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Heptagonic symmetry for quarks and leptons

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The non-Abelian discrete symmetry D_7 of the heptagon is successfully applied to both quark and lepton mass matrices, including CP violation.

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I. INTRODUCTION

The structure of quark and lepton mass matrices has been under theoretical study for many years. Whereas the six quark masses and the three mixing angles and one CP violating phase in the quark sector are now measured with some precision, the lepton sector is still missing some crucial information. Recently, the neutrino mixing angle θ_{13} has been measured by the Daya Bay [1] and RENO [2] collaborations. The fact that $\sin^2 2\theta_{13}$ is now centered at around 0.1 means that the previously favored tribimaximal mixing pattern ($\sin^2\theta_{23} = 1/2$, $\sin^2\theta_{12} = 1/3$, $\theta_{13} = 0$) is invalid, although the A_4 symmetry [3–5] used to obtain it [6] is still applicable with some simple modifications [7–9]. On the other hand, in the simplest application [3,5] of A_4, all the quark mixing angles are zero. The question is whether there exists another symmetry which successfully yields both quark and lepton mass matrices, with good fits of all masses, mixing angles, and phases. The answer is yes, as elaborated below.

Using the non-Abelian discrete symmetry D_7 of the heptagon, it has been shown [10] that the CP violating phase of the quark mixing matrix may be predicted, whereas D_7 also yields a pattern [11] for the neutrino mass matrix consistent with what is observed. This pattern is previously derived using the symmetry Q_8 [12], and realizes a specific conjecture [13] that the neutrino mass matrix has two texture zeros in the basis that charged-lepton masses are diagonal.

In Sec. II the symmetry D_7 is explained. In Sec. III the assignments of quarks under D_7 are given with the accompanying Higgs structure and the resulting mass matrices. In Sec. IV numerical fits to the quark masses and mixing angles are given, with a prediction of the CP violating phase. In Sec. V the assignments of leptons under D_7 are given with the accompanying Higgs structure and the resulting mass matrices. In Sec. VI the neutrino mass matrix is analyzed to show that it allows for nonzero θ_{13} and a specific correlation between it and θ_{23} as well as δ_{CP}. Given that θ_{12} is close to the tribimaximal value, it prefers an inverted hierarchy of neutrino masses although a quasidegenerate pattern with either normal or inverted ordering cannot be ruled out. In Sec. VII there are some concluding remarks.

II. HEPTAGONIC SYMMETRY D_7

The group D_7 is the symmetry group of the regular heptagon with 14 elements, five equivalence classes, and five irreducible representations. Its character table is shown in Table I.

Here n is the number of elements and h is the order of each element. The numbers a_k are given by $a_k = 2\cos(2k\pi/7)$. The character of each representation is its trace and must satisfy the following two orthogonality conditions:

$$\sum_{C_i} n_i \chi_{ai}\chi_{bi}^* = n\delta_{ab}, \qquad \sum_{\chi_a} n_i \chi_{ai}\chi_{aj}^* = n\delta_{ij}, \tag{1}$$

where $n = \sum_i n_i$ is the total number of elements. The number of irreducible representations must be equal to the number of equivalence classes.

The three irreducible two-dimensional representations of D_7 may be chosen as follows. For $\mathbf{2}_1$, let

$$\begin{aligned}
&C_1: \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\
&C_2: \begin{pmatrix} 0 & \omega^k \\ \omega^{7-k} & 0 \end{pmatrix}, \quad (k = 0, 1, 2, 3, 4, 5, 6), \\
&C_3: \begin{pmatrix} \omega & 0 \\ 0 & \omega^6 \end{pmatrix}, \begin{pmatrix} \omega^6 & 0 \\ 0 & \omega \end{pmatrix}, \\
&C_4: \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega^5 \end{pmatrix}, \begin{pmatrix} \omega^5 & 0 \\ 0 & \omega^2 \end{pmatrix}, \\
&C_5: \begin{pmatrix} \omega^4 & 0 \\ 0 & \omega^3 \end{pmatrix}, \begin{pmatrix} \omega^3 & 0 \\ 0 & \omega^4 \end{pmatrix},
\end{aligned} \tag{2}$$

where $\omega = \exp(2\pi i/7)$, then $\mathbf{2}_{2,3}$ are simply obtained by the cyclic permutation of $C_{3,4,5}$.

