Compressible quantum phases from conformal field theories in 2 + 1 dimensions

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Conformal field theories (CFTs) with a globally conserved U(1) charge $\mathcal{Q}$ can be deformed into compressible phases by modifying their Hamiltonian, $\mathcal{H}$, by a chemical potential $\mathcal{H} \to \mathcal{H} - \mu \mathcal{Q}$. We study 2 + 1 dimensional CFTs upon which an explicit S duality mapping can be performed. We find that this construction leads naturally to compressible phases which are superfluids, solids, or non-Fermi liquids which are more appropriately called ''Bose metals'' in the present context. The Bose metal preserves all symmetries and has Fermi surfaces of gauge-charged fermions, even in cases where the parent CFT can be expressed solely by bosonic degrees of freedom. Monopole operators are identified as order parameters of the solids, and the product of their magnetic charge and $\mathcal{Q}$ determines the area of the unit cell. We present implications for holographic theories on asymptotically AdS$_4$ spacetimes: S duality and monopole/dyon fields play important roles in this connection.

I. INTRODUCTION

A powerful method of analyzing correlated systems of interest to condensed matter physics is the application of a chemical potential to (i.e., doping) conformal field theories in 2 + 1 dimensions (CFT3s) [1]. This opens up a route to applying the advanced technology of the AdS/CFT correspondence [2–4]. In the latter approach, the charge density conjugate to the chemical potential is equated to an electric flux emanating from the boundary of a spacetime which is asymptotically AdS$_4$. A central question in then the nature of the stable ground state in the presence of a boundary electric flux.

Gubser [5] pointed out that the ground state of a doped CFT could be a superfluid. He described the condensation of a bulk charged scalar in the presence of a AdS-Reissner-Nördtrom black brane solution with a charged horizon [6]. It is important to note, however, that a conventional superfluid does not have low energy excitations associated with such a horizon above its ground state. Such a superfluid only appears when the infrared geometry is confining, there is no horizon, and the boundary electric flux is fully neutralized by the condensate [7,8].

A second class of compressible ground states of doped CFTs, known variously as ''non-Fermi liquids,'' ''strange metals'' or ''Bose metals,'' were proposed in Ref. [9] using arguments mainly from the CFT side. These states have Fermi surfaces of fermions carrying both the global U(1) charge of the doped CFT, and the charges of deconfined gauge fields (in the condensed matter context, the latter gauge fields are invariably ''emergent''). Because of the gauge charges, the single-fermion Green's function is not a gauge-invariant observable, and so the Fermi surface is partially ''hidden'' [10]. In the holographic context, these gauge-charged Fermi surfaces are possibly linked to electric flux that goes past the horizon in the zero temperature limit [9,11–19]. Particular holographic duals

for these states [10,20–22], were supported by evidence which matched the entropy density, numerous features of the entanglement entropy, and an inequality on the critical exponents [10,22].

Finally, from a condensed matter perspective, a natural ground state of a doped CFT is a solid (or a ''crystal'' or a ''striped state''), in which the doped charges localize in a regular periodic arrangement. Spatially modulated states have been found in the context of the AdS/CFT correspondence [23–28] but in situations with parity violating terms or in the presence of magnetic fields. We will see here that a version of such instabilities, after a S dual mapping of CFTs in 2 + 1 dimensions [29,30], will yield solid phases of parity preserving CFTs in a chemical potential. The solid will choose its periodic density modulation so that there are an integer number of doped charges per unit cell.

An important motivation for the present work was provided by the recent analysis by Faulkner and Iqbal [31]. They examined holographic duals of finite density quantum systems in 1 + 1 dimensions, and showed that monopole tunneling events in the bulk led to oscillatory density-density correlations on the boundary. They identified these oscillatory correlations as the Friedel oscillations of an underlying Fermi surface. Similar oscillations also appeared in deconfined phases of gauge theories coupled to fermionic matter at nonzero density [32]. However, it should be noted that in one spatial dimension such oscillatory correlations are present also in superfluids and solids, neither of which breaks any symmetry.

We are interested here in examining the role of monopoles on CFT3s in 2 + 1 dimensions, and in the corresponding doped CFT3s. The monopole and dyon operators of such CFT3s are closely linked to their properties under S duality transformations [29,33–40]. We will therefore present a reasonably complete description of two CFTs with global U(1) symmetries upon which the S duality transformation can be explicitly carried out. In the absence

of supersymmetry, such explicit transformations are only possible in theories with abelian gauge fields, abelian global symmetries, and bosonic matter; our CFTs are two of the simplest examples with such restrictions: the *XY* model and the abelian $\mathbb{CP}^{N-1}$ model.

After describing these CFT3s, we will dope them into compressible states by applying a chemical potential. In both cases, we easily find that such CFT3s can exhibit superfluid and solid phases. Naturally, the superfluid phases break the global U(1) symmetry. We will show that the monopole operators serve as order parameters for the solid phases: condensation of monopoles implies broken translational symmetry; this is similar to phenomena in insulating phases [41–43]. Furthermore, the magnetic charge of the monopole condensate will determine the size of the unit cell so that there are an integer number of doped electric charges per unit cell.

However, our primary interest is in phases of doped CFTs which do not break any symmetries. Such phases appeared in the previous analysis of CFTs with fermionic degrees of freedom [9] as non-Fermi liquid states with Fermi surfaces of gauge-charged fermions. We will show here that essentially identical compressible phases also appear upon doping CFT3s whose local Lagrangian contains only bosonic degrees of freedom. These compressible phases also contain gauge-charged Fermi surfaces of emergent fermionic degrees of freedom. The advantage of our present bosonic starting point is that it will shed new light on the role of monopoles, dyons, and *S* duality on such phases, and this information is surely crucial in setting up a complete holographic theory. Given our bosonic formulation, we will call these non-Fermi liquid states "Bose metals." This appellation is also apropos given the similarity of our analysis to the Bose metal phases of lattice spin and boson models [44–47].

