Perturbative expansion of the QCD Adler function improved by renormalization-group summation and analytic continuation in the Borel plane

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We examine the large-order behavior of a recently proposed renormalization-group-improved expansion of the Adler function in perturbative QCD, which sums in an analytically closed form the leading logarithms accessible from renormalization-group invariance. The expansion is first written as an effective series in powers of the one-loop coupling, and its leading singularities in the Borel plane are shown to be identical to those of the standard "contour-improved" expansion. Applying the technique of conformal mappings for the analytic continuation in the Borel plane, we define a class of improved expansions, which implement both the renormalization-group invariance and the knowledge about the large-order behavior of the series. Detailed numerical studies of specific models for the Adler function indicate that the new expansions have remarkable convergence properties up to high orders. Using these expansions for the determination of the strong coupling from the hadronic width of the τ lepton we obtain, with a conservative estimate of the uncertainty due to the nonperturbative corrections, $\alpha_s(M_\tau^2) = 0.3189^{+0.0173}_{-0.0151}$, which translates to $\alpha_s(M_Z^2) = 0.1184^{+0.0021}_{-0.0018}$.

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I. INTRODUCTION

The determination of the QCD coupling constant to increasing precision is one of the most important goals of the Standard Model of particle physics (for a review see Ref. [1]). The nonstrange hadronic decays of the τ lepton are an important source of information on this quantity and have been exploited now for a couple of decades. The recent calculation of the Adler function to four loops in massless QCD [2] renewed the interest in the extraction of the strong coupling α_s at the scale of the τ mass from the treatment of these processes [3–14]. In this context, several modifications of the perturbative expansion of the relevant observables have been proposed. The main ambiguities affecting perturbation theory are related to the implementation of renormalization-group invariance and to the large-order behavior of the series. The differences between the specific ways of accounting for these properties are the main source of theoretical error on the extraction of $\alpha_s(M_\tau^2)$.

In a recent work, Ref. [14], we applied to the analysis of the hadronic τ decay width a method of improving the perturbative expansions in QCD by summing the leading logarithms accessible from renormalization-group invariance, proposed in Refs. [15,16] and developed in Refs. [17,18]. The properties of the new expansion, which has been referred to as "improved fixed-order perturbation theory," in the complex energy plane were investigated and were compared with those of the "contour-improved perturbation theory" (CIPT), and the standard "fixed-order perturbation theory" (FOPT). The new expansion has the advantage of being written in an analytically closed form, while CIPT, the alternative approach of implementing renormalization-group (RG) invariance, requires the numerical solution of the renormalization-group equation for the strong coupling.

It is known that the perturbative expansions of QCD correlators are divergent in many physically interesting situations, with the coefficients growing as $n!$ at large orders n [19–22]. Alternatively, the divergent character of the series is inferred from the fact that the expanded correlators, like the Adler function, are singular at the origin of the coupling constant plane [23]. These problems are present also in perturbative QED [24], whose phenomenological success is explained by the fact that the fine structure constant is numerically very small. By contrast, for a relatively large coupling like $\alpha_s(M_\tau^2)$ in QCD the consequences are nontrivial.

Special mathematical techniques for divergent series are available, like Borel summation, which under certain conditions recovers the expanded function from its increasing expansion coefficients. For QCD the problem was studied for many years, and it is known that the conditions of Borel summability are not satisfied [19,23,25] (see also the reviews Ref. [26]). In particular, the Borel transform of the Adler function has singularities in the Borel plane, known as ultraviolet (UV) and infrared (IR) renormalons, the latter producing an ambiguity in the reconstruction of the function. However, if one adopts a certain prescription (e.g., the principal value prescription), it is possible to exploit the available knowledge on the large-order behavior of the coefficients for defining a new expansion, in which the divergent pattern is considerably tamed. Such an approach was proposed in Refs. [27–29] and developed recently in Refs. [6,8,11], using techniques of series acceleration based on conformal mappings and "singularity softening" (these techniques were also applied by other authors; for a discussion and references see Ref. [11]). We recall that the method of conformal mapping was introduced and applied in particle physics in Ref. [30] for extending the convergence region of an expansion beyond the circle of convergence and for increasing the convergence rate at points lying inside the circle. As discussed in Ref. [11], the method is not applicable to the (formal) perturbative series in powers of α_s because the expanded correlators are singular at the point of expansion, $\alpha_s = 0$, but can be applied in the Borel plane. It leads to a modified perturbative expansion in terms of a new set of functions, which have the advantage of resembling the expanded correlator (the Adler function), in particular by sharing its known singularities in the coupling and the Borel complex planes.

In Refs. [6,8,11] the method was applied to the two standard versions of perturbation theory, CIPT and FOPT. As argued in Ref. [11], the new expansions are particularly suitable in the contour-improved version, since they make simultaneously the RG summation and the Borel large-order summation of the Adler function. Detailed numerical studies [6,11] established the good convergence properties of the latter expansions for several exact models which simulate the known properties of the Adler function.

In this work, we consider the large-order properties of the expansion discussed in Ref. [14], which we shall henceforth refer to as "renormalization-group-summed" (RGS) expansion. We investigate the properties of this scheme in the Borel plane and, using the techniques discussed in Refs. [6,8,11], we define a new class of expansions, which simultaneously implement the renormalization-group and the large-order summation by the analytic continuation in the Borel plane. We shall refer to these as "Borel and renormalization-group-summed" (BRGS) expansions.

The plan of the paper is as follows: We briefly review in Sec. II the perturbative expansion of the Adler function and its connection to the hadronic decay width of τ lepton. In Sec. III we review the derivation of the RGS expansion following Ref. [14] and show further that it can be expressed as an expansion in powers of the one-loop solution of the RG equation for the coupling, which is needed for rendering this expansion suitable for convergence acceleration. In Sec. IV we discuss the properties of the expansion in the Borel plane and show that it has the same dominant singularities as the CI expansion. In Sec. V we define a set of new Borel and RG-improved expansions, by using the technique of singularity softening and conformal mappings of the Borel plane [11]. In Sec. VI we investigate the properties of the new expansions in the complex energy plane and illustrate their convergence for the physical observable relevant for the hadronic width of the τ lepton, using a class of models for the Adler function considered in Refs. [5,6,11,31,32]. In Sec. VII we report a new determination of $\alpha_s(M_\tau^2)$ based on the new BRGS expansions and in Sec. VIII we summarize our results and conclusions.

II. ADLER FUNCTION AND THE HADRONIC τ-DECAY WIDTH

The Adler function plays a crucial role in the determination of $\alpha_s(M_\tau^2)$ from hadronic τ decays. The method is discussed in the seminal paper [33] and is reviewed in several recent articles [3,5,7,9].

The inclusive character of the total τ hadronic width makes possible an accurate calculation of the ratio

$$R_\tau \equiv \frac{\Gamma[\tau^- \to \nu_\tau \,\text{hadrons}]}{\Gamma[\tau^- \to \nu_\tau e^- \bar{\nu}_e]}. \tag{1}$$

Of interest is the Cabibbo allowed component $R_{\tau,V/A}$ proceeding either through a vector or an axial vector current, which can be expressed theoretically in the form

$$R_{\tau,V/A} = \frac{N_c}{2} S_{\mathrm{EW}} |V_{ud}|^2 \left[1 + \delta^{(0)} + \delta'_{\mathrm{EW}} + \sum_{D\geq 2} \delta_{ud}^{(D)} \right], \tag{2}$$

where $N_c = 3$ is the number of quark colours, S_{EW} and δ'_{EW} are electroweak corrections, $\delta^{(0)}$ is the dominant perturbative QCD correction, and $\delta_{ud}^{(D)}$ denote quark mass and higher D-dimensional operator corrections (condensate contributions) arising in the operator product expansion (OPE). The decay width is suitable for the precise extraction of the strong coupling, since the (less-known) higher terms in the OPE bring a very small contribution to (2). Therefore, a fairly accurate phenomenological determination of the QCD perturbative part $\delta^{(0)}$ is possible [3,5,9,10].

The theoretical calculation of $\delta^{(0)}$ is based on unitarity, which implies that the inclusive hadronic decay rate can be written as a weighted integral along the timelike axis of the spectral function of a polarization function. As shown in Ref. [33], the analytic properties of the polarization function and the Cauchy theorem allow one to write equivalently this quantity as an integral along a contour in the complex s plane (chosen for convenience to be the circle $|s| = M_\tau^2$). After an integration by parts, the quantity of interest $\delta^{(0)}$ is expressed as the contour integral:

$$\delta^{(0)} = \frac{1}{2\pi i} \oint_{|s|=M_\tau^2} \frac{ds}{s} \left(1 - \frac{s}{M_\tau^2}\right)^3 \left(1 + \frac{s}{M_\tau^2}\right) \hat{D}_{\mathrm{pert}}(s), \tag{3}$$

where the reduced Adler function $\hat{D}(s) \equiv D^{(1+0)}(s) - 1$ is obtained by subtracting the dominant term from the logarithmic derivative of the polarization function, $D^{(1+0)}(s) \equiv -s\mathrm{d}\Pi^{(1+0)}(s)/\mathrm{d}s$, where the superscript denotes the spin [33]. The perturbative expansion of $\hat{D}(s)$ reads [5]

$$\hat{D}_{\rm pert}(s) = \sum_{n=1}^{\infty}(a_s(\mu^2))^n \sum_{k=1}^{n} k c_{n,k}(\ln(-s/\mu^2))^{k-1}, \quad (4)$$

where $a_s(\mu^2) \equiv \alpha_s(\mu^2)/\pi$ is the strong coupling at the renormalization scale μ. The leading coefficients $c_{n,1}$ are obtained from Feynman diagrams, the known coefficients $c_{n,1}$ calculated to four loops in the $\overline{\rm MS}$-renormalization scheme being (see Ref. [2] and references therein)

$$c_{1,1} = 1, \qquad c_{2,1} = 1.640, \qquad c_{3,1} = 6.371, \qquad c_{4,1} = 49.076. \quad (5)$$

Several estimates for the next coefficient $c_{5,1}$ were made recently [3,5,9,10].

