=\frac{2}{ar^2}-9a^5r^4.$$

We intend to adapt the above procedure in quantum mechanics applied to Schrödinger problems to the *Bethe-Salpeter approach*. To this end, we first identify, in Sec. II, examples of Bethe-Salpeter equations with internal structure sufficiently simple to allow for the kind of *inversion* we have in mind. For these tractable bound-state equations, we then, in Sec. III, relate postulated candidate solutions to confining and nonconfining interaction potentials.

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[1]Here, the eigenvalue $E=0$ is located in the continuous spectrum of the Hamiltonian; thus, it is not discrete.

II. BETHE-SALPETER FORMALISM IN INSTANTANEOUS LIMIT

Motivated by the needs of physics, the aim of the present discussion is to construct analytic solutions to equations of motion describing relativistic bound states, which we assume to be composed of some fermion, with momentum p_1, and some antifermion, with momentum p_2; the total momentum of the bound state is $P \equiv p_1 + p_2$, and its mass M is given by $M^2 = P^2$.

A. Homogeneous Bethe-Salpeter equation

In principle, within the framework of relativistic quantum field theory, the adequate tool for the (Poincaré-covariant) description of bound states is the Bethe-Salpeter formalism [1–3]. In this approach, a bound state $\mathrm{B}(P)$ of momentum P and mass $M \equiv \sqrt{P^2}$ is described by a Bethe-Salpeter amplitude defined in configuration-space representation as a matrix element of the time-ordered product of the field operators of the bound-state constituents evaluated between the vacuum state $|0\rangle$ and the bound state $|\mathrm{B}(P)\rangle$. In momentum-space representation, the Bethe-Salpeter amplitude, after splitting off the center-of-momentum motion of the bound state and suppressing all indices generically denoted by $\Phi(p, P)$, encodes the distribution of the relative momentum p of the two bound-state constituents; it satisfies the formally exact homogeneous Bethe-Salpeter equation, which involves two kinds of dynamical ingredients: the full propagators $S_i(p_i)$ of the constituents with individual momenta p_i, $i = 1, 2$; and the Bethe-Salpeter interaction kernel $K(p, q, P)$, by construction a fully amputated four-point Green function of the bound-state constituents defined perturbatively by summation of the countable infinity of all Bethe-Salpeter-irreducible Feynman diagrams for two-particle into two-particle scattering. Skipping all indices, this Bethe-Salpeter equation generically reads

$$\Phi(p, P) = \frac{\mathrm{i}}{(2\pi)^4} S_1(p_1) \int \mathrm{d}^4 q K(p, q, P) \Phi(q, P) S_2(-p_2). \tag{1}$$

Unfortunately, attempts to apply the Bethe-Salpeter formalism to relativistic bound-state problems are doomed to face various grave obstacles, such as the impossibility of determining the Bethe-Salpeter interaction kernel beyond the tight limits of perturbation theory, or the appearance of excitations in the relative time variable of the bound-state constituents—that is, of solutions called abnormal, difficult to interpret in the framework of quantum physics.[2] They may be remedied in *three-dimensional reductions* of the Bethe-Salpeter equation; the most well known of the resulting bound-state equations is the one proposed by Salpeter [4].

B. (Full) Salpeter equation

In order to facilitate the formulation of instantaneous Bethe-Salpeter equations, let us recall the standard definitions of one-particle energy $E_i(\boldsymbol{p})$, the one-particle Dirac Hamiltonian $H_i(\boldsymbol{p})$, and the energy projection operators $\Lambda_i^{\pm}(\boldsymbol{p})$ for positive and negative energy of a particle $i = 1, 2$:

$$E_i(\boldsymbol{p}) \equiv \sqrt{\boldsymbol{p}^2 + m_i^2}, \qquad H_i(\boldsymbol{p}) \equiv \gamma_0(\boldsymbol{\gamma} \cdot \boldsymbol{p} + m_i),$$
$$\Lambda_i^{\pm}(\boldsymbol{p}) \equiv \frac{E_i(\boldsymbol{p}) \pm H_i(\boldsymbol{p})}{2E_i(\boldsymbol{p})}.$$

The projection operators $\Lambda_i^{\pm}(\boldsymbol{p})$ satisfy, in terms of the charge conjugation Dirac matrix C,

$$[\Lambda_i^{\pm}(\boldsymbol{p})]^{\mathrm{c}} \equiv [C^{-1} \Lambda_i^{\pm}(\boldsymbol{p}) C]^{\mathrm{T}} = \Lambda_i^{\mp}(\boldsymbol{p}), \qquad i = 1, 2.$$

The reduction of the Bethe-Salpeter equation to the Salpeter equation relies on exactly two simplifying assumptions (brief reviews of such reductions can be found in Refs. [9–11]):

First, the instantaneous approximation assumes that in the center-of-momentum frame of the bound states, fixed by $P = (M, 0)$, all interactions between bound-state constituents are instantaneous. In this ("static") limit, the Bethe-Salpeter interaction kernel $K(p, q, P)$ depends only on the spatial components $\boldsymbol{p}$ and $\boldsymbol{q}$ of the relative momenta p and q involved; that is, it takes the form $K(p, q, P) = K(\boldsymbol{p}, \boldsymbol{q})$. If, furthermore, the propagators $S_i(p_i)$ of both constituents may be assumed to be entirely free from any nontrivial dependence on the zero component p_0 of the relative momentum p, integrating with respect to p_0 reduces the Bethe-Salpeter equation [Eq. (1)] to a kind of instantaneous Bethe-Salpeter equation for the Salpeter amplitude

$$\phi(\boldsymbol{p}) \equiv \frac{1}{2\pi} \int \mathrm{d}p_0 \Phi(p).$$

For various attempts in this direction, see, for instance, Refs. [12,13] and references therein.