TABLE I. Character table of D_7.

Class	n	h	χ_1	χ_2	χ_3	χ_4	χ_5
C_1	1	1	1	1	2	2	2
C_2	7	2	-1	1	0	0	0
C_3	2	7	1	1	a_1	a_2	a_3
C_4	2	7	1	1	a_2	a_3	a_1
C_5	2	7	1	1	a_3	a_1	a_2

For D_n with n prime, there are $2n$ elements divided into $(n+3)/2$ equivalence classes: C_1 contains just the identity, C_2 has the n reflections, C_k from $k=3$ to $(n+3)/2$ has two elements each of order n. There are two one-dimensional representations and $(n-1)/2$ two-dimensional ones.

The group multiplication rules of D_7 are

$$\mathbf{1}' \times \mathbf{1}' = \mathbf{1}, \qquad \mathbf{1}' \times \mathbf{2}_i = \mathbf{2}_i, \tag{3}$$

$$\mathbf{2}_i \times \mathbf{2}_i = \mathbf{1} + \mathbf{1}' + \mathbf{2}_{i+1}, \quad \mathbf{2}_i \times \mathbf{2}_{i+1} = \mathbf{2}_i + \mathbf{2}_{i+2}, \tag{4}$$

where $\mathbf{2}_{4,5}$ means $\mathbf{2}_{1,2}$. In particular, let (a_1, a_2), $(b_1, b_2) \sim \mathbf{2}_1$; then

$$\begin{aligned} a_1 b_2 + a_2 b_1 &\sim \mathbf{1}, \\ a_1 b_2 - a_2 b_1 &\sim \mathbf{1}', \\ (a_1 b_1, a_2 b_2) &\sim \mathbf{2}_2. \end{aligned} \tag{5}$$

In the decomposition of $\mathbf{2}_1 \times \mathbf{2}_2$, we have instead

$$(a_2 b_1, a_1 b_2) \sim \mathbf{2}_1, \qquad (a_2 b_2, a_1 b_1) \sim \mathbf{2}_3. \tag{6}$$

III. QUARK SECTOR

We assign quarks as shown in Table II and Higgs doublets as shown in Table III, together with an extra $Z_2^d \times Z_2^u$ symmetry.

As a result, the (u, c, t) mass matrix is diagonal, coming from the Yukawa terms $uu^c\phi_7^0 + cc^c\phi_8^0$ and $tt^c\phi_1^0$. As for the (d, s, b) mass matrix, the allowed Yukawa terms are $(ds^c + sd^c)\bar\phi_2^0$, $bb^c\bar\phi_2^0$, $b(d^c\bar\phi_4^0 + s^c\bar\phi_3^0)$, and $(d\bar\phi_4^0 + s\bar\phi_3^0)b^c$. The resulting mass matrix is thus of the form [10]

$$\mathcal{M}_d = \begin{pmatrix} 0 & a & \xi b \\ a & 0 & b \\ \xi c & c & d \end{pmatrix}, \tag{7}$$

where $\xi = \langle \bar\phi_4^0 \rangle / \langle \bar\phi_3^0 \rangle$.

TABLE II. Quark assignments under $D_7 \times Z_2^d \times Z_2^u$.

Symmetry	$[(u,d),(c,s)]$	(t,b)	(d^c,s^c)	b^c	(u^c,c^c)	t^c
D_7	$\mathbf{2}_1$	$\mathbf{1}$	$\mathbf{2}_1$	$\mathbf{1}$	$\mathbf{2}_2$	$\mathbf{1}$
Z_2^d	+	+	−	−	+	+
Z_2^u	+	+	+	+	−	+

TABLE III. Higgs doublet assignments under $D_7 \times Z_2^d \times Z_2^u$.