The remaining coefficients $c_{n,k}$ for $k > 1$ are determined from renormalization-group invariance: the function $\hat{D}_{\rm pert}$, calculated in a fixed renormalization scheme, is scale independent and therefore satisfies the equation

$$\mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2}[\hat{D}_{\rm pert}(s)] = 0, \quad (6)$$

which can be written equivalently as

$$\beta(a_s)\frac{\partial \hat{D}_{\rm pert}}{\partial a_s} - \frac{\partial \hat{D}_{\rm pert}}{\partial \ln(-s/\mu^2)} = 0, \quad (7)$$

where

$$\beta(a_s) \equiv \mu^2 \frac{\mathrm{d}a_s(\mu^2)}{\mathrm{d}\mu^2} = -(a_s(\mu^2))^2 \sum_{k=0}^{\infty} \beta_k (a_s(\mu^2))^k \quad (8)$$

is the β function governing the scale dependence of the coupling. From (7) one can express $c_{n,k}$ for $k > 1$ in terms of $c_{n,1}$ and the coefficients β_j of the perturbation expansion (8). The known β_j coefficients, calculated to four loops in the $\overline{\rm MS}$ scheme, are (see Refs. [34,35] for the calculation of β_3 and earlier references)

$$\beta_0 = 9/4, \qquad \beta_1 = 4, \qquad \beta_2 = 10.0599, \qquad \beta_3 = 47.228. \quad (9)$$

In the fixed-order perturbation theory calculation of $\delta^{(0)}$, the choice $\mu^2 = M_\tau^2$ is adopted, when the expansion (4) reads

$$\hat{D}_{\rm FOPT}(s) = \sum_{n=1}^{\infty}(a_s(M_\tau^2))^n \times \left[c_{n,1} + \sum_{k=2}^{n} k c_{n,k}(\ln(-s/M_\tau^2))^{k-1}\right]. \quad (10)$$

As remarked in Ref. [36], due to the large imaginary part of the logarithm $\ln(-s/M_\tau^2)$ along the circle $|s| = M_\tau^2$, the series (10) is badly behaved, especially near the timelike axis. The CIPT [37,38] is defined by the choice $\mu^2 = -s$, when (4) reduces to

$$\hat{D}_{\rm CIPT}(s) = \sum_{n=1}^{\infty} c_{n,1}(a_s(-s))^n, \quad (11)$$

where the running coupling $a_s(-s)$ is determined by solving the renormalization-group equation (8) numerically in an iterative way along the circle, starting with the input value $a_s(M_\tau^2)$ at $s = -M_\tau^2$. This expansion avoids the appearance of large logarithms along the circle $|s| = M_\tau^2$.

The expansions (10) and (11) coincide formally as long as all the terms in the series are retained. In fact, the coefficients $c_{n,1}$ are known to increase as $n!$, so the series are divergent. We shall turn to this property in Sec. IV. If the series are truncated at some order N, the expansions (10) and (11) differ by terms of order a_s^{N+1}, this being the main theoretical error in the the determination of $\alpha_s(M_\tau^2)$ from the measured τ hadronic width.

III. RENORMALIZATION-GROUP SUMMATION

As proposed in Refs. [17,18], the expansion (4) can be written in the RGS form

$$\hat{D}_{\rm RGS}(s) = \sum_{n=1}^{\infty}(a_s(\mu^2))^n D_n(y), \quad (12)$$

where the functions $D_n(y)$, depending on a single variable

$$y \equiv 1 + \beta_0 a_s(\mu^2)\ln(-s/\mu^2), \quad (13)$$

are defined as

$$D_n(y) \equiv \sum_{k=n}^{\infty}(k-n+1)c_{k,k-n+1}\left(\frac{y-1}{\beta_0}\right)^{k-n}. \quad (14)$$

As seen from the definition, the function D_1 sums the leading logarithms in the series (4), the function D_2 sums the next-to-leading logarithms, and so on. The attractive feature pointed out in Refs. [17,18] is that these functions can be obtained in a closed analytical form. The derivation is based on renormalization-group invariance: after inserting the expansion (12) into the condition (6), a straightforward calculation leads to the following system of differential equations for $D_n(y)$, for $n \geq 1$:

$$\beta_0 \frac{\mathrm{d}D_n(y)}{\mathrm{d}y} + \sum_{\ell=0}^{n-1}\beta_\ell\left((y-1)\frac{\mathrm{d}}{\mathrm{d}y} + n - \ell\right)D_{n-\ell}(y) = 0, \quad (15)$$

with the initial conditions $D_n(1) = c_{n,1}$ which follow from (14).

The solution of the system (15) can be found iteratively in an analytical form. The expressions of $D_n(y)$ for $n \leq 5$, written in terms of the coefficients $c_{k,1}$ with $k \leq n$ and β_j

with $0 \le j \le n-1$, are

$$D_1(y) = \frac{c_{1,1}}{y}, \tag{16}$$

$$D_2(u) = \frac{1}{y^2}(c_{2,1} + c_{1,1}d_{2,1}), \qquad d_{2,1} = -\frac{\beta_1}{\beta_0}\ln y, \tag{17}$$

$$D_3(y) = \frac{1}{y^3}(c_{3,1} + c_{2,1}d_{3,2} + c_{1,1}d_{3,1}), \tag{18}$$

$$\begin{aligned} d_{3,2} &= -\frac{2\beta_1}{\beta_0}\ln y, \\ d_{3,1} &= -\frac{\beta_1^2}{\beta_0^2}(1 - y + \ln y - \ln^2 y) + \frac{\beta_2}{\beta_0}(1-y), \end{aligned} \tag{19}$$

$$D_4(u) = \frac{1}{y^4}(c_{4,1} + c_{3,1}d_{4,3} + c_{2,1}d_{4,2} + c_{1,1}d_{4,1}), \tag{20}$$

$$\begin{aligned} d_{4,3} &= -3\frac{\beta_1}{\beta_0}\ln y, \\ d_{4,2} &= -2\frac{\beta_2}{\beta_0}(-1+y) + \frac{\beta_1^2}{\beta_0^2}(-2 - 2\ln y + 3\ln^2 y + 2y), \\ d_{4,1} &= -\frac{\beta_1^3}{2\beta_0^3}(-5\ln^2 y + 2\ln^3 y + 4\ln y(-1+y) \\ &\quad + (-1+y)^2) - \frac{\beta_1\beta_2}{\beta_0^2}(3\ln y + y - 2y\ln y - y^2) \\ &\quad - \frac{\beta_3}{2\beta_0}(-1+y^2), \end{aligned} \tag{21}$$

$$D_5(y) = \frac{1}{y^5}(c_{5,1} + c_{4,1}d_{5,4} + c_{3,1}d_{5,3} + c_{2,1}d_{5,2} + c_{1,1}d_{5,1}), \tag{22}$$

$$\begin{aligned} d_{5,4} &= -4\frac{\beta_1}{\beta_0}\ln y, \qquad d_{5,3} = -3\left[\frac{\beta_1^2}{\beta_0^2}(-2\ln^2 y + \ln y - y + 1) + \frac{\beta_2}{\beta_0}(y-1)\right], \\ d_{5,2} &= -\left[\frac{\beta_1^3}{\beta_0^3}(4\ln^3 y - 7\ln^2 y + 6(y-1)\ln y + (y-1)^2) + 2\frac{\beta_2\beta_1}{\beta_0^2}(-y^2 - 3y\ln y + y + 4\ln y) + \frac{\beta_3}{\beta_0}(y^2-1)\right], \\ d_{5,1} &= \frac{\beta_1^4}{6\beta_0^4}(6\ln^4 y - 26\ln^3 y + 9(2y-1)\ln^2 y + 6(y^2 - 5y + 4)\ln y + (y-1)^2(2y+7)) \\ &\quad - \frac{\beta_1^2\beta_2}{\beta_0^3}(3(y-2)\ln^2 y + (2y^2 - 5y + 3)\ln y + (y-1)^2(y+3)) + \frac{\beta_1\beta_3}{6\beta_0^2}(4y^3 - 3y^2 + 6\ln y(y^2-2) - 1) \\ &\quad + \frac{\beta_2^2}{3\beta_0^2}(y-1)^2(y+5) - \frac{\beta_4}{3\beta_0}(y^3-1). \end{aligned} \tag{23}$$

The expressions of $D_n(y)$ for $6 \le n \le 10$, which depend also on the coefficients $c_{n,1}$ for $6 \le n \le 10$ and β_j for $5 \le j \le 9$, are given in a somewhat different form[1] in the Appendix of Ref. [14]. In the numerical applications presented in Sec. VI, we shall use the expressions of $D_n(y)$ up to $n = 18$, which can be obtained easily by solving the system (15) with a MATHEMATICA program.