Second, any bound-state constituent is assumed to propagate as a free particle with some effective mass m_i required to encompass appropriately all dynamical

[2]There are, however, very sound arguments [8] claiming that the presence of timelike excitations might be just an artifact of the *ladder approximation* to the Bethe-Salpeter kernel, incorporating the interactions responsible for the bound states only by an iteration of a single-particle exchange between the constituents.

self-energy effects.[3] In other words, any fermion propagator $S_i(p)$ is approximated by its corresponding free form[4]:

$$S_{i,0}(p, m_i) = \frac{\mathrm{i}}{\not{p} - m_i + \mathrm{i}\varepsilon} \equiv \mathrm{i}\frac{\not{p} + m_i}{p^2 - m_i^2 + \mathrm{i}\varepsilon}, \qquad \varepsilon \downarrow 0, \qquad i = 1, 2.$$

Imposing both restrictions above on the Bethe-Salpeter equation [Eq. (1)] yields, by contour integration in the complex p_0 plane and the residue theorem, the Salpeter equation (where, trivially, $\boldsymbol{p} \equiv \boldsymbol{p}_1 = -\boldsymbol{p}_2$ and $P_0 = M$ in the center-of-momentum frame of the bound state) [4]:

$$\phi(\boldsymbol{p}) = \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \left(\frac{\Lambda_1^+(\boldsymbol{p}_1)\gamma_0[K(\boldsymbol{p}, \boldsymbol{q})\phi(\boldsymbol{q})]\gamma_0\Lambda_2^-(\boldsymbol{p}_2)}{P_0 - E_1(\boldsymbol{p}_1) - E_2(\boldsymbol{p}_2)} - \frac{\Lambda_1^-(\boldsymbol{p}_1)\gamma_0[K(\boldsymbol{p}, \boldsymbol{q})\phi(\boldsymbol{q})]\gamma_0\Lambda_2^+(\boldsymbol{p}_2)}{P_0 + E_1(\boldsymbol{p}_1) + E_2(\boldsymbol{p}_2)} \right). \tag{2}$$

Multiplying the Salpeter equation [Eq. (2)] from the left by $\Lambda_1^{\pm}(\boldsymbol{p}_1)$ and from the right by $\Lambda_2^{\pm}(\boldsymbol{p}_2)$ and using $\Lambda_i^{\pm}(\boldsymbol{p})\Lambda_i^{\mp}(\boldsymbol{p}) = 0$ reveals that any of its solutions $\phi(\boldsymbol{p})$ will satisfy the constraints

$$\Lambda_1^+(\boldsymbol{p}_1)\phi(\boldsymbol{p})\Lambda_2^+(\boldsymbol{p}_2) = \Lambda_1^-(\boldsymbol{p}_1)\phi(\boldsymbol{p})\Lambda_2^-(\boldsymbol{p}_2) = 0.$$

These constraints halve the number of independent components of any solution $\phi(\boldsymbol{p})$ of the Salpeter equation [Eq. (2)]; obviously, this has important implications for practical calculations. Generally, the Bethe-Salpeter interaction kernel $K(\boldsymbol{p}, \boldsymbol{q})$ may be decomposed into a sum of products of Lorentz-scalar potentials $V(\boldsymbol{p}, \boldsymbol{q})$ and associated Lorentz structures. Assuming the latter to be represented by identical Dirac matrices, generically labelled Γ, the action of the kernel $K(\boldsymbol{p}, \boldsymbol{q})$ on the Salpeter amplitude $\phi(\boldsymbol{q})$ requested by the Salpeter equation [Eq. (2)] is

$$[K(\boldsymbol{p}, \boldsymbol{q})\phi(\boldsymbol{q})] = \sum_\Gamma V_\Gamma(\boldsymbol{p}, \boldsymbol{q})\Gamma\phi(\boldsymbol{q})\Gamma.$$

C. Reduced Salpeter equation

By the charge conjugation properties of the energy projection operators $\Lambda_i^{\pm}(\boldsymbol{p})$, the second term on the right-hand side of the Salpeter equation [Eq. (2)] corresponds to the negative-energy components of the Salpeter amplitude $\phi(\boldsymbol{p})$: $\Lambda_1^-(\boldsymbol{p}_1)\phi(\boldsymbol{p})\Lambda_2^+(\boldsymbol{p}_2) \equiv \Lambda_1^-(\boldsymbol{p}_1)\phi(\boldsymbol{p}) \times [\Lambda_2^-(\boldsymbol{p}_2)]^{\mathrm{c}}$. Assuming that it is justifiable to neglect the contribution of this latter term relative to that of the first term on the right-hand side of Eq. (2) leads to the *reduced Salpeter equation* [5,6]:

$$[P_0 - E_1(\boldsymbol{p}_1) - E_2(\boldsymbol{p}_2)]\phi(\boldsymbol{p}) = \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \Lambda_1^+(\boldsymbol{p}_1)\gamma_0[K(\boldsymbol{p}, \boldsymbol{q})\phi(\boldsymbol{q})]\gamma_0\Lambda_2^-(\boldsymbol{p}_2), \tag{3}$$

forming, unlike Eq. (2), an *explicit eigenvalue problem* for the bound-state masses $P_0 = M$. Formally, this reduction of the (full) Salpeter equation [Eq. (2)] to the reduced Salpeter equation [Eq. (3)] may be effected by imposing as a further constraint $\Lambda_1^-(\boldsymbol{p}_1)\phi(\boldsymbol{p}) = 0$ or $\phi(\boldsymbol{p})\Lambda_2^+(\boldsymbol{p}_2) = 0$; as a trivial consequence of this, all solutions $\phi(\boldsymbol{p})$ of the *reduced* Salpeter equation [Eq. (3)] involve only the *positive-energy* components $\phi(\boldsymbol{p}) = \Lambda_1^+(\boldsymbol{p}_1)\phi(\boldsymbol{p})\Lambda_2^-(\boldsymbol{p}_2) \equiv \Lambda_1^+(\boldsymbol{p}_1)\phi(\boldsymbol{p})[\Lambda_2^+(\boldsymbol{p}_2)]^{\mathrm{c}}$. Physically, this simplification supposes that $P_0 - E_1(\boldsymbol{p}_1) - E_2(\boldsymbol{p}_2) \ll P_0 + E_1(\boldsymbol{p}_1) + E_2(\boldsymbol{p}_2)$ or $[P_0 + E_1(\boldsymbol{p}_1) + E_2(\boldsymbol{p}_2)]^{-1} \ll [P_0 - E_1(\boldsymbol{p}_1) - E_2(\boldsymbol{p}_2)]^{-1}$ holds at the level of expectation values; this assumption may be justified for semirelativistic and weakly bound heavy constituents. In the center-of-momentum frame of the bound state, $\boldsymbol{P} = 0$ clearly implies $\boldsymbol{p} = \boldsymbol{p}_1 = -\boldsymbol{p}_2$.