Symmetry	Φ_1	Φ_2	$\Phi_{3,4}$	$\Phi_{5,6}$	$\Phi_{7,8}$
D_7	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{2}_1$	$\mathbf{2}_2$	$\mathbf{2}_3$
Z_2^d	+	−	−	+	+
Z_2^u	+	+	+	+	−

IV. PREDICTION OF *CP* PHASE

As in Ref. [10], we can redefine the phases of $\mathcal{M}_d$ so that a, b, c, d are real, but ξ is complex. Since $\mathcal{M}_u$ is diagonal, we have

$$\begin{aligned} V_L^\dagger \mathcal{M}_d V_R &= \begin{pmatrix} m_d & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b \end{pmatrix}, \\ V_L^\dagger \mathcal{M}_d \mathcal{M}_d^\dagger V_L &= \begin{pmatrix} m_d^2 & 0 & 0 \\ 0 & m_s^2 & 0 \\ 0 & 0 & m_b^2 \end{pmatrix}, \end{aligned} \tag{8}$$

where V_L is the observed quark mixing matrix up to phase conventions. The structure of $\mathcal{M}_d \mathcal{M}_d^\dagger$ allows us to obtain the following first approximations:

$$\begin{aligned} m_b \simeq \sqrt{c^2 + d^2}, \qquad V_{cb} &\simeq \frac{bd + \xi^* ac}{(1 + |\xi|^2)c^2 + d^2}, \\ V_{ub} &\simeq \frac{ac + \xi bd}{c^2 + d^2}, \end{aligned} \tag{9}$$

where $a^2 \ll b^2$ and $|\xi|^2 \ll 1$ are assumed. We now rotate $\mathcal{M}_d \mathcal{M}_d^\dagger$ using

$$V_3 = \begin{pmatrix} 1 & 0 & V_{ub} \\ 0 & 1 & V_{cb} \\ -V_{ub}^* & -V_{cb}^* & 1 \end{pmatrix} \tag{10}$$

to obtain the 2×2 matrix

$$\mathcal{M}_2 \mathcal{M}_2^\dagger = \begin{pmatrix} A & C \\ C^* & B \end{pmatrix}, \tag{11}$$

where

$$A = a^2 + |\xi|^2 b^2 - |V_{ub}|^2 m_b^2, \tag{12}$$

$$B = a^2 + b^2 - |V_{cb}|^2 m_b^2, \tag{13}$$

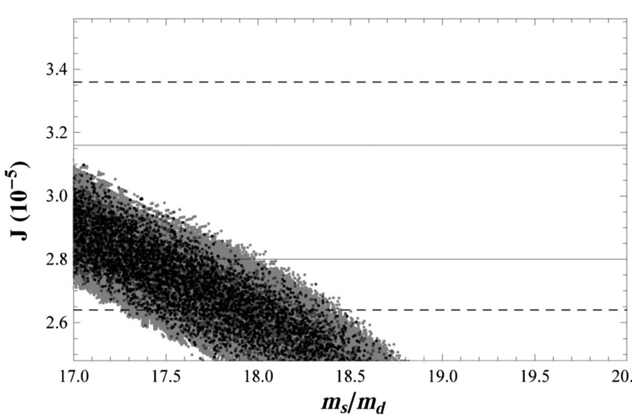

FIG. 1. The *CP* violating parameter J versus m_s/m_d. The solid (dashed) lines indicate the one (two) standard-deviation bounds of J.

TABLE IV. D_7 parameter fits of quark masses and mixing.

a (GeV) m_d (MeV)	b (GeV) m_s (MeV)	c (GeV) m_b (GeV)	d (GeV) $\lvert V_{us}\rvert$	Re(ξ) $\lvert V_{ub}\rvert$	Im(ξ) $\lvert V_{cb}\rvert$	m_s/m_d J
0.0125	0.138	1.32	−2.60	0.053	−0.084	17.00
3.89	66.2	2.92	0.22534	0.00355	0.0420	2.95×10^{-5}
0.0124	0.139	1.34	−2.60	0.058	−0.084	17.25
3.91	67.4	2.93	0.22532	0.00358	0.0420	2.89×10^{-5}
0.0123	0.138	1.40	−2.60	0.064	−0.087	17.50
3.96	69.2	2.96	0.22519	0.00363	0.0409	2.76×10^{-5}
0.0122	0.138	1.39	−2.55	0.068	−0.084	17.75
3.94	69.9	2.91	0.22501	0.00359	0.0415	2.70×10^{-5}

$$C = \xi b^2 - V_{ub}V_{cb}^* m_b^2, \qquad (14)$$

yielding

$$m_s^2 = \frac{1}{2}(B + A) + \frac{1}{2}\sqrt{(B - A)^2 + 4|C|^2}, \qquad (15)$$

$$|V_{us}|^2 = \frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{4|C|^2}{(B - A)^2 + 4|C|^2}}, \qquad (16)$$

where the phase of V_{us} is that of C, and

$$m_d = |2abc\xi - a^2 d|/m_s m_b. \qquad (17)$$

Using $|V_{us}| = 0.22534$, we find $|C|^2/(B - A)^2 = 0.05971$, and $m_s^2 \gg m_d^2$ implies $A \simeq 0.05351B$, hence $m_s^2 \simeq 1.05349B$. Using these formulas, the six parameters a, b, c, d, Re(ξ), Im(ξ) may then be determined and the CP violating parameter J is predicted.