Using Eqs. (16)–(23) and the expressions of higher $D_n(y)$ derived analytically, we note that these functions can be written as

$$D_n(y) = \frac{1}{y^n}\left[c_{n,1} + \sum_{j=1}^{n-1} c_{j,1}d_{n,j}(y)\right], \tag{24}$$

where the coefficients $d_{n,j}$ are functions of y, which depend also on the coefficients β_j of the β function. One can check that $d_{n,j}(y)$ vanish identically for $y = 1$ or if $\beta_j = 0$ for $j \ge 1$.

By inserting (24) into (12), we note that the denominators y^n can be combined in each term with the powers of $a_s(\mu^2)$, so that (12) can be written as

$$\hat{D}_{\rm RGS}(s) = \sum_{n=1}^{\infty}(\tilde{a}_s(-s))^n\left[c_{n,1} + \sum_{j=1}^{n-1} c_{j,1}d_{n,j}(y)\right], \tag{25}$$

where

$$\tilde{a}_s(-s) = \frac{a_s(\mu^2)}{1 + \beta_0 a_s(\mu^2)\ln(-s/\mu^2)} \tag{26}$$

is the solution of the RG equation (8) to one loop at the scale $-s$, written in terms of $a_s(\mu^2) \equiv \alpha_s(\mu^2)/\pi$. The terms $c_{n,1}$ in the series (25) yield the all-order summation of the one-loop coupling (26), while the remaining sums yield the corrections accounting for the higher-order terms in the expansion of the β function.

IV. BOREL TRANSFORM

In this section we discuss the properties of the expansion (25) in the Borel plane. We start by recalling the

[1]For simplicity, in Ref. [14] we presented the expressions obtained by inserting the known numerical values of β_j for $j \le 3$ from (9) and setting $\beta_j = 0$ for $j \ge 4$.

standard definition [5] of the Borel transform $B(u)$ of the expansion (11):

$$B(u) = \sum_{n=0}^{\infty} c_{n+1,1} \frac{u^n}{\beta_0^n n!}. \tag{27}$$

The original function $\hat{D}_{\mathrm{CIPT}}(s)$ is recovered from $B(u)$ by a Laplace-Borel integral. Actually, the function $B(u)$ has singularities on the positive axis of the u plane, so the Laplace-Borel integral requires a regularization. Following Refs. [5,6,11] we shall use the principal value (PV) prescription. Note that as argued in Refs. [39,40], the PV prescription preserves the reality of the correlators in the s plane and is therefore more consistent than other prescriptions with the analyticity properties imposed by causality and unitarity. Thus, we write

$$\hat{D}_{\mathrm{CIPT}}(s) = \frac{1}{\beta_0} \mathrm{PV} \int_0^{\infty} \exp\left(\frac{-u}{\beta_0 a_s(-s)}\right) B(u) \mathrm{d}u. \tag{28}$$

Similarly, we can define the Borel transform $B_{\mathrm{FO}}(u,s)$ of the FOPT expansion (10). The structure of the coefficients of this expansion implies that we can write $B_{\mathrm{FO}}(u,s)$ as

$$B_{\mathrm{FO}}(u,s) = B(u) + \sum_{n=0}^{\infty} \frac{u^n}{\beta_0^n n!} \sum_{k=2}^{n+1} k c_{n+1,k} \left(\ln\frac{-s}{M_\tau^2}\right)^{k-1}. \tag{29}$$

The function $\hat{D}_{\mathrm{FOPT}}(s)$ is obtained from its Borel transform by

$$\hat{D}_{\mathrm{FOPT}}(s) = \frac{1}{\beta_0} \mathrm{PV} \int_0^{\infty} \exp\left(\frac{-u}{\beta_0 a_s(M_\tau^2)}\right) B_{\mathrm{FO}}(u,s) \mathrm{d}u. \tag{30}$$

We introduce now the Borel transform $B_{\mathrm{RGS}}(u,y)$ of the expansion (25), which can be written as

$$B_{\mathrm{RGS}}(u,y) = B(u) + \sum_{n=0}^{\infty} \frac{u^n}{\beta_0^n n!} \sum_{j=1}^{n} c_{j,1} d_{n+1,j}(y). \tag{31}$$

The function $\hat{D}_{\mathrm{RGS}}(s)$ is recovered by the similar Laplace-Borel integral

$$\hat{D}_{\mathrm{RGS}}(s) = \frac{1}{\beta_0} \mathrm{PV} \int_0^{\infty} \exp\left(\frac{-u}{\beta_0 \tilde{a}_s(-s)}\right) B_{\mathrm{RGS}}(u,y) \mathrm{d}u, \tag{32}$$

written in terms of the one-loop coupling (26).

As we already mentioned, the function $B(u)$ defined in (27) has singularities on the real axis in the u plane, namely along the rays $u \geq 2$ and $u \leq -1$ [19,25]. Moreover, the nature of the dominant singularities can be described exactly: they are branch points, near which $B(u)$ behaves, respectively, as

$$B(u) \sim (1+u)^{-\gamma_1}, \qquad B(u) \sim (1-u/2)^{-\gamma_2}, \tag{33}$$

where the exponents γ_1 and γ_2, calculated using renormalization-group invariance, have known positive values [5,19,41]:

$$\gamma_1 = 1.21, \qquad \gamma_2 = 2.58. \tag{34}$$

From (29) and (31) it follows that these singularities are present also in the Borel transforms $B_{\mathrm{FO}}(u,s)$ and $B_{\mathrm{RGS}}(u,y)$. In principle, these functions might have also other singularities, due to the additional infinite sums appearing in (29) and (31), respectively. However, as we shall argue below, the dominant singularities of these functions, i.e., the singularities closest to the origin $u=0$, are those at $u=-1$ and $u=2$ contained in $B(u)$.

We first present some evidence which results from the inspection of the next-to-leading terms in the expression (31) of $B_{\mathrm{RGS}}(u,y)$. Thus, from the expressions (16)–(23) we note that

$$d_{n,n-1}(y) = -(n-1)\frac{\beta_1}{\beta_0} \ln y. \tag{35}$$

By inserting this expression in (31), we obtain by a straightforward calculation the contribution to $B_{\mathrm{RGS}}(u,y)$ of the term with $j=n$:

$$B_{\mathrm{RGS}}(u,y)|_{j=n} = -\frac{\beta_1}{\beta_0^2} u B(u) \ln y. \tag{36}$$

Similarly, we note that

$$d_{n,n-2}(y) = -(n-2)\xi(y) + \frac{(n-1)(n-2)}{2}\frac{\beta_1^2}{\beta_0^2}\ln^2 y, \tag{37}$$

where

$$\xi(y) = \frac{\beta_1^2}{\beta_0^2}\ln y + \frac{\beta_1^2 - \beta_0\beta_2}{\beta_0^2}(1-y). \tag{38}$$

Then we obtain by a straightforward calculation the contribution of the term with $j=n-1$ in (31) as

$$B_{\mathrm{RGS}}(u,y)|_{j=n-1} = -\frac{\xi(y)}{\beta_0^2}\int_0^u u' B(u')\mathrm{d}u' + \frac{\beta_1^2}{2\beta_0^4}\ln^2 y\, u^2 B(u), \tag{39}$$

where the integral is defined along a contour from the origin to the point u, which does not reach the singularities of $B(u)$.

The next coefficients in the second term of (31) exhibit a similar pattern: $d_{n,n-l}$ for $1<l<n-1$ contains polynomials of the variables $(1-y)$ and $\ln y$, of degree $(l-1)$ and l, respectively, with coefficients depending on β_j, n and l. For instance, the term proportional to $\ln^l y$ in $d_{n,n-l}$ has the expression

$$d_{n,n-l} \sim \frac{(-1)^l}{l!}\prod_{k=1}^{l}(n-k)\frac{\beta_1^l}{\beta_0^l}\ln^l y, \tag{40}$$

bringing a contribution to (31) of the form

$$B_{\rm RGS}(u,y)|_{j=n+1-l} \sim \frac{(-1)^l}{l!}\frac{\beta_1^l}{\beta_0^{2l}} y u^l B(u) \ln^l y. \qquad (41)$$

This reproduces (36) and the second term in (39) for $l=1$ and $l=2$, respectively.

The other terms appearing in $d_{n,n-l}$ may contribute also with integrals of $B(u)$ multiplied by powers of u, as in (39). Thus, in general, the second term in the expression (31) of $B_{\rm BRG}(u,y)$ is expected to contain either $B(u)$, multiplied by factors which vanish in the limit $y\longrightarrow 1$, or integrals of $B(u)$, in which the singularities have the same positions as in (33) but are milder. Therefore, the dominant singularities of the Borel transform (31) coincide with the dominant singularities of the Borel transform $B(u)$ defined in (27).

One may invoke also the general argument that Mueller [19] used for concluding that the dominant singularities of the Borel transform $B_{\rm FO}(u,s)$ defined in (29) are the same as those of $B(u)$. The crucial observation is that the positions of the dominant singularities of the Borel transform are determined from the behavior of the correlators in the limit of a small coupling.[2] Since in this limit the running coupling $a_s(-s)$ entering (28), the fixed scale coupling $a_s(M_\tau^2)$ entering (30), and the one-loop coupling $\tilde{a}_s(-s)$ entering (32) are close to each other, it follows that the positions of the dominant singularities in the u plane of the corresponding Borel transforms, $B(u)$, $B_{\rm FO}(u,s)$ and $B_{\rm RGS}(u,y)$, must be the same.