D. Pseudoscalar bound states

To start with, let us focus our interest on those fermion-antifermion bound states which are represented by a Salpeter amplitude with the least number of *independent* components: the pseudoscalar states. These are bound states composed of some fermion and its antiparticle, and are thus characterized by $m_1 = m_2 = m$ and well-defined behavior under charge conjugation; the quantum number assignment common to all pseudoscalar states is $J = 0$ for their total spin, $P = -1$ for their parity, and $C = +1$ for their charge conjugation parity: $J^{PC} = 0^{-+}$. Such states are realized in nature in the form of, e.g., the pion, as a quark-antiquark bound state. Henceforth, all indices $i = 1, 2$ distinguishing the bound-state constituents can be dropped, since now, for instance, $E_1(\boldsymbol{p}) = E_2(\boldsymbol{p}) = E(\boldsymbol{p}) = E(p) \equiv \sqrt{p^2 + m^2}$, with $p \equiv |\boldsymbol{p}| \equiv \sqrt{\boldsymbol{p}^2}$.

The most general expansion of the Salpeter amplitude $\phi(\boldsymbol{p})$ over a complete set of Dirac matrices would, of course, introduce 16 Salpeter components. However, as a consequence of the peculiar projector structure of the Salpeter equation [Eq. (2)], manifesting in the constraints mentioned in Sec. II B, precisely eight of these components are independent. Specifically, for describing pseudoscalar states, just two of the latter, labeled $\varphi_1(\boldsymbol{p})$ and $\varphi_2(\boldsymbol{p})$, are relevant:

[3] This forms the example par excellence of the trivial p_0 dependence of the propagators requested above.

[4] The assumption of free propagators for the bound-state constituents may, however, encounter a serious conceptual problem for the following reason: in quantum field theory, the infinite tower of Dyson-Schwinger equations relates any n-point Green function to at least one $(m > n)$-point Green function; this entails that propagators, being two-point Green functions, and the four-point Green function entering as an interaction kernel, cannot be viewed as independent. In particular, in quantum chromodynamics (the theory describing the strong interactions), free propagators for any of its fundamental quark and gluon degrees of freedom are incompatible with the feature of color confinement, exhibited by this unbroken non-Abelian gauge theory.

$$\phi(\boldsymbol{p}) = \left[\varphi_1(\boldsymbol{p})\frac{H(\boldsymbol{p})}{E(\boldsymbol{p})} + \varphi_2(\boldsymbol{p})\right]\gamma_5.$$

In fact, the above decomposition of $\phi(\boldsymbol{p})$ applies to *all* bound states of a spin-$\frac{1}{2}$ fermion and a spin-$\frac{1}{2}$ antifermion for which the quantum number s of the sum of the two fermion spins is zero; i.e., to all spin singlet states carrying $s = 0$, irrespective of the relative orbital angular momentum ℓ of the constituents. Pseudoscalar bound states are just the special case $\ell = 0$. The one additional constraint truncating the full to the reduced Salpeter equation enforces the equality of the Salpeter components $\varphi_1(\boldsymbol{p})$ and $\varphi_2(\boldsymbol{p})$: $\varphi_1(\boldsymbol{p}) = \varphi_2(\boldsymbol{p}) \equiv \varphi(\boldsymbol{p})$. Thus, all spin singlet—notably, all pseudoscalar—solutions of a reduced Salpeter equation [Eq. (3)] will read

$$\phi(\boldsymbol{p}) = \varphi(\boldsymbol{p})\frac{H(\boldsymbol{p}) + E(\boldsymbol{p})}{E(\boldsymbol{p})}\gamma_5 \equiv 2\varphi(\boldsymbol{p})\Lambda^+(\boldsymbol{p})\gamma_5.$$

E. Radial eigenvalue equations

In order to follow the more-or-less standard route of nonrelativistic reduction, let us assume that the Bethe-Salpeter interaction kernel $K(\boldsymbol{p}, \boldsymbol{q})$ is of *convolution type*; i.e., it is of the form $K(\boldsymbol{p}, \boldsymbol{q}) = K(\boldsymbol{p} - \boldsymbol{q})$, which entails for the potentials $V_\Gamma(\boldsymbol{p}, \boldsymbol{q}) = V_\Gamma(\boldsymbol{p} - \boldsymbol{q})$; and let us assume that $K(\boldsymbol{p}, \boldsymbol{q})$ respects spherical symmetry; i.e., $K(\boldsymbol{p}, \boldsymbol{q}) = K((\boldsymbol{p} - \boldsymbol{q})^2)$, and thus $V_\Gamma(\boldsymbol{p}, \boldsymbol{q}) = V_\Gamma((\boldsymbol{p} - \boldsymbol{q})^2)$; trivially, this means that each momentum-space potential $V_\Gamma(\boldsymbol{p}, \boldsymbol{q})$ is the Fourier transform of a spherically symmetric configuration-space potential $V_\Gamma(r)$ depending just on the radial coordinate $r \equiv |\boldsymbol{x}|$. In this case, any dependence on the angular variables, encoded either in spherical harmonics or in vector spherical harmonics, can be split off; this factorization effects the reductions of both the Salpeter equation [Eq. (2)] [14,15] and its reduced counterpart [Eq. (3)] [16] to equivalent systems of coupled equations for the radial factors of all independent Salpeter components. For a particular Lorentz structure of the Bethe-Salpeter kernel (such that the index Γ identifying the Lorentz structure may be suppressed), the interactions experienced by bound-state constituents enter the radial eigenvalue equations in the form of Fourier-Bessel transforms $V_L(p, q)$ $(L = 0, 1, 2, \ldots)$ of the radial static configuration-space potential $V(r)$, defined in terms of spherical Bessel functions of the first kind [17], $j_n(z)$, $n = 0, \pm 1, \pm 2, \ldots$:

$$V_L(p, q) \equiv 8\pi \int_0^\infty \mathrm{d}r r^2 j_L(pr) j_L(qr) V(r),$$
$$p \equiv |\boldsymbol{p}|, \qquad q \equiv |\boldsymbol{q}|, \qquad L = 0, 1, 2, \ldots.$$