For our numerical analysis, we start with the approximate solutions, then diagonalize $\mathcal{M}_d\mathcal{M}_d^\dagger$ directly. We scan for solutions consistent with data on the three masses and three mixing angles, within one standard deviation of each parameter. We then obtain J numerically from the resulting $V_{\rm CKM}$. This is then the prediction of our model. In Fig. 1 we plot J versus m_s/m_d, which shows good agreement with data. We use the 2008 updated values [14] of $m_{d,s,b}$ evaluated at M_W,

$$m_d(M_W) = 2.93(+1.25/-1.21)\ \text{MeV}, \qquad (18)$$

$$m_s(M_W) = 56 \pm 16\ \text{MeV}, \qquad (19)$$

$$m_b(M_W) = 2.92 \pm 0.09\ \text{GeV}, \qquad (20)$$

and the 2012 Particle Data Group [15] values of the mixing angles

$$|V_{us}| = 0.22534 \pm 0.00065, \qquad (21)$$

$$|V_{cb}| = 0.0412(+0.0011/-0.0005), \qquad (22)$$

$$|V_{ub}| = 0.00351(+0.00015/-0.00014). \qquad (23)$$

Note that Particle Data Group also lists the condition $17 < m_s/m_d < 22$ and the value of the CP violating parameter is

$$J = 2.96(+0.20/-0.16) \times 10^{-5}. \qquad (24)$$

We show in Table IV sample values of a, b, c, d, Re(ξ), Im(ξ) with the corresponding values of m_d, m_s, m_b, $|V_{us}|$, $|V_{ub}|$, $|V_{cb}|$, and J as well as m_s/m_d.

V. LEPTON SECTOR

Using again $D_7 \times Z_2^d \times Z_2^u$, we assign leptons as shown in Table V and Higgs triplets as shown in Table VI.

As a result, the (e, μ, τ) mass matrix is diagonal, coming from the Yukawa terms $ee^c\bar{\phi}_1^0$ and $\mu\mu^c\bar{\phi}_5^0 + \tau\tau^c\bar{\phi}_6^0$. As for the Majorana $(\nu_e, \nu_\mu, \nu_\tau)$ mass matrix, the allowed Yukawa terms are $\nu_e\nu_e\xi_1^0$, $(\nu_\mu\nu_\tau + \nu_\tau\nu_\mu)\xi_1^0$, and $\nu_e(\nu_\mu\xi_3^0 + \nu_\tau\xi_2^0)$. The resulting mass matrix is thus of the form [11]

$$\mathcal{M}_\nu = \begin{pmatrix} a & c & d \\ c & 0 & b \\ d & b & 0 \end{pmatrix}, \qquad (25)$$

which was first derived using Q_8 [12], and realizes one of the conjectures of Ref. [13].

TABLE V. Lepton assignments under $D_7 \times Z_2^d \times Z_2^u$.

Symmetry	(ν_e, e)	$[(\nu_\mu, \mu), (\nu_\tau, \tau)]$	e^c	$[(\mu^c, \tau^c)]$
D_7	**1**	$\mathbf{2}_1$	**1**	$\mathbf{2}_3$
Z_2^d	+	+	+	+
Z_2^u	+	+	+	+

TABLE VI. Higgs triplet assignments under $D_7 \times Z_2^d \times Z_2^u$.