V. ANALYTIC CONTINUATION IN THE BOREL PLANE AND NEW PERTURBATIVE RGS EXPANSIONS

As discussed above, the RGS expansion (25) is divergent, the coefficients $c_{n,1}$ increasing as $n!$ at large n. In fact, as we shall show in the next section, the divergence is quite bad for expanded functions supposed to resemble the physical Adler function. A procedure to tame this divergent behavior is therefore mandatory. In this section we shall improve the large-order behavior of the RGS expansion by applying a method based on analytic continuation in the Borel plane, applied to the standard CIPT and FOPT in Refs. [6,11].

The starting remark is that the Taylor expansion (27) of $B(u)$ is convergent only inside the disk $|u|<1$, limited by the nearest singularity at $u=-1$. The region of convergence can be enlarged if the series in powers of u is replaced by a series in powers of another variable. As shown in Refs. [6,11,27], the "optimal" variable according to the definition proposed in Ref. [30] is the function $w\equiv\tilde{w}(u)$ that conformally maps the assumed holomorphy domain of $B(u)$, i.e., the whole u-plane cut along $u\ge 2$ and $u\le -1$, onto the unit disk $|w|<1$ in the w complex plane. The expansion of $B(u)$ in powers of w is convergent in the whole complex u plane except for the cuts. Moreover, this optimal mapping provides the fastest large-order convergence rate, compared to other variables that conformally map onto the unit disk only parts of the u plane [30]. The detailed proof of these statements is given in Ref. [11].

An additional improvement, discussed in detail in Ref. [11], is obtained by exploiting the known behavior at the first singularities, presented in (33) and (34). The main idea of the procedure, denoted as singularity softening [43], is to multiply $B(u)$ with a suitable factor $S(u)$, such that in the product $S(u)B(u)$ the dominant singularities are compensated or are replaced by milder singularities. Moreover, after compensating the leading singularities, one can expand the resulting function in powers of variables that take into account only the higher, i.e., more distant, renormalons.

As shown in the previous section, the Borel transform $B_{\rm RGS}(u,y)$ of the RGS expansion (25) has the same dominant singularities as $B(u)$. Therefore we can apply the techniques of improving the convergence mentioned above. Following Ref. [11], we consider the functions

$$\tilde{w}_{lm}(u) = \frac{\sqrt{1+u/l}-\sqrt{1-u/m}}{\sqrt{1+u/l}+\sqrt{1-u/m}}, \qquad (42)$$

where l, m are positive integers satisfying $l\ge 1$ and $m\ge 2$. The function $\tilde{w}_{lm}(u)$ maps the u-plane cut along $u\le -l$ and $u\ge m$ onto the disk $|w_{lm}|<1$ in the plane $w_{lm}\equiv\tilde{w}_{lm}(u)$. The optimal mapping defined above is $\tilde{w}(u)\equiv\tilde{w}_{12}(u)$, for which the entire holomorphy domain of the Borel transform is mapped onto the interior of the unit circle in the plane w_{12}.

We define further the class of compensating factors of the simple form [11]

$$S_{lm}(u) = \left(1-\frac{\tilde{w}_{lm}(u)}{\tilde{w}_{lm}(-1)}\right)^{\gamma_1^{(l)}}\left(1-\frac{\tilde{w}_{lm}(u)}{\tilde{w}_{lm}(2)}\right)^{\gamma_2^{(m)}}, \qquad (43)$$

where the exponents, written in terms of the powers γ_l defined in (33) and (34) and the Kronecker symbol δ_{lm}, as

$$\gamma_1^{(l)} = \gamma_1(1+\delta_{l1}), \qquad \gamma_2^{(m)} = \gamma_2(1+\delta_{m2}), \qquad (44)$$

are chosen such that $S_{lm}(u)$ cancel the dominant singularities defined in (33). Following Ref. [11], we further expand the product $S_{lm}(u)B_{\rm RGS}(u,y)$ in powers of the variable $\tilde{w}_{lm}(u)$, as

$$S_{lm}(u)B_{\rm RGS}(u,y) = \sum_{n\ge 0} c_{n,{\rm RGS}}^{(lm)}(y)(\tilde{w}_{lm}(u))^n. \qquad (45)$$

For the optimal mapping $\tilde{w}_{12}$ this expansion converges in the whole disk $|w_{12}|<1$, i.e., in the whole u-plane cut

[2] The connection between the behavior of the Borel-summed correlators in the a_s plane and the position and nature of the dominant singularities of the Borel transform in the u plane is given explicitly in the case of a finite number of renormalons in Ref. [42].

along $u \geq 2$ and $u \leq -1$, and has the best asymptotic convergence rate [11,30]. Moreover, since the first singularities of the Borel transform are compensated by the softening factor, a good convergence is expected also at finite orders. For other mappings, with either $l > 1$ or $m > 2$, the expansions would converge in the unit disks $|w_{lm}| < 1$ if the singularities situated between $-l \leq u \leq -1$ and $2 \leq u \leq m$ were completely removed by the compensating factors. In practice, however, only the first singularities at $u = -1$ and $u = 2$ are compensated by the factors $S_{lm}(u)$, and after the compensation they may survive as "mild" branch points, where the function vanishes instead of becoming infinite. The presence of the residual cuts sets a limit on the convergence domain of the expansions (45), but for a mild singularity the effect is expected to become important only at large orders [11].

While the optimal mapping is based on a mathematical theorem [11,30], there is no such rigorous result for the form of the softening factors. They are arbitrary to a large extent and can be chosen empirically. With the choice (43), the compensating factors used in different expansions are parametrized in different way. By this we reduce the bias related to the choice of these factors.

By combining the expansion (45) with the definition (32), we are led to the class of BRGS expansions

$$\hat{D}_{\rm BRGS}(s) = \sum_{n\geq 0} c_{n,\rm RGS}^{(lm)}(y)\, \mathcal{W}_{n,\rm RGS}^{(lm)}(s), \qquad (46)$$

where

$$\mathcal{W}_{n,\rm RGS}^{(lm)}(s) = \frac{1}{\beta_0}\, {\rm PV}\int_0^\infty \exp\left(\frac{-u}{\beta_0 \tilde{a}_s(-s)}\right) \frac{(\tilde{w}_{lm}(u))^n}{S_{lm}(u)}\, {\rm d}u, \qquad (47)$$

and the coefficients $c_{n,\rm RGS}^{(lm)}(y)$ are defined by the expansion (45).

For completeness we write below the similar "Borel and contour-improved" (BCI) expansions [6,11]

$$\hat{D}_{\rm BCI}(s) = \sum_n c_{n,\rm CI}^{(lm)}\, \mathcal{W}_{n,\rm CI}^{(lm)}(s), \qquad (48)$$

where the expansion functions are expressed in terms of the running coupling $a_s(s)$:

$$\mathcal{W}_{n,\rm CI}^{(lm)}(s) = \frac{1}{\beta_0}\, {\rm PV}\int_0^\infty {\rm e}^{-u/(\beta_0 a_s(s))} \frac{(\tilde{w}_{lm}(u))^n}{S_{lm}(u)}\, {\rm d}u, \qquad (49)$$

and the coefficients $c_{n,\rm CI}^{(lm)}$ are defined by the expansion

$$S_{lm}(u) B(u) = \sum_{n\geq 0} c_{n,\rm CI}^{(lm)} (\tilde{w}_{lm}(u))^n. \qquad (50)$$

Finally, the "Borel improved fixed-order" (BFO) expansions are written as [6,11]

$$\hat{D}_{\rm BFO}(s) = \sum_n c_{n,\rm FO}^{(lm)}(s)\, \mathcal{W}_{n,\rm FO}^{(lm)}, \qquad (51)$$

where the expansion functions involve the fixed-scale coupling $a_s(M_\tau^2)$:

$$\mathcal{W}_{n,\rm FO}^{(lm)} = \frac{1}{\beta_0}\, {\rm PV}\int_0^\infty {\rm e}^{-u/(\beta_0 a_s(M_\tau^2))} \frac{(\tilde{w}_{lm}(u))^n}{S_{lm}(u)}\, {\rm d}u, \qquad (52)$$

and the coefficients $c_{n,\rm FO}^{(lm)}(s)$ are defined by

$$S_{lm}(u) B_{\rm FO}(u, s) = \sum_{n\geq 0} c_{n,\rm FO}^{(lm)}(s) (\tilde{w}_{lm}(u))^n. \qquad (53)$$

The properties of the new expansions were discussed in detail in Refs. [28,29] in the particular case of the optimal mapping w_{12}. Their definition is based on a prescription for the infrared ambiguity of perturbation theory which is consistent with analyticity in the energy plane. When reexpanded in powers of a_s, the expansions reproduce the coefficients $c_{n,1}$ known from Feynman diagrams, up to any order N. The remarkable feature is that the expansion functions $\mathcal{W}_n^{(lm)}$ resemble the expanded function $\hat{D}(s)$, being singular at $a_s = 0$ and admitting divergent expansions in powers of the coupling. Therefore, the divergent pattern of the expansion of $\hat{D}(s)$ in terms of these new functions is expected to be tamed. The numerical studies reported in the next section confirm this expectation.[3] As in Ref. [11], we shall use in these analyses the expansions based on the optimal mappings w_{12} and the alternative mappings w_{13}, $w_{1\infty}$ and w_{23}.