In Sec. III, we intend to exemplify our technique of constructing exact solutions to the reduced Salpeter equation by considering only a few of those Lorentz structures $\Gamma \otimes \Gamma$ of the interaction kernel $K(\boldsymbol{p}, \boldsymbol{q})$ that are commonly used in phenomenological descriptions of hadrons—notably, of mesons, as bound states of quarks confined by the strong interactions.[5]

1. (Full) Salpeter equation

The Salpeter equation for bound states with the spin-sum quantum number $s = 0$ is equivalent to a system of two equations for the two independent radial Salpeter components $\varphi_1(p)$ and $\varphi_2(p)$. For pure time-component Lorentz-vector kernels ($\Gamma \otimes \Gamma = \gamma^0 \otimes \gamma^0$), this system reads

$$2E(p)\varphi_2(p) + \int_0^\infty \frac{\mathrm{d}qq^2}{(2\pi)^2} V_0(p, q)\varphi_2(q) = M\varphi_1(p),$$
$$2E(p)\varphi_1(p) + \int_0^\infty \frac{\mathrm{d}qq^2}{(2\pi)^2}\left[\frac{m^2}{E(p)E(q)} V_0(p, q) + \frac{pq}{E(p)E(q)} V_1(p, q)\right]\varphi_1(q) = M\varphi_2(p).$$

2. Reduced Salpeter equation

The (compared to the full Salpeter equation) simple and unique energy projector structure of the reduced Salpeter equation [Eq. (3)] guarantees that each spin-singlet bound-state solution involves just one independent radial Salpeter component $\varphi(p)$, since $\varphi_1(p) = \varphi_2(p) \equiv \varphi(p)$. The system of coupled radial equations thus collapses to a single radial eigenvalue equation which reads, for a kernel of time-component Lorentz-vector Dirac structure $\Gamma \otimes \Gamma = \gamma^0 \otimes \gamma^0$,

$$2E(p)\varphi(p) + \frac{1}{2}\int_0^\infty \frac{\mathrm{d}qq^2}{(2\pi)^2}\left[\left(1 + \frac{m^2}{E(p)E(q)}\right)V_0(p, q) + \frac{pq}{E(p)E(q)} V_1(p, q)\right]\varphi(q) = M\varphi(p);$$

for a kernel of Lorentz-vector Dirac structure $\Gamma \otimes \Gamma = \gamma_\mu \otimes \gamma^\mu$, describing all the interactions by an effective

[5] Since the first encounter [15,18] with unstable solutions of Salpeter equations with confining interactions, the stability of solutions of different relativistic bound-state equations has been an issue of concern [19–25]. For confining interactions of harmonic oscillator form in configuration space (in which case all bound-state integral equations reduce to easier-to-handle differential equations in momentum space), the conditions for stability of bound-state solutions within the instantaneous Bethe-Salpeter framework (in the sense of their spectrum being real, discrete, and bounded from below) have been analyzed for those Lorentz structures of the Bethe-Salpeter interaction kernel which are most frequently employed for the description of hadrons as bound states of quarks. A rigorous analytic proof of the stability of the bound-state energy spectra entailed has been constructed for the reduced Salpeter equation [26–28] and a particular generalization of it [28,29], formulated by taking into account the full propagators of the bound-state constituents [12,13,30], obtained as the solutions of the Dyson-Schwinger equations for the corresponding two-point Green's functions of the bound-state constituents, and this problem has also been discussed for the (full) Salpeter equation [31–33].

vector-boson exchange between the two fermionic bound-state constituents,

$$2E(p)\varphi(p) + \int_0^\infty \frac{\mathrm{d}qq^2}{(2\pi)^2}\left(2 - \frac{m^2}{E(p)E(q)}\right)V_0(p,q)\varphi(q) = M\varphi(p);$$

or, for the *linear combination* $2\Gamma\otimes\Gamma = \gamma_\mu\otimes\gamma^\mu + \gamma_5\otimes\gamma_5 - 1\otimes 1$ of the kernel's Dirac structure,

$$2E(p)\varphi(p) + \int_0^\infty \frac{\mathrm{d}qq^2}{(2\pi)^2}V_0(p,q)\varphi(q) = M\varphi(p). \quad (4)$$

F. Special case: Massless bound-state constituents ($m = 0$)

In the limit of vanishing masses of the two bound-state constituents, i.e., in the case $m = 0$, the bound-state equations of Sec. II E simplify, of course, still further: the Salpeter equation with a time-component Lorentz-vector kernel $\Gamma\otimes\Gamma = \gamma^0\otimes\gamma^0$ reduces to the set of equations

$$\begin{aligned} 2p\varphi_2(p) + \int_0^\infty \frac{\mathrm{d}qq^2}{(2\pi)^2}V_0(p,q)\varphi_2(q) &= M\varphi_1(p), \\ 2p\varphi_1(p) + \int_0^\infty \frac{\mathrm{d}qq^2}{(2\pi)^2}V_1(p,q)\varphi_1(q) &= M\varphi_2(p), \end{aligned} \quad (5)$$

whereas the single radial eigenvalue equation emerging from the reduced Salpeter equation reads, for interactions with the time-component Lorentz-vector Dirac structure $\Gamma\otimes\Gamma = \gamma^0\otimes\gamma^0$,

$$2p\varphi(p) + \frac{1}{2}\int_0^\infty \frac{\mathrm{d}qq^2}{(2\pi)^2}[V_0(p,q) + V_1(p,q)]\varphi(q) = M\varphi(p); \quad (6)$$

for interactions with the Lorentz-vector Dirac structure $\Gamma\otimes\Gamma = \gamma_\mu\otimes\gamma^\mu$,

$$2p\varphi(p) + 2\int_0^\infty \frac{\mathrm{d}qq^2}{(2\pi)^2}V_0(p,q)\varphi(q) = M\varphi(p); \quad (7)$$

or, for kernels of the particularly favorable Dirac structure $2\Gamma\otimes\Gamma = \gamma_\mu\otimes\gamma^\mu + \gamma_5\otimes\gamma_5 - 1\otimes 1$,

$$2p\varphi(p) + \int_0^\infty \frac{\mathrm{d}qq^2}{(2\pi)^2}V_0(p,q)\varphi(q) = M\varphi(p). \quad (8)$$

Although we focus on reduced Salpeter equations, we need Eq. (5) at an intermediate stage.