Symmetry	ξ_1	$\xi_{2,3}$
D_7	**1**	$\mathbf{2}_1$
Z_2^d	+	+
Z_2^u	+	+

VI. ANALYSIS OF NEUTRINO MASS MATRIX

Rotating $\mathcal{M}_\nu$ to the tribimaximal basis using

$$\begin{pmatrix}\nu_1\\ \nu_2\\ \nu_3\end{pmatrix} = U_{\rm TB}^\dagger \begin{pmatrix}\nu_e\\ \nu_\mu\\ \nu_\tau\end{pmatrix} = \begin{pmatrix}\sqrt{2/3} & -\sqrt{1/6} & -\sqrt{1/6}\\ \sqrt{1/3} & \sqrt{1/3} & \sqrt{1/3}\\ 0 & -\sqrt{1/2} & \sqrt{1/2}\end{pmatrix}\begin{pmatrix}\nu_e\\ \nu_\mu\\ \nu_\tau\end{pmatrix}, \quad (26)$$

it becomes

$$\mathcal{M}_\nu^{(1,2,3)} = \begin{pmatrix} m_1 & m_6 & m_4\\ m_6 & m_2 & m_5\\ m_4 & m_5 & m_3\end{pmatrix}, \quad (27)$$

where

$$m_1 = \frac{1}{3}(2a + b - 2c - 2d), \quad (28)$$

$$m_2 = \frac{1}{3}(a + 2b + 2c + 2d), \quad (29)$$

$$m_3 = -b, \quad (30)$$

$$m_4 = \frac{1}{\sqrt{3}}(-c + d), \quad (31)$$

$$m_5 = \frac{1}{\sqrt{6}}(-c + d) = \frac{m_4}{\sqrt{2}}, \quad (32)$$

$$m_6 = \frac{1}{3\sqrt{2}}(2a - 2b + c + d) = \frac{1}{2\sqrt{2}}(m_1 + 2m_2 + 3m_3). \quad (33)$$

If $m_4 = m_5 = m_6 = 0$, tribimaximal mixing is recovered. In particular, $m_4 \neq 0$ or $m_5 \neq 0$ means that $\theta_{13} \neq 0$. In previous studies, the special cases $m_4 \neq 0$, $m_5 = m_6 = 0$ [16,17] and $m_5 \neq 0$, $m_4 = m_6 = 0$ [8,9,18] have been explored. The requirement from D_7 that $m_5 = m_4/\sqrt{2}$ is a new condition which will predict a special correlation between θ_{13} and θ_{23} as well as δ_{CP}.

Consider the unitary matrix U_ϵ such that

$$U_\epsilon^\dagger \mathcal{M}_\nu^{(1,2,3)}(\mathcal{M}_\nu^{(1,2,3)})^\dagger U_\epsilon = \begin{pmatrix} |m_1'|^2 & 0 & 0\\ 0 & |m_2'|^2 & 0\\ 0 & 0 & |m_3'|^2\end{pmatrix}; \quad (34)$$

then $U'_{\alpha i} = U_{\rm TB}U_\epsilon$ is the lepton mixing matrix up to phases. Let U_ϵ be approximately given by

$$U_\epsilon = \begin{pmatrix} 1 & \epsilon_{12} & \epsilon_{13}\\ \epsilon_{21} & 1 & \epsilon_{23}\\ \epsilon_{31} & \epsilon_{32} & 1\end{pmatrix}; \quad (35)$$

then for $|m_1'|^2 \simeq |m_1|^2$ we have

$$\epsilon_{21} \simeq \frac{-(m_6 m_1^* + m_2 m_6^*)}{|m_2|^2 - |m_1|^2}. \quad (36)$$

In addition, since the effective neutrino mass m_{ee} in neutrinoless double beta decay is given by

$$m_{ee} = |a| = |m_1 + m_2 + m_3|, \quad (37)$$

whereas

$$m_3 = \frac{1}{3}(2\sqrt{2}m_6 - m_1 - 2m_2), \quad (38)$$

we have the relationship

$$|m_3|^2 - m_{ee}^2 = \frac{1}{3}(|m_2|^2 - |m_1|^2)[1 + 4\sqrt{2}Re(\epsilon_{21})]. \quad (39)$$

Since $|m_2|^2 - |m_1|^2 \simeq \Delta m_{21}^2$ is very small, this model predicts $m_{ee} = |m_3|$ to a very good approximation. The structure of Eq. (38) also shows that an inverted ordering of neutrino masses is expected, although the quasidegenerate limit is also possible for this texture as fully analyzed in Ref. [19], in which case either inverted or normal ordering may occur. In the following we focus on the inverted case, i.e., $|m_3| < |m_1| < |m_2|$.