VI. MODELS FOR THE ADLER FUNCTIONS

In order to test numerically the convergence properties of the BRGS expansions defined in the previous section, we consider a class of physical models discussed recently in the literature [5,6,11,31,32].

We first consider the model proposed in Ref. [5], where the Adler function $\hat{D}_{BJ}(s)$ is defined as the PV-regulated Laplace-Borel integral

$$\hat{D}_{BJ}(s) = \frac{1}{\beta_0}\, {\rm PV}\int_0^\infty \exp\left(\frac{-u}{\beta_0 a_s(-s)}\right) B_{BJ}(u) {\rm d}u, \qquad (54)$$

in terms of a Borel transform $B_{BJ}(u)$ parametrized in terms of a few UV and IR renormalons and a regular, polynomial part:

$$\frac{B_{BJ}(u)}{\pi} = B_1^{\rm UV}(u) + B_2^{\rm IR}(u) + B_3^{\rm IR}(u) + d_0^{\rm PO} + d_1^{\rm PO} u. \qquad (55)$$

In the expressions of the renormalons

[3] A formal proof for the optimal mapping w_{12} was given in Ref. [28], where it was shown that under some special conditions the expansion (48) is convergent in a domain of the complex s plane.

$$B_p^{\mathrm{IR}}(u) = \frac{d_p^{\mathrm{IR}}}{(p-u)^{\gamma_p}}\left[1 + \tilde{b}_1(p-u) + \cdots\right],$$
$$B_p^{\mathrm{UV}}(u) = \frac{d_p^{\mathrm{UV}}}{(p+u)^{\bar{\gamma}_p}}\left[1 + \bar{b}_1(p+u) + \cdots\right], \tag{56}$$

most of the parameters were fixed by imposing renormalization-group invariance at four loops. Finally, the free parameters of the model, namely the residues d_1^{UV}, d_2^{IR} and d_3^{IR} of the first renormalons and the coefficients d_0^{PO}, d_1^{PO} of the polynomial in (55), were determined by the requirement of reproducing the perturbative coefficients $c_{n,1}$ for $n \leq 4$ from (5) and the estimate $c_{5,1} = 283$. Their numerical values are [5]

$$d_1^{\mathrm{UV}} = -1.56 \times 10^{-2}, \qquad d_2^{\mathrm{IR}} = 3.16, \qquad d_3^{\mathrm{IR}} = -13.5, \tag{57}$$

$$d_0^{\mathrm{PO}} = 0.781, \qquad d_1^{\mathrm{PO}} = 7.66 \times 10^{-3}. \tag{58}$$

Then all the higher-order coefficients $c_{n,1}$ are fixed and exhibit a factorial increase. The numerical values up to $n = 18$ are listed in Refs. [5,6].

The magnitude of the residue d_2^{IR} of the first IR renormalon in the above model was questioned by some authors [7,31]. In order to avoid any bias, we have also investigated alternative models, in which a smaller residue at $u = 2$ was imposed (an extreme case of this type of alternative models, in which the singularity at $u = 2$ is completely removed, was investigated recently in Ref. [32]). In one example, we have retained the same expressions as in Ref. [5] for the first three singularities and the same values of the residues at $u = -1$ and $u = 3$, while choosing a smaller residue at $u = 2$, $d_2^{\mathrm{IR}} = 1$. In order to reproduce the first five coefficients $c_{n,1}$, the model must contain then three additional free parameters, which were introduced by a quadratic term in the polynomial and two additional IR singularities, at $u = 4$ and $u = 5$. For convenience, the nature of these additional singularities, which is not known, was taken to be the same as that of the $u = 3$ singularity. Thus, we have considered the alternative model:

$$\frac{B_{\mathrm{alt}}(u)}{\pi} = B_1^{\mathrm{UV}}(u) + B_2^{\mathrm{IR}}(u) + B_3^{\mathrm{IR}}(u) + \frac{d_4^{\mathrm{IR}}}{(4-u)^{3.37}} + \frac{d_5^{\mathrm{IR}}}{(5-u)^{3.37}} + d_0^{\mathrm{PO}} + d_1^{\mathrm{PO}} u + d_2^{\mathrm{PO}} u^2, \tag{59}$$

where, as discussed above, we have taken as input d^{UV}, d_2^{IR} and d_3^{IR} from (57) and determined the remaining five parameters by matching the coefficients $c_{n,1}$ for $n \leq 5$, which gives

$$d_0^{\mathrm{PO}} = 3.2461, \qquad d_1^{\mathrm{PO}} = 1.3680, \qquad d_2^{\mathrm{PO}} = 0.2785,$$
$$d_4^{\mathrm{IR}} = 560.614, \qquad d_5^{\mathrm{IR}} = -1985.73. \tag{60}$$

The physical plausibility of this type of models was discussed in several recent works [5,9,31,32]. In particular, arguments in favor of the first model presented above were brought in Refs. [5,32]. In the present work we adopted these models as a mathematical framework for testing the convergence properties of the various expansions.

We illustrate first the properties of the new expansions by the approximation they provide to the expanded function along the circle $s = M_\tau^2 \exp(i\theta)$. In Fig. 1, we show the real part of the Adler function for the model [5] defined in (55)–(58), calculated with the new BRGS expansions (46) and (47), with $N = 5$ terms in the perturbative series. As in Ref. [11], we considered the expansion functions for the optimal mapping w_{12} and the alternative mappings w_{13}, $w_{1\infty}$ and w_{23}. For the comparison with previous studies [5,6,11], we have used $\alpha_s(M_\tau^2) = 0.3156$ in this calculation. In Fig. 2, we show the same curves for the imaginary part of the Adler function. In Figs. 3 and 4, we repeat our calculations with $N = 18$ terms.

As shown in Figs. 1–4, the BRGS expansions provide a good description of the exact function along the whole circle, including the points near the timelike axis, which correspond to $\theta = 0$, and near the spacelike axis, where $\theta = \pi$. The worse approximation provided by the mapping w_{23} for $N = 18$ can be explained by the effect of the residual mild cut between $u = -1$ and $u = -2$, which limits the convergence radius of the expansion (45) in powers of w_{23} to $u < 1$. For other mappings, the divergence due to the residual cuts is manifest only for $u > 2$, and this region is more suppressed by the exponent in the Laplace-Borel integrals (47) defining the expansion functions (for more explanations see Ref. [11]). In all the cases, however, the conformal mappings improve the convergence rate for small values of u, which bring the most important contribution to the Laplace-Borel integral.

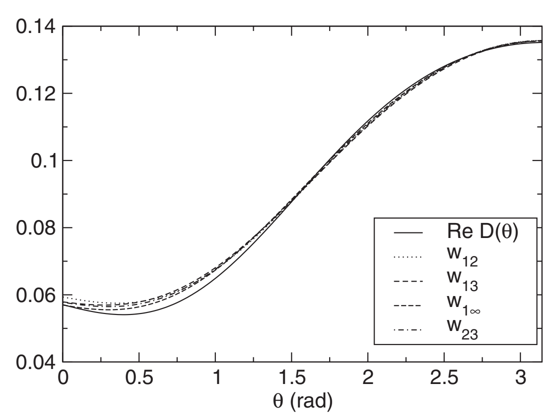

FIG. 1. Real part of the Adler function of the model [5] defined in (55)–(58), calculated along the circle $s = M_\tau^2 \exp(i\theta)$ for $\alpha_s(M_\tau^2) = 0.3156$, using the BRGS perturbative expansions (46) and (47) with $N = 5$ terms. The exact function is represented by the solid line.

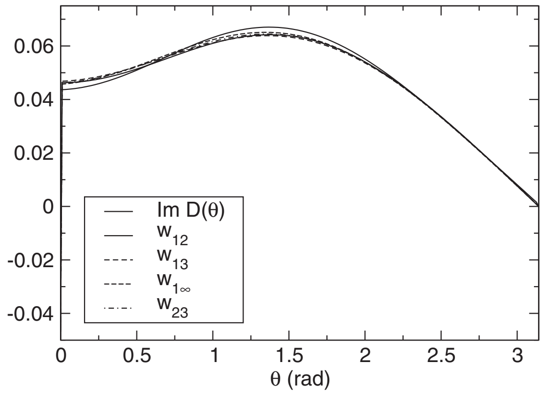

FIG. 2. As in Fig. 1 for the imaginary part of the Adler function.

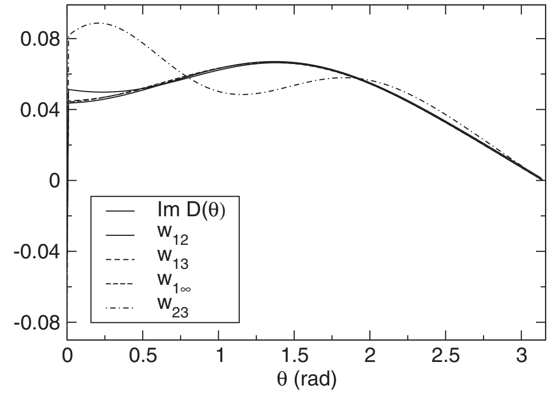

FIG. 4. As in Fig. 2 for $N = 18$ terms in the expansions.