III. CONFIGURATION-SPACE POTENTIALS $V(r)$ BY INVERSION

All the radial bound-state eigenvalue equations recalled in Secs. II E and II F emerging from the reduced Salpeter equation [Eq. (3)] under our reasonable assumption of spherical symmetry, are for $M \neq 2E(p)$ homogeneous linear Fredholm integral equations of the second kind but are still simple enough that, for sufficiently sophisticated choices of the Salpeter solutions $\varphi(p)$, their underlying configuration-space potentials $V(r)$ can be extracted by analytical means. For notational convenience, we define the Fourier-Bessel transforms to configuration space,

$$\varphi_L(r) \equiv \mathrm{i}^L\sqrt{\frac{2}{\pi}}\int_0^\infty \mathrm{d}pp^2 j_L(pr)\varphi(p), \qquad L = 0, 1,$$

$$T_L(r) \equiv \mathrm{i}^L\sqrt{\frac{2}{\pi}}\int_0^\infty \mathrm{d}pp^2 j_L(pr)E(p)\varphi(p), \qquad L = 0, 1,$$

of both radial Salpeter amplitude $\varphi(p)$ and free-energy part $E(p)\varphi(p)$ in momentum space. In the following subsections, we illustrate the idea of analytical extraction of configuration-space radial potentials $V(r)$ by examining explicit examples: For an appropriate ansatz for the Salpeter solution $\varphi(p)$, by application of the Fourier-Bessel transformation to the momentum-space bound-state equation considered, we would like to get its *configuration-space representation* in terms of $\varphi_L(r)$ and $T_L(r)$; from the latter formulation, we should be able to read off $V(r)$. Since in Eqs. (4) and (6)–(8) the potential may absorb any mass $M \neq 0$, we assume $M = 0$. Of course, the simple procedure sketched above can only be followed if the interaction term in the bound-state equation under study contains merely a single Fourier-Bessel transform $V_L(p,q)$ of the radial potential $V(r)$; i.e., it involves a unique value of L. If, on the other hand, both $V_0(p,q)$ and $V_1(p,q)$ enter in the interaction term, as happens, e.g., for any interaction of time-component Lorentz-vector structure, a different line of reasoning has to be devised. For good reason, we first invert equations for massless bound-state constituents (Sec. III A). Then we turn to the more delicate case of bound-state constituents of finite mass (Sec. III B).

A. Massless bound-state constituents: $m = 0$

In order to be able to deal simultaneously with interaction kernels of Lorentz-vector nature and of the (simplifying) linear combination yielding Eq. (4), we introduce a parameter η by

$$\eta = \begin{cases} 2 & \text{for } \Gamma\otimes\Gamma = \gamma_\mu\otimes\gamma^\mu, \\ 1 & \text{for } \Gamma\otimes\Gamma = \frac{1}{2}(\gamma_\mu\otimes\gamma^\mu + \gamma_5\otimes\gamma_5 - 1\otimes 1). \end{cases}$$

By this definition, two of the reduced Salpeter equations [Eqs. (7) and (8)] may be subsumed in the form

$$2p\varphi(p) + \eta\int_0^\infty \frac{\mathrm{d}qq^2}{(2\pi)^2}V_0(p,q)\varphi(q) = M\varphi(p). \quad (9)$$

For this equation of motion, its $L = 0$ Fourier-Bessel transform is straightforwardly found:

$$2T_0(r) + \eta V(r)\varphi_0(r) = M\varphi_0(r).$$

Thus, the configuration-space potential associated to mass eigenvalue $M = 0$ is found to be

$$V(r) = -\frac{2T_0(r)}{\eta\varphi_0(r)}. \tag{10}$$

In the case of interaction kernels of the time-component Lorentz-vector Dirac structure, i.e., $\Gamma \otimes \Gamma = \gamma^0 \otimes \gamma^0$, the reduced Salpeter equation involves, even for zero-mass bound-state constituents, Fourier-Bessel transforms $V_L(p, q)$ of $V(r)$ for more than one L—namely, both $V_0(p, q)$ *and* $V_1(p, q)$. From this type of bound-state equation the potential $V(r)$ *cannot* be recovered by applying to Eq. (6) a Fourier-Bessel transformation of a particular value of L. In order to overcome this adverse observation, we recall that, as a consequence of the *equality* $\varphi_1(p) = \varphi_2(p) \equiv \varphi(p)$ of the two independent radial components of Salpeter amplitudes for spin-singlet bound states enforced by the reduced Salpeter constraint discussed in Sec. II C, for a definite Lorentz structure $\Gamma \otimes \Gamma$ of the interaction kernel, the radial eigenvalue equation resulting from the *reduced* Salpeter equation [Eq. (3)] can be found by adding, for $\varphi_1(p) = \varphi_2(p)$, the two equations that constitute the set of (originally coupled) radial eigenvalue equations related to the corresponding Salpeter equation [Eq. (2)]; see, for instance, footnote 2 of Ref. [27]. Bearing these findings in mind, we seek, for tentative solutions $\varphi(p)$ of the reduced Salpeter equation [Eq. (6)] for pure time-component Lorentz-vector interaction, the responsible potential $V(r)$, with the help of the *full*-Salpeter "precursor" Eq. (5) of Eq. (6), via a two-step procedure:

(1) After equating the two independent components $\varphi_1(p) = \varphi_2(p) \equiv \varphi(p)$, we represent the decoupled relations arising from the set of equations [Eq. (5)] in configuration space by application of the appropriate Fourier-Bessel transformation to each of the relations:

$$2T_0(r) + V_0(r)\varphi_0(r) = M\varphi_0(r),$$
$$2T_1(r) + V_1(r)\varphi_1(r) = M\varphi_1(r).$$

At this stage we must take into account, by an index $L = 0, 1$, the possibility that the potentials $V_L(r)$ derived from each of these relations by analogy to Eq. (10) can differ:

$$V_L(r) \equiv -\frac{2T_L(r)}{\varphi_L(r)}, \qquad L = 0, 1. \tag{11}$$

(2) We *assume* that the unique configuration-space potential $V(r)$ we are seeking may be expressed as a *linear combination* $V(r) = c_0V_0(r) + c_1V_1(r)$ of the auxiliary functions $V_0(r)$ and $V_1(r)$, with yet-to-be-determined (of course, constant) coefficients c_0 and c_1. We attempt to find these coefficients by inserting our ansatz for $V(r)$ into the slightly more complex and intricate reduced Salpeter equation [Eq. (6)] for $M = 0$; if we manage to deduce thereby a solution for c_0 and c_1, our quest for the potential $V(r)$ is completed.