For $m_4 \neq 0$, ν_3 is rotated to ν_3' according to

$$\epsilon_{13} \simeq \frac{m_1 m_4^* + m_4 m_3^*}{|m_3|^2 - |m_1|^2}, \qquad \epsilon_{23} \simeq \frac{m_2 m_4^* + m_4 m_3^*}{\sqrt{2}(|m_3|^2 - |m_1|^2)}. \quad (40)$$

As a result,

$$U'_{e3} \simeq \sqrt{\frac{2}{3}}\epsilon_{13} + \sqrt{\frac{1}{3}}\epsilon_{23} \simeq \frac{-m_4(m_1 + 2m_2)^* + m_4^*(2m_1 + m_2)}{\sqrt{6}(|m_1|^2 - |m_3|^2)}, \quad (41)$$

$$U'_{\mu 3} \simeq -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{6}}\epsilon_{13} + \frac{1}{\sqrt{3}}\epsilon_{23} \simeq -\frac{1}{\sqrt{2}} - \frac{(m_1 - m_2)m_4^*}{\sqrt{6}(|m_1|^2 - |m_3|^2)}, \quad (42)$$

$$U'_{\tau 3} \simeq \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{6}}\epsilon_{13} + \frac{1}{\sqrt{3}}\epsilon_{23} \simeq \frac{1}{\sqrt{2}} - \frac{(m_1 - m_2)m_4^*}{\sqrt{6}(|m_1|^2 - |m_3|^2)}. \quad (43)$$

If all parameters are real, then for $U'_{e3} = 0.16$, $\sin^2 2\theta_{23}$ would be 0.80, which is ruled out by present data, i.e., $\sin^2 2\theta_{23} > 0.92$. However, a fit may be obtained for complex values.

We go back to Eq. (25) and observe that a, c, d may be chosen real, so only b is complex. This means that m_4 is real as well as $2m_1 - m_2$, and for $m_6 = 0$, $m_3 = -(m_1 + 2m_2)/3$. Writing $m_{1,2}$ as $m_{1,2}e^{i\phi_{1,2}}$ with $m_2 \simeq m_1$ and $\sin\phi_2 = 2\sin\phi_1$, we obtain

$$U'_{e3} \simeq \frac{m_1 m_4}{\sqrt{6}\Delta m^2_{32}}[-\cos\phi_1 + \cos\phi_2 - 9i\sin\phi_1], \qquad (44)$$

$$U'_{\mu 3} \simeq -\frac{1}{\sqrt{2}} + \frac{m_1 m_4}{\sqrt{6}\Delta m^2_{32}}[\cos\phi_1 - \cos\phi_2 - i\sin\phi_1], \qquad (45)$$

$$U'_{\tau 3} \simeq \frac{1}{\sqrt{2}} + \frac{m_1 m_4}{\sqrt{6}\Delta m^2_{32}}[\cos\phi_1 - \cos\phi_2 - i\sin\phi_1], \qquad (46)$$

where $\cos\phi_2 = \pm\sqrt{1 - 4\sin^2\phi_1}$. We then have

$$\sin^2\theta_{13} = \frac{|U'_{e3}|^2}{1 + |\epsilon_{13}|^2 + |\epsilon_{23}|^2}, \qquad \tan^2\theta_{23} = \frac{|U'_{\mu 3}|^2}{|U'_{\tau 3}|^2}. \qquad (47)$$

Since

$$|m_3| \simeq \frac{\sqrt{\Delta m^2_{32}}\sqrt{5 + 4\cos(\phi_2 - \phi_1)}}{2\sqrt{1 - \cos(\phi_2 - \phi_1)}}, \qquad (48)$$

the above equations relate $|m_3| = m_{ee}$ with θ_{13} and θ_{23}. If we fix θ_{13}, we then obtain $|m_3|$ as a function of θ_{23}. We plot in Fig. 2 our model predictions for $|m_{1,2}|$ and $|m_3| = m_{ee}$ versus $\sin^2 2\theta_{23}$. The other data points are taken to be their experimental central values.

If we rotate $\mathcal{M}_\nu^{1,2,3}(\mathcal{M}_\nu^{1,2,3})^\dagger$ by

$$U'_\epsilon = \begin{pmatrix} 1 & 0 & \epsilon_{13} \\ 0 & 1 & \epsilon_{23} \\ -\epsilon^*_{13} & -\epsilon^*_{23} & 1 \end{pmatrix}, \qquad (49)$$

we obtain the 2×2 mass-squared matrix spanning $\nu'_{1,2}$. This differs from the 2×2 submatrix in the tribimaximal basis by terms quadratic in m_4 which are important in obtaining the correct Δm^2_{21}, and Eq. (36) becomes modified. However, we can adjust $|m_2|$ versus $|m_1|$ as well as m_6 to fit the data. These adjustments will have negligible effects on $|m_3|$.