Similar representations along the circle $|s| = M_\tau^2$ using the standard CI and FO expansions are given in Refs. [5,6]. For the standard RGS expansion the results reported in Ref. [14] show that its predictions are quite close to those of the standard CIPT. The standard expansions exhibit bigger and bigger oscillations with increasing N, failing to reproduce with accuracy the exact function along the circle.

The BRGS expansions should be compared with the Borel improved CI and FO expansions, defined in (48), (49), (51), and (52), respectively, which were investigated in Ref. [11] (the particular expansion based on the optimal mapping w_{12} was treated also in Ref. [6]). As illustrated in the figures given in Refs. [6,11], the Borel improved CI expansions reproduce very well the exact function, much like the new BRGS expansions, while the Borel improved FO expansions provide a very good approximation near the spacelike axis, where the powers of the logarithm $\ln(-s/M_\tau^2)$ present in the coefficients are small, but the approximation becomes worse near the timelike axis, where the logarithm acquires a large imaginary part.

In order to assess the physical relevance of the convergence acceleration of the perturbative expansions, we considered the behavior of the new BRGS schemes in the

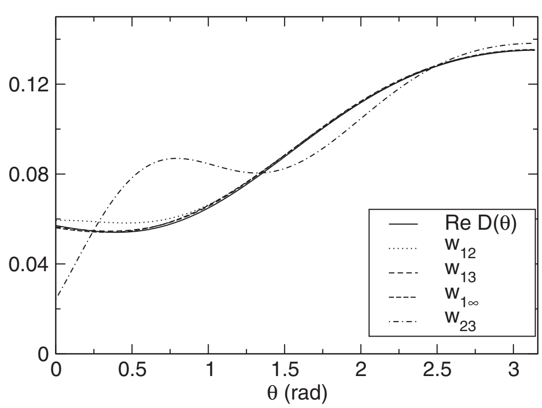

FIG. 3. As in Fig. 1 for $N = 18$ terms in the expansions.

context of τ-lepton hadronic width, which requires the theoretical calculation of the quantity $\delta^{(0)}$ defined in (3). In Tables I, II, III, and IV we give the differences $\delta^{(0)} - \delta^{(0)}_{\rm exact}$ order by order in perturbation theory for the models discussed above, using various perturbative expansions. The tendency of this quantity to flatten out to 0 would indicate that a particular scheme is efficient and reliable.

In Table I we show these differences for the model proposed in Ref. [5] and reviewed in Eqs. (55)–(58), and in Table II we present the results for the alternative model specified in Eqs. (59) and (60). For a consistent comparison with previous results reported in Refs. [5,6,11], we performed the calculations with $\alpha_s(M_\tau^2) = 0.34$.

The first three columns of Tables I and II show that at low truncation orders the standard FO expansion provides a more precise approximation for the model presented in Table I, while the standard CI expansion describes better alternative models of the type shown in Table II, characterized by a smaller residue of the first IR renormalon. These features were discussed also in Refs. [5,32]. At larger orders, the standard FO expansion exhibits in both cases a milder divergence, explained [10] by the cancellations between the contributions of the coefficients $c_{n,1}$ and the remaining terms in the series (10).

As concerns the RGS expansion, it provides at low orders an approximation comparable to the standard CI expansion for the first model and slightly better for the second model. However, the description deteriorates beyond $N = 10$ where large oscillations in the results appear, the RGS expansion exhibiting in a more dramatic way than CIPT and FOPT the divergent pattern of the QCD perturbation theory. An improvement of its large-order behavior by the techniques discussed in this paper is therefore mandatory.

The last four columns of Table I show that for the first model the new BRGS expansions provide a very good approximation already at low orders, and the accuracy increases with the truncation order N. According to the recent work [32], this model is a solid candidate for the physical Adler function. For the alternative model,

TABLE I. The difference $\delta^{(0)} - \delta^{(0)}_{\rm exact}$ for the model $B_{\rm BJ}$ proposed in Ref. [5] and specified in (55)–(58), calculated for $\alpha_s(M_\tau^2) = 0.34$ with the standard CI, FO and RGS expansions, and the new BRGS expansions (46) and (47) for various conformal mappings w_{lm}, truncated at order N. Exact value $\delta^{(0)}_{\rm exact} = 0.2371$.

N	CI	FO	RGS	BRGS w_{12}	BRGS w_{13}	BRGS $w_{1\infty}$	BRGS w_{23}
2	−0.0595	−0.0679	−0.0574	−0.0347	−0.0239	−0.0417	−0.0177
3	−0.0473	−0.0345	−0.0440	−0.0333	−0.0301	−0.0349	−0.0303
4	−0.0388	−0.0171	−0.0347	−0.0089	−0.0142	−0.0067	−0.0132
5	−0.0349	−0.0083	−0.0315	−0.0070	−0.0086	−0.0058	−0.0070
6	−0.0325	−0.0043	−0.0284	−0.0073	−0.0071	−0.0064	−0.0072
7	−0.0325	−0.0029	−0.0298	−0.0059	−0.0057	−0.0056	−0.0044
8	−0.0354	−0.0018	−0.0309	−0.0041	−0.0035	−0.0041	−0.0011
9	−0.0367	−0.0004	−0.0363	−0.0023	−0.0019	−0.0028	−0.0010
10	−0.0529	0.0019	−0.0483	0.0014	−0.0012	−0.0020	0.0004
11	−0.0409	0.0031	−0.0458	0.0036	−0.0008	−0.0016	−0.0009
12	−0.1248	0.0065	−0.1335	0.0031	−0.0006	−0.0015	0.0005
13	0.0258	0.0037	0.0534	0.0026	−0.0004	−0.0015	−0.0005
14	−0.5286	0.0204	−0.7850	0.0018	−0.0003	−0.0015	−0.0011
15	0.8640	−0.0201	1.7734	0.0006	−0.0002	−0.0015	0.0044
16	−3.5991	0.1447	−7.7043	0.0001	-7×10^{-6}	−0.0015	−0.0131
17	9.3560	−0.4252	24.8586	−0.0004	4×10^{-6}	−0.0014	0.0238
18	−31.76	1.907	−94.26	−0.0013	−0.0001	−0.0013	−0.0310

the results shown in Table II indicate a slightly worse approximation at low orders. However, the good convergence of the new expansions at large orders, in contrast with the big divergencies of the standard expansions, is visible also in this case.

As we mentioned, the standard RGS expansion is rather similar to the standard CI expansion up to relatively large orders, beyond which the RGS expansion starts to exhibit much wilder oscillations. It is of interest to compare these schemes also in the Borel improved versions given in Eqs. (46)–(49), respectively. This comparison is presented in Tables III and IV. The results show that, with small variations, the approximation provided by the BCI and BRGS expansions is similar up to large truncation orders N, for both models considered. Thus, using the technique of series acceleration by conformal mappings of the Borel plane, the strong divergence of the standard RGS was considerably tamed. We recall that an advantage of the

TABLE II. As in Table I for the modified model $B_{\rm alt}$ specified in (59) and (60). Exact value $\delta^{(0)}_{\rm exact} = 0.2102$.

N	CI	FO	RGS	BRGS w_{12}	BRGS w_{13}	BRGS $w_{1\infty}$	BRGS w_{23}
2	−0.0326	−0.0410	−0.0305	−0.0078	0.0030	−0.0148	0.0092
3	−0.0204	−0.0076	−0.0171	−0.0064	−0.0033	−0.0080	−0.0034
4	−0.0119	0.0098	−0.0078	0.0180	0.0127	0.0202	0.0137
5	−0.0080	0.0186	−0.0046	0.0199	0.0183	0.0211	0.0110
6	−0.0061	0.0216	−0.0026	0.0175	0.0175	0.0182	0.0197
7	−0.0061	0.0188	−0.0047	0.0193	0.0150	0.0153	0.0225
8	−0.0079	0.0111	−0.0052	0.0201	0.0132	0.0131	0.0258
9	−0.0065	0.0008	−0.0078	0.0106	0.0101	0.0109	0.0259
10	−0.0178	−0.0070	−0.0142	−0.0012	0.0047	0.0083	0.0273
11	0.0022	−0.0098	−0.0020	−0.0118	−0.0010	0.0058	0.0260
12	−0.0690	−0.0031	−0.0737	−0.0231	−0.0054	0.0037	0.0274
13	0.1019	0.0015	0.1397	−0.0310	−0.0081	0.0023	0.0264
14	−0.4207	0.0242	−0.6549	−0.0339	−0.0093	0.0014	0.0258
15	1.0234	−0.0168	1.9784	−0.0347	−0.0086	0.0008	0.0313
16	−3.3572	0.1398	−7.3731	−0.0316	−0.0062	0.0004	0.0139
17	9.7378	−0.4435	25.4225	−0.0239	−0.0028	0.0002	0.0507
18	−31.15	1.874	−93.316	−0.0156	0.0003	-2×10^{-5}	−0.0041

TABLE III. The difference $\delta^{(0)} - \delta^{(0)}_{\rm exact}$ for the model $B_{\rm BJ}$ proposed in Ref. [5] and specified in (55) and (59), calculated for $\alpha_s(M_\tau^2) = 0.34$ with the improved BCI expansions (48) and (49) and the BRGS expansions (46) and (47), for various conformal mappings w_{lm}, truncated at order N. Exact value $\delta^{(0)}_{\rm exact} = 0.2371$.