1. Nonconfining interaction potentials

Presumably, the first guess that comes to one's mind for the momentum-space bound-state amplitude $\varphi(p)$ is the exponential, involving a parameter λ with the dimension of inverse mass:

$$\varphi(p) = 2\lambda^{3/2}\exp(-\lambda p), \qquad \lambda > 0,$$
$$\| \varphi \|^2 \equiv \int_0^\infty \mathrm{d}p\, p^2 |\varphi(p)|^2 = 1. \tag{12}$$

For this choice, the $L = 0, 1$ Fourier-Bessel transforms of $\varphi(p)$ and $m = 0$ kinetic term read

$$\varphi_0(r) = \sqrt{\frac{2}{\pi}}\frac{4\lambda^{5/2}}{(r^2+\lambda^2)^2}, \qquad \varphi_1(r) = \sqrt{\frac{2}{\pi}}\frac{4\mathrm{i}\lambda^{3/2}r}{(r^2+\lambda^2)^2},$$
$$T_0(r) = \sqrt{\frac{2}{\pi}}4\lambda^{3/2}\frac{3\lambda^2 - r^2}{(r^2+\lambda^2)^3}, \qquad T_1(r) = \sqrt{\frac{2}{\pi}}\frac{16\mathrm{i}\lambda^{5/2}r}{(r^2+\lambda^2)^3}.$$

If the Lorentz structure of the interaction kernel is either a pure vector ($\Gamma \otimes \Gamma = \gamma_\mu \otimes \gamma^\mu$) or a linear combination of vector, pseudoscalar, and scalar ($2\Gamma \otimes \Gamma = \gamma_\mu \otimes \gamma^\mu + \gamma_5 \otimes \gamma_5 - 1 \otimes 1$) as summarized by the reduced Salpeter equation in Eq. (9), our findings [Eq. (10)] immediately entail, for a Salpeter amplitude $\varphi(p)$ of the *exponential* form [Eq. (12)], in configuration space the potential

$$V(r) = \frac{2}{\eta\lambda}\left(1 - \frac{4\lambda^2}{r^2+\lambda^2}\right), \qquad V(0) = -\frac{6}{\eta\lambda},$$
$$V(r) \underset{r\to\infty}{\longrightarrow} \frac{2}{\eta\lambda}.$$

In the limit $r \to \infty$, this potential $V(r)$ approaches a finite value. It is thus a representative of the class of nonconfining interactions, with dependence on the radial variable r (in units of λ, i.e., for $\lambda = 1$) for Lorentz-vector Bethe-Salpeter kernels ($\eta = 2$) as depicted in Fig. 1.

For the reduced Salpeter equation [Eq. (6)] with a time-component Lorentz-vector Dirac structure $\Gamma \otimes \Gamma = \gamma^0 \otimes \gamma^0$ and an exponential ansatz [Eq. (12)] for $\varphi(p)$, the auxiliary functions [Eq. (11)] become

$$V_0(r) = \frac{2}{\lambda}\left(1 - \frac{4\lambda^2}{r^2+\lambda^2}\right), \qquad V_1(r) = -\frac{8\lambda}{r^2+\lambda^2}.$$

Inserting these potentials into Eq. (6) fixes the coefficients $c_{0,1}$ to $c_0 = c_1 = \frac{1}{2}$, which entails a configuration-space potential $V(r)$ with behavior clearly similar to that shown in Fig. 1:

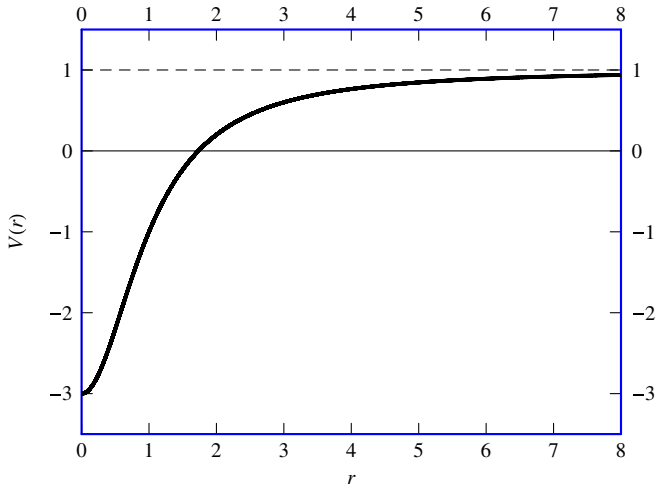

FIG. 1 (color online). Configuration-space *nonconfining* potential $V(r) = 1 - 4/(r^2 + 1)$, determined by inversion of the *reduced* Salpeter equation [Eq. (9)] with kernels of *Lorentz-vector* nature ($\eta = 2$) when assuming an exponential form $\varphi(p) \propto \exp(-p)$ of its momentum-space solution $\varphi(p)$. Starting at $V(0) = -3$, this potential $V(r)$ behaves for $r \to \infty$ like $V(r) \to 1$ (dashed line).