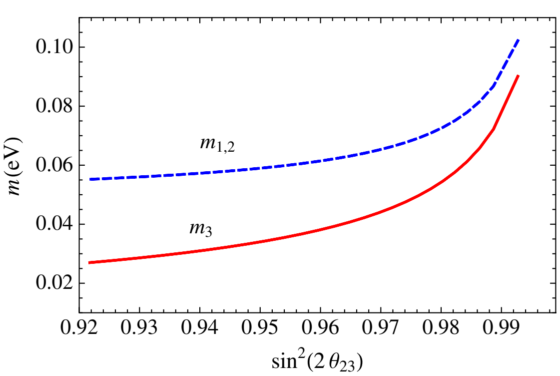

FIG. 2 (color online). Neutrino masses $m_{1,2}$ and $m_3 = m_{ee}$ versus $\sin^2 2\theta_{23}$.

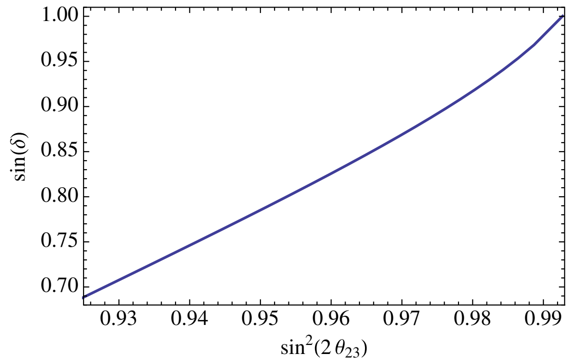

FIG. 3 (color online). The CP violating parameter $|\sin\delta_{CP}|$ versus $\sin^2 2\theta_{23}$.

We plot in Fig. 3 our model prediction for $|\sin\delta_{CP}|$ versus $\sin^2 2\theta_{23}$. To obtain $\sin\delta_{CP}$, we use

$$U'_{e2} \simeq \frac{1}{\sqrt{3}}, \qquad U'_{\mu 2} \simeq \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{2}}\epsilon^*_{23}, \qquad (50)$$
$$J = \mathrm{Im}(U'_{e2}U'_{\mu 3}U'^{\,*}_{\mu 2}U'^{\,*}_{e3}),$$

from which we find (using $U'_{\mu 2} = |U'_{\mu 2}|e^{i\theta_{\mu 2}}$, etc.)

$$\sqrt{\frac{2}{3}}\cos\theta_{23}\sin\delta \simeq |U'_{\mu 2}|\sin(\theta_{\mu 3} - \theta_{\mu 2} - \theta_{e3}). \qquad (51)$$

VII. CONCLUDING REMARKS

We have studied a specific pattern for both quark and lepton mass matrices. In both cases, one mass matrix is diagonal ($\mathcal{M}_u$ and $\mathcal{M}_e$), whereas the other has two zeros ($\mathcal{M}_d$ and $\mathcal{M}_\nu$). In the case of $\mathcal{M}_\nu$, the assumption that it is Majorana corresponds to one of the conjectures of Ref. [13], whereas the Dirac mass matrix $\mathcal{M}_d$ requires further restrictions to make it predictive, as first proposed in Ref. [10] using the non-Abelian discrete symmetry D_7. The conjectured form of $\mathcal{M}_\nu$ was first derived [12] using Q_8, but it may also be obtained [11] using D_5 or D_7. Here we consider D_7 as the unifying symmetry for both quarks and leptons.

The CP violating parameter J in the quark sector is constrained in this model by m_d, m_s, m_b, $|V_{us}|$, $|V_{ub}|$, $|V_{cb}|$. Within one standard deviation of all six measurements, we obtain J in agreement with data. In the neutrino sector, we obtain $|m_{1,2}|$ as well as $|m_3| = m_{ee}$ as functions of $\sin^2 2\theta_{23}$ and also predict $\sin\delta_{CP}$ as a function of $\sin^2 2\theta_{23}$.

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