N	BCI w_{12}	BRGS w_{12}	BCI w_{13}	BRGS w_{13}	BCI $w_{1\infty}$	BRGS $w_{1\infty}$	BCI w_{23}	BRGS w_{23}
2	−0.0394	−0.0347	−0.0301	−0.0239	−0.0488	−0.0417	−0.0248	−0.0177
3	−0.0362	−0.0333	−0.0341	−0.0301	−0.0396	−0.0349	−0.0343	−0.0303
4	−0.0108	−0.0089	−0.0177	−0.0142	−0.0083	−0.0067	−0.0165	−0.0132
5	−0.0081	−0.0070	−0.0103	−0.0086	−0.0061	−0.0058	−0.0079	−0.0070
6	−0.0047	−0.0073	−0.0065	−0.0071	−0.005	−0.0064	−0.0052	−0.0072
7	−0.0032	−0.0059	−0.004	−0.0057	−0.0038	−0.0056	−0.0026	−0.0044
8	−0.0032	−0.0041	−0.0028	−0.0035	−0.003	−0.0041	−0.0024	−0.0011
9	−0.0030	−0.0023	−0.0023	−0.0019	−0.0025	−0.0028	−0.0024	−0.0010
10	−0.0020	0.0014	−0.0023	−0.0012	−0.0023	−0.0020	−0.0018	0.0004
11	−0.0012	0.0036	−0.0023	−0.0008	−0.0022	−0.0016	−0.0023	−0.0009
12	−0.0009	0.0031	−0.002	−0.0006	−0.0022	−0.0015	0.0003	0.0005
13	−0.0009	0.0026	−0.0016	−0.0004	−0.0022	−0.0015	−0.0023	−0.0005
14	−0.0007	0.0018	−0.001	−0.0003	−0.0022	−0.0015	0.0024	−0.0011
15	−0.0004	0.0006	−0.0005	−0.0002	−0.0021	−0.0015	−0.0015	0.0044
16	−0.0003	0.0001	−0.0002	-7×10^{-6}	−0.002	−0.0015	−0.0028	−0.0131
17	−0.0003	−0.0004	0.0001	4×10^{-6}	−0.0019	−0.0014	0.0162	0.0238
18	−0.0003	−0.0013	0.0002	−0.0001	−0.0017	−0.0013	−0.0445	−0.0310

BRGS expansion is that it does not require the numerical determination of the running coupling along the integration circle $|s| = M_\tau^2$, involving only analytical expressions.

VII. DETERMINATION OF $\alpha_s(M_\tau^2)$

In this section we shall use the new BRGS expansions defined in the present paper for a new determination of $\alpha_s(M_\tau^2)$ in the $\overline{\rm MS}$ scheme. The determination of this fundamental parameter is one of the important goals of this work and significant care has to be exercised in adopting proper values of input along with the experimental and theoretical uncertainties.

We use as input the recent phenomenological value of the pure perturbative correction to the hadronic τ width [10]

$$\delta^{(0)}_{\rm phen} = 0.2037 \pm 0.0040_{\rm exp} \pm 0.0037_{\rm PC}, \qquad (61)$$

where the first error is experimental and the second reflects the uncertainty of the higher-order terms

TABLE IV. As in Table III for the modified model $B_{\rm alt}$ specified in (59) and (60). Exact value $\delta^{(0)}_{\rm exact} = 0.2102$.

N	BCI w_{12}	BRGS w_{12}	BCI w_{12}	BRGS w_{13}	BCI $w_{1\infty}$	BRGS $w_{1\infty}$	BCI w_{23}	BRGS w_{23}
2	−0.0125	−0.0078	−0.0032	0.0030	−0.0219	−0.0148	0.0021	0.0092
3	−0.0093	−0.0064	−0.0072	−0.0033	−0.0127	−0.0080	−0.0074	−0.0034
4	0.0161	0.0180	0.0092	0.0127	0.0186	0.0202	0.0104	0.0137
5	0.0188	0.0199	0.0166	0.0183	0.0208	0.0211	0.0190	0.0110
6	0.0161	0.0175	0.0169	0.0175	0.0182	0.0182	0.0158	0.0197
7	0.0099	0.0193	0.0118	0.0150	0.0128	0.0153	0.0072	0.0225
8	0.0100	0.0201	0.0062	0.0132	0.0080	0.0131	0.0034	0.0258
9	0.0073	0.0106	0.0041	0.0101	0.0052	0.0109	0.0036	0.0259
10	−0.0047	−0.0012	0.0042	0.0047	0.0044	0.0083	0.0013	0.0273
11	−0.0120	−0.0118	0.0034	−0.0010	0.0044	0.0058	−0.0034	0.0260
12	−0.0095	−0.0231	0.0009	−0.0054	0.0046	0.0037	−0.0021	0.0274
13	−0.0080	−0.0310	−0.0016	−0.0081	0.0047	0.0023	−0.0042	0.0264
14	−0.0101	−0.0339	−0.0028	−0.0093	0.0044	0.0014	0.0022	0.0258
15	−0.0093	−0.0347	−0.0023	−0.0086	0.0040	0.0008	−0.0015	0.0313
16	−0.0058	−0.0316	−0.0011	−0.0062	0.0034	0.0004	−0.0029	0.0139
17	−0.0043	−0.0239	0.0	−0.0028	0.0028	0.0002	0.0173	0.0507
18	−0.0044	−0.0156	0.0005	0.0003	0.0022	-2×10^{-5}	−0.0485	−0.0041

("power corrections") in the OPE estimated by reasonable theoretical assumptions. We emphasize that our calculation is not based on the models discussed in the previous section, but relies only on the known coefficients $c_{n,1}$ given in (5), and the conservative choice $c_{5,1} = 283 \pm 283$ for the next coefficient [5,10].

Using this input, the values of $\alpha_s(M_\tau^2)$ obtained with the new BRGS expansions defined in (46) and (47), with the expansion functions $\mathcal{W}_{n,\mathrm{RGS}}^{(12)}$, $\mathcal{W}_{n,\mathrm{RGS}}^{(13)}$, $\mathcal{W}_{n,\mathrm{RGS}}^{(1\infty)}$ and $\mathcal{W}_{n,\mathrm{RGS}}^{(23)}$, respectively, are

$$\begin{aligned}
&0.3189 \pm 0.0034_{\mathrm{exp}} \pm 0.0031_{\mathrm{PC}}\ {}^{+0.0162}_{-0.0121}(\mathrm{c}_{51})^{+0.0014}_{-0.0013}(\beta_4),\\
&0.3198 \pm 0.0034_{\mathrm{exp}} \pm 0.0031_{\mathrm{PC}}\ {}^{+0.0112}_{-0.0088}(\mathrm{c}_{51})^{+0.0007}_{-0.0007}(\beta_4),\\
&0.3180 \pm 0.0034_{\mathrm{exp}} \pm 0.0031_{\mathrm{PC}}\ {}^{+0.0134}_{-0.0103}(\mathrm{c}_{51})^{+0.0010}_{-0.0009}(\beta_4),\\
&0.3188 \pm 0.0034_{\mathrm{exp}} \pm 0.0031_{\mathrm{PC}}\ {}^{+0.0143}_{-0.0107}(\mathrm{c}_{51})^{+0.0010}_{-0.0009}(\beta_4).
\end{aligned} \tag{62}$$

The first two errors are due to the uncertainties of the phenomenological value of $\delta^{(0)}$ given in (61). The third one, produced by the range adopted for the coefficient $c_{5,1}$, brings the most important contribution to the total error. The last uncertainty accounts for the higher terms in the expansion of the β function, simulated as in Ref. [7] by an additional coefficient $\beta_4 = \pm\beta_3^2/\beta_2$ in this expansion. We explored also the influence of the scale variation, choosing it as $\mu^2 = \xi M_\tau^2$ with $\xi = 1 \pm 0.5$, but the effects are very small, so we show only the result corresponding to the choice $\mu^2 = M_\tau^2$ in (26).

A very small sensitivity of $\alpha_s(M_\tau^2)$ to the variation of the scale is specific also to the standard CIPT analyses [7,9], the RGS expansion [14], and the Borel improved CI expansions [6,11]. The uncertainty related to the coefficient $c_{5,1}$ is bigger in the case of the Borel improved expansions than in the standard CI and RGS. However, as discussed in Refs. [6,11], having in view the divergent character of the series, the truncation error in the latter versions is certainly underestimated. The calculation of the five-loop coefficient $c_{5,1}$ is therefore of great interest, as it would reduce considerably the total error of $\alpha_s(M_\tau^2)$ determined from the Borel improved perturbation schemes.