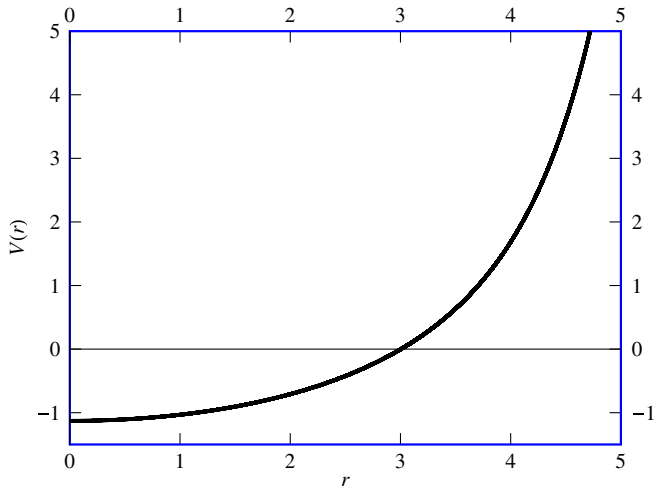

FIG. 2 (color online). Configuration-space *confining* potential $V(r) = (\frac{r}{2} - \frac{1}{r})\mathrm{erfi}(\frac{r}{2}) - \exp(r^2/4)/\sqrt{\pi}$, found by inversion of the *reduced* Salpeter equation [Eq. (9)] with *Lorentz-vector* kernels ($\eta = 2$) if trying a Gaussian form $\varphi(p) \propto \exp(-p^2)$ for its momentum-space solution $\varphi(p)$. Starting at $V(0) = -2/\sqrt{\pi} \approx -1.12837\ldots$, this potential $V(r)$ behaves like $V(r) \to \infty$ for $r \to \infty$.

$$V(r) = \frac{1}{\lambda}\left(1 - \frac{8\lambda^2}{r^2 + \lambda^2}\right), \qquad V(0) = -\frac{7}{\lambda},$$
$$V(r) \underset{r\to\infty}{\to} \frac{1}{\lambda}.$$

2. Confining interaction potentials

In order to present also an example for a confining potential, we next consider an amplitude of normalized *Gaussian* form, using a parameter λ with the dimension of inverse mass squared:

$$\varphi(p) = 2\left(\frac{8\lambda^3}{\pi}\right)^{1/4} \exp(-\lambda p^2), \qquad \lambda > 0,$$
$$\| \varphi \|^2 \equiv \int_0^\infty \mathrm{d}p\, p^2 |\varphi(p)|^2 = 1. \tag{13}$$

Trivially, the $L = 0$ Fourier-Bessel transform of this function $\varphi(p)$ is also of Gaussian form:

$$\varphi_0(r) = \left(\frac{2}{\pi\lambda^3}\right)^{1/4} \exp\left(-\frac{r^2}{4\lambda}\right).$$

The configuration-space potential entailed by Eq. (9) involves the imaginary error function $\mathrm{erfi}(z)$ defined in terms of the error function $\mathrm{erf}(z)$ [17] by $\mathrm{erfi}(z) \equiv -i\,\mathrm{erf}(iz)$; for large r, it rises like $16\lambda^{3/2}\exp(r^2/4\lambda)/\eta\sqrt{\pi}r^4$ and thus realizes confinement, as illustrated by Fig. 2:

$$V(r) = \frac{1}{\eta\lambda}\left[\left(r - \frac{2\lambda}{r}\right)\mathrm{erfi}\left(\frac{r}{2\sqrt{\lambda}}\right) - 2\sqrt{\frac{\lambda}{\pi}}\exp\left(\frac{r^2}{4\lambda}\right)\right],$$
$$V(0) = -\frac{4}{\eta\sqrt{\pi\lambda}}.$$

A time-component Lorentz-vector Dirac structure can be dealt with along the same lines as in Sec. III A 1; however, the resulting expressions are lengthy and not really enlightening.

B. Bound-state constituents with nonzero mass: $m > 0$

The extension of the particular inversion technique framed here to the case of *nonvanishing* masses m of the bound-state constituents clearly requires somewhat more careful choices of momentum-space amplitudes $\varphi(p)$. One's first attempt might employ the rational function

$$\varphi(p) = \sqrt{\frac{2}{\pi}}\frac{4\mu^{5/2}}{(p^2 + \mu^2)^2}, \qquad \mu > 0,$$
$$\| \varphi \|^2 \equiv \int_0^\infty \mathrm{d}p\, p^2 |\varphi(p)|^2 = 1, \tag{14}$$

with a mass parameter μ. Its $L = 0$ Fourier-Bessel transform is, for $\mu \leq m$, an exponential:

$$\varphi_0(r) = 2\mu^{3/2}\exp(-\mu r), \qquad 0 < \mu \leq m.$$

For illustrative purposes, we apply this Salpeter amplitude to invert the particularly simple reduced Salpeter equation [Eq. (4)] with the linear combination $\Gamma \otimes \Gamma = \frac{1}{2}(\gamma_\mu \otimes \gamma^\mu + \gamma_5 \otimes \gamma_5 - 1 \otimes 1)$ of Lorentz structures.

In order to evaluate the relevant Fourier-Bessel transform of Eq. (4),

$$\begin{aligned} &2T_0(r) + V(r)\varphi_0(r) = M\varphi_0(r) \\ \Rightarrow &V(r) = -\frac{2T_0(r)}{\varphi_0(r)} \quad \text{for } M = 0, \end{aligned} \tag{15}$$

we need, for $m \neq 0$, the $L = 0$ Fourier-Bessel transform $T_0(r)$ of the kinetic part $E(p)\varphi(p)$.

1. Case $0 < \mu = m$

For the rational Salpeter amplitude [Eq. (14)], the $L = 0$ Fourier-Bessel transform of the kinetic part is easily pinned down if the parameter μ is chosen to be equal to the constituents' mass m; the outcome for $T_0(r)$ is basically the modified Bessel function $K_\nu(z)$ [17] of order $\nu = 0$:

$$T_0(r) = \frac{8m^{5/2}}{\pi} K_0(mr).$$

Hence, the instant reply of our inversion procedure [Eq. (15)] is the configuration-space potential

$$\begin{aligned} &V(r) = -\frac{8m}{\pi} K_0(mr)\exp(mr), \\ &V(r) \underset{r\to 0}{\to} \frac{8m}{\pi}\ln(mr), \qquad V(r) \underset{r\to\infty}{\to} 0. \end{aligned}$$

The monotonic increase of this potential $V(r)$, from its logarithmic singularity at the origin $r = 0$ to its asymptotic value 0 for $r \to \infty$, is depicted for constituent mass $m = 1$ in Fig. 3. The singularity is milder than Coulombic, and hence is compatible with Herbst's findings [34].

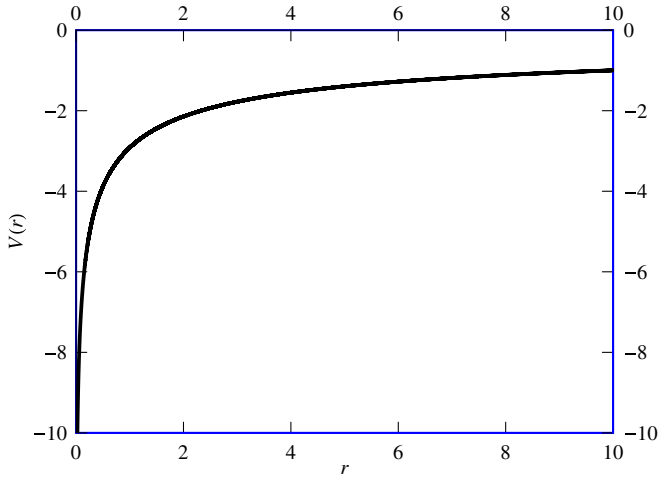

FIG. 3 (color online). Configuration-space *nonconfining* potential $V(r) = -(8/\pi)K_0(r)\exp(r)$, found by studying the *reduced* Salpeter equation [Eq. (4)] for bound-state constituents of *nonvanishing* mass m with a kernel of Lorentz structure $\Gamma \otimes \Gamma = \frac{1}{2}(\gamma_\mu \otimes \gamma^\mu + \gamma_5 \otimes \gamma_5 - 1 \otimes 1)$ when relying on a rational function $\varphi(p) \propto (p^2 + 1)^{-2}$ as an ansatz for the momentum-space solution $\varphi(p)$. This $V(r)$ vanishes in the limit $r \to \infty$ but exhibits a logarithmic singularity at the origin $r = 0$.

2. Case $0 < \mu < m$

Allowing the parameter μ in our ansatz [Eq. (14)] for the Salpeter amplitude $\varphi(p)$ to be less than the constituents' mass m, i.e., $\mu < m$, performing a contour integration and adopting the residue theorem enables us to cast the resulting configuration-space potential into the form

$$\begin{aligned} V(r) = -2\Bigg\{&\sqrt{m^2-\mu^2} + \frac{\mu}{\sqrt{m^2-\mu^2}}\frac{1}{r} - \frac{4\mu}{\pi r} \\ &\times \int_m^\infty \mathrm{d}\rho\,\rho\exp[-r(\rho-\mu)]\frac{\sqrt{\rho^2-m^2}}{(\rho^2-\mu^2)^2}\Bigg\}. \end{aligned}$$

For $r \to \infty$, the r-dependent terms of the potential vanish and $V(r)$ approaches a constant:

$$V(r) \underset{r\to\infty}{\to} -2\sqrt{m^2-\mu^2}.$$

Examining the r-dependent portion of this potential $V(r)$ for $r \to 0$ by L'Hôpital's rule, we encounter a logarithmically divergent integral. Hence, since for $\mu \uparrow m$ the asymptotic value of $V(r)$ reduces to zero, the behavior of $V(r)$ resembles that found for the case $\mu = m$.

IV. SUMMARY, CONCLUSIONS, AND OUTLOOK

In this study, we showed how to establish, by analytical means, exact relationships between solutions of Bethe-Salpeter equations and the underlying—instantaneous—interactions: for elaborate assumptions about the nature of both interactions and resulting bound states culminating in a manageable structure of the equations governing such bound states, this is accomplished by postulating particular solutions and reading off the interaction potentials. Among other benefits, having these rigorous solutions at one's disposal obviously provides a useful or even decisive test when solving Bethe-Salpeter equations numerically by conversion into equivalent *matrix eigenvalue problems* (as, e.g., in Refs. [14–16,35–40]) or when attempting to construct *approximate models* for Bethe-Salpeter solutions, as proposed in Refs. [41,42].

Clearly, the three-dimensional Fourier transform of any function which depends only on a radial coordinate is just the $L = 0$ Fourier-Bessel transform of this function. Accordingly, due to the simplicity of the bound states inspected, some but not all of our reduced Salpeter equations—more precisely, Eqs. (4), (7), and (8)—are, in fact, equivalent to so-called spinless Salpeter equations with, where necessary, appropriately adjusted overall coupling strength. (For reviews on this latter bound-state equation, see, e.g., Refs. [43–45].) Earlier attempts to construct exact solutions of spinless Salpeter equations may be found in, e.g., Refs. [46–50].

In order to provide a kind of "proof of feasibility" of the inversion technique constructed here, this formalism has been elaborated only for the simplest conceivable problem;

that is, the one posed by the *reduced* Salpeter equation. There exist, however, exceptional cases for which the above findings apply directly, without changes, also to the *full* Salpeter equation: As recalled in Sec. II D, bound states with a vanishing sum of the spins of their constituents, such as pseudoscalar states, are represented by only two independent Salpeter components, the minimal number of independent components for solutions of the *full* Salpeter equation. Correspondingly, for these states the full Salpeter equation becomes equivalent to a system of merely two—in general, coupled—equations. For a Lorentz structure of the Bethe-Salpeter interaction kernel of, for example, the form $2\Gamma \otimes \Gamma = \gamma_\mu \otimes \gamma^\mu + \gamma_5 \otimes \gamma_5 - 1 \otimes 1$, one of the latter equations does not contain any interactions and is therefore of purely algebraic nature [27]. For vanishing bound-state mass, the two equations decouple, and the inversion problem for the full Salpeter equation thus becomes identical to that for the reduced Salpeter equation.

The intention behind this study was to carry out an analysis of purely academic nature. Nevertheless, one may ask the legitimate question: to which physical bound states observed in nature do the above considerations apply? Section II C confines the validity of the *reduced* Salpeter equation to semirelativistic, weakly bound, heavy constituents; this precludes, for instance, the pion, but not necessarily pseudoscalar mesons composed of heavy quarks, such as η_c and η_b, or h_c and h_b. The range of application to be expected for any *full* Salpeter equation is, of course, much wider. However, a thorough study of the latter equation, as well as the extraction of a realistic potential, is definitely beyond the scope of the present work.

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