By taking the average of the central values and of the errors given in Eq. (62) we obtain the prediction

$$\begin{aligned}
\alpha_s(M_\tau^2) = 0.3189 \pm 0.0034_{\mathrm{exp}} \pm 0.0031_{\mathrm{PC}}\ {}^{+0.0138}_{-0.0105}(\mathrm{c}_{5,1})\\
\pm 0.0010_{\beta_4},
\end{aligned} \tag{63}$$

which becomes, after adding the errors in quadrature,

$$\alpha_s(M_\tau^2) = 0.3189^{+0.0145}_{-0.0115}. \tag{64}$$

We emphasize that the error quoted above was obtained as the average of the errors of the individual determinations (62). Much lower uncertainties would have been obtained if standard statistical procedures for combining independent determinations were applied. In practice, although the values given in (62) may be considered independent theoretical determinations, we prefer the conservative errors given in (63), which avoid any bias. We note however the remarkable consistency of the theoretical determinations given in (62), which is a strong argument in favor of our predictions. It is remarkable also that the central value of our prediction (64) practically coincides with the world average $\alpha_s(M_\tau^2) = 0.3186 \pm 0.0056$ [1].

By evolving (64) to the scale of M_Z, using the CRunDec package [44], our prediction reads

$$\alpha_s(M_Z^2) = 0.1184^{+0.0018}_{-0.0015}, \tag{65}$$

where the central value coincides with the 2012 world average, $\alpha_s(M_Z^2) = 0.1184 \pm 0.0007$ [1].

It is of interest to compare the result (64) obtained with the BRGS expansions defined in the present work with other recent determinations of $\alpha_s(M_\tau^2)$. The values reported in the recent works [3–14] are not all based on the same input. Therefore, for a consistent comparison of different perturbative schemes, we use in what follows the same input for the phenomenological value of $\delta^{(0)}_{\mathrm{phen}}$, given in (61), and the five-loop coefficient $c_{5,1} = 283 \pm 283$. Then the standard FO, CI and RGS expansions lead to the predictions

$$\begin{aligned}
\alpha_s(M_\tau^2) &= 0.3199^{+0.0118}_{-0.0074} \quad \mathrm{FO},\\
\alpha_s(M_\tau^2) &= 0.3419^{+0.0084}_{-0.0085} \quad \mathrm{CI},\\
\alpha_s(M_\tau^2) &= 0.3378^{+0.0088}_{-0.0095} \quad \mathrm{RGS},
\end{aligned} \tag{66}$$

where the FO result is quoted in Ref. [10] and the RGS one in Ref. [14]. The FO and CI expansions improved by conformal mappings of the Borel plane (denoted as BFO and BCI, respectively) give [6,11]

$$\begin{aligned}
\alpha_s(M_\tau^2) &= 0.3109^{+0.0114}_{-0.0049} \quad \mathrm{BFO},\\
\alpha_s(M_\tau^2) &= 0.3195^{+0.0189}_{-0.0138} \quad \mathrm{BCI}.
\end{aligned} \tag{67}$$

One can see that the BRGS and BCI expansions, which implement both Borel and RG summation, lead to similar values for $\alpha_s(M_\tau^2)$, which actually are also close to the prediction of standard FOPT, where neither of the two summations is performed. On the other hand, the standard CI and RGS expansions, where only the summation related to RG is implemented, lead to values larger by about 0.02, while the BFO expansions, which improve only the large-order behavior without summing the terms known from the RG, lead to a value smaller by about 0.01 than the predictions of BRGS, BCI and FOPT.

In the recent work, Ref. [13], it has been pointed out, on the basis of a detailed analysis of moments of the spectral functions measured by OPAL experiment, that the uncertainty of the nonperturbative contributions to the hadronic τ width may be larger than usually assumed. To account for this possibility, we adopt a more conservative value for the uncertainty of $\delta^{(0)}_{\mathrm{phen}}$ due to power corrections, viz. replacing ± 0.0037 in (61) by the estimate ± 0.012 of this uncertainty quoted in Ref. [13].

Using this input, the value ± 0.0031 of the error on $\alpha_s(M_\tau^2)$ produced by the power corrections, quoted in (63), increases to ${}^{+0.0099}_{-0.0103}$, leading to

$$\alpha_s(M_\tau^2) = 0.3189^{+0.0173}_{-0.0151}. \tag{68}$$

We shall adopt this result, having the same central value and a more conservative error than in (64), as our final prediction.

By evolving (68) to the scale of M_Z our prediction reads

$$\alpha_s(M_Z^2) = 0.1184^{+0.0021}_{-0.0018}. \tag{69}$$

VIII. DISCUSSION AND CONCLUSIONS

The nonstrange hadronic decays of the τ lepton provide one of the most important ways of extracting the strong coupling α_s. The perturbative schemes of the Adler function that are used in this extraction continue, however, to be a significant source of uncertainty. There are two major ambiguities that affect the QCD perturbation expansions: one is related to the implementation of the renormalization-group invariance, and the other regards the large-order behavior of the series.

The first ambiguity is usually illustrated by the significant difference between the predictions of the FO and CI expansions of the Adler function. In a recent work, Ref. [14], we used for the analysis of the τ hadronic width another renormalization-group-improved expansion, proposed in Refs. [17,18], which sums in an analytically closed form the logarithms accessible by RG invariance.

The second ambiguity is associated with the fact that the perturbative series are divergent, due to a factorial growth of the coefficients calculated from Feynman diagrams [19–22]. In QCD this ambiguity is even more dramatic than in QED due to the IR renormalons, singularities in the Borel plane situated on the positive axis, which prevent the unique reconstruction of the function by means of a Laplace-Borel integral. However, if one adopts a suitable prescription (principal value), it is possible to exploit the available knowledge on the large-order behavior of the coefficients for defining a new expansion, in which the divergent pattern is tamed to a great extent. Such a method was proposed some time ago in Refs. [27–29] and was applied recently in Refs. [6,8,11] to both the FO and CI expansions of the Adler function. It is based on series acceleration by analytic continuation in the Borel plane, achieved by conformal mappings and the "softening" of the dominant singularities of the Borel transform.

In the present work, we used this method in order to improve the large-order behavior of the RGS expansion discussed in Ref. [14]. That it should be possible to have a straightforward application of the technique is not obvious, as the RGS expansion (12) involves a set of complicated functions $D_n(y)$ determined iteratively by the differential equations (15). However, after expressing the series in the alternative form (25), i.e., as an expansion in powers of the one-loop coupling (26), it was possible to show in Sec. IV that the dominant singularities of this expansion in the Borel plane coincide with those of the standard Borel transform $B(u)$ of the CI expansion. As a result, we are able to apply the techniques discussed in Refs. [6,8,11], defining the class of improved expansions (46) and (47), denoted as BRGS expansions.

As discussed in earlier works [11,29], the expansions based on conformal mappings of the Borel plane have a number of remarkable properties. In particular, the divergent pattern of the expansion of $\hat{D}(s)$ in terms of these new functions is expected to be tamed. The detailed numerical studies of two representative models for the Adler function, presented in Sec. VI, show that indeed the expansions improved by both RG summation and analytic continuation in the Borel plane, i.e., BCI and BRGS, provide a very good approximation of the true functions in the complex energy plane up to high orders. This is seen from Figs. 1–4 of Sec. VI and the similar figures presented in Refs. [6,11]. As a consequence, as shown in Tables I, II, III, and IV, the approximation of the integral $\delta^{(0)}$ provided by these expansions is very good and improves with increasing N. In contrast, the standard expansions shown in the first three columns of Tables I and II exhibit a divergent pattern.

As argued in Refs. [5,10] and is seen also from our results, the FO expansion of the observable $\delta^{(0)}$ exhibits a better behavior at high orders, which results from suitable cancellations between the increasing coefficients $c_{n,1}$ and the remaining terms in (10). This cancellation is spoiled in the standard CI expansion (11), which sums only the RG part, leaving the increase of $c_{n,1}$ uncompensated. The better large-order behavior of FOPT is one of the arguments used in favor of this expansion compared to CIPT [10]. As concerns the standard RGS expansion, some cancellations are taking place at low orders, but above $N = 10$ this expansion exhibits a divergence pattern even more dramatic than CIPT.

It should be noted however that the better behavior of FOPT is restricted only to integrals like $\delta^{(0)}$ or some special moments of the spectral function: as shown in previous studies of specific models [5,6,11], the pointwise description of the true Adler function along the circle $|s| = M_\tau^2$ provided by the FO expansion is quite poor. The good convergence obtained for some integrals along the circle is due to fortuitous cancellations of large contributions from different integration regions. On the other hand, as we mentioned, the BCI and BRGS expansions yield a good pointwise approximation of the true function along the whole circle.

For the determination of $\alpha_s(M_\tau^2)$ presented in Sec. VII we used as input the phenomenological value of the QCD correction $\delta^{(0)}$ to the τ hadronic width evaluated in Ref. [10]. Our analysis shows that the BRGS and BCI expansions, improved by both RG and high-order Borel summation, lead to similar results for $\alpha_s(M_\tau^2)$, which are actually also very close to the standard FOPT prediction obtained with the same input. It is remarkable that the central value of the prediction (64) obtained with the new BRGS expansions coincides with the world average quoted in Ref. [1]. We also considered the possibility that the uncertainty related to nonperturbative contributions might be larger, as follows from the recent analysis [13]. The error of our final prediction (68) is a conservative estimate that takes into account this possibility.

In conclusion, we advocate the use of the BCI and BRGS expansions of the QCD Adler functions, which implement simultaneously the RG invariance and the available knowledge about the large-order behavior of perturbation theory. In particular, the BRGS expansions proposed in the present work have the advantage that RG summation is implemented through analytically closed expressions. Therefore, these expansions are suitable for more detailed analyses of τ hadronic decays, based on the moments of the hadronic spectral function.

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