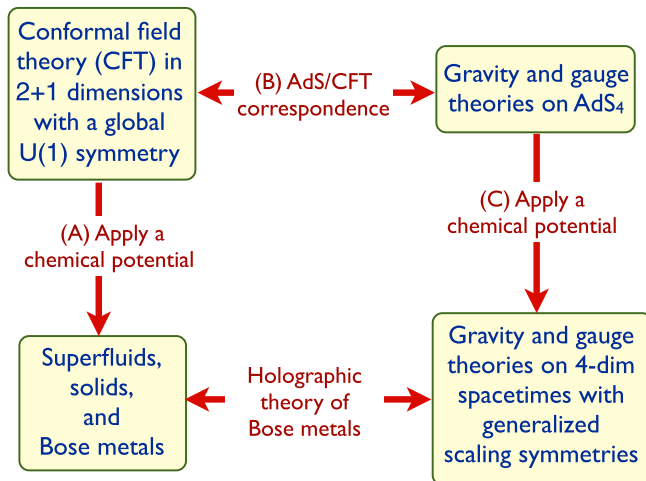It is useful to summarize the relationships between ideas discussed here in the flowchart in Fig. 1. We have so far

FIG. 1 (color online). Connections between CFT3s and holography.

discussed step A in Fig. 1. Armed with this understanding of the *S* duality of CFT3s, and the possible phases of doped CFT3s, we move onto holographic considerations. For the case of CFT3s, as in step B of Fig. 1, we will discuss features of the holographic theory on AdS$_4$, building on ideas of Witten [29] on the role of *S* duality. We propose a bulk theory with fields corresponding to the electric, magnetic, and dyon operators of the CFT3, and these couple to $3 + 1$ dimensional U(1) Maxwell fields and their electromagnetic duals. We will check our bulk theory by a comparison of its predictions for 3-point correlators with those of boundary CFT3s in Appendix B. Then we apply a chemical potential, as in step C, by fixing the value of a Maxwell vector potential at the boundary of AdS$_4$. We will discuss aspects of the resulting holographic theory, and note connections to the phases obtained from a direct analysis of the doped CFT3s.

We note that we will restrict our attention to models which preserve time-reversal and parity symmetries. So external magnetic fields coupling to the global U(1) charge will not be allowed. Although after *S* duality transformations some of our holographic solutions will contain background "magnetic" fields, there is always a frame in which background is purely electric, and parity and time-reversal are preserved. We do not consider situations in which there is both a chemical potential and a magnetic field, leading to possible quantum Hall states.

We will begin by describing two model CFT3s with global U(1) symmetries in Sec. II: the *XY* model and the abelian $\mathbb{CP}^{N-1}$ model. We will describe some of their properties, including identification of their primary operators with electric, magnetic, and dyonic charges: this will be important for the holographic formulation. We will apply a chemical potential to these CFT3s in Sec. III, as in step A of Fig. 1. This will allow us to identify classes of phases which are possible in such situations. Finally, in Sec. IV, we will discuss feature of the holographic realizations of these CFT3s (step B) and their compressible descendants (step C).

We will follow the convention of using indices $\mu, \nu \ldots$ for the 3 spacetime components, $a, b \ldots$ for the 4 directions of AdS$_4$, and $i, j \ldots$ for the 2 spatial components. We use the Euclidean time signature throughout.

II. CONFORMAL FIELD THEORIES

A. *XY* model

We begin with the simplest possible interacting CFT3 (in $2 + 1$ dimensions) with a conserved U(1) charge: the *XY* model, described by the Wilson-Fisher fixed point of the ϕ^4 field theory

$$\mathcal{L}_{XY} = |\partial_\mu \phi|^2 + s|\phi|^2 + u|\phi|^4. \tag{2.1}$$

This fixed point has one relevant operator, and we assume that either *s* or *u* has been tuned to place the field theory at

the conformally invariant point. The complex field ϕ will serve as our superfluid order parameter. We can also consider it as an operator carrying unit "electric" charge, and we define its correlator as

$$G_e(y) = \langle \phi^*(y)\phi(0) \rangle_{\mathcal{L}_{XY}}. \tag{2.2}$$

This correlation function has a power-law decay for the CFT3, and this defines the scaling dimension of the electric operator. The CFT3 has a conserved charge $\mathcal{Q}$ associated with the current

$$J_\mu = i(\phi^*\partial_\mu\phi - \phi\partial_\mu\phi^*), \tag{2.3}$$

where

$$\mathcal{Q} = \int d^2x J_\tau \tag{2.4}$$

and $\mathcal{Q} = 1$ is the electric charge of quanta of ϕ. This current has the correlator (after subtracting a contact term)

$$\langle J_\mu(k)J_\nu(-k) \rangle_{\mathcal{L}_{XY}} = -\frac{1}{g^2}|k|\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right), \tag{2.5}$$

where g is a universal number characteristic of the XY CFT3, and k_μ is a 3-momentum.

We can also define a monopole operator for this CFT3 as follows. First, couple ϕ to a background gauge field α_μ

$$\mathcal{L}_{XY}[\alpha] = |(\partial_\mu - i\alpha_\mu)\phi|^2 + s|\phi|^2 + u|\phi|^4, \tag{2.6}$$

where we follow the convention of indicating source/background fields which are not integrated over as arguments of the Lagrangian. We choose the magnetic field

$$\beta_\mu = \epsilon_{\mu\nu\lambda}\partial_\nu\alpha_\lambda \tag{2.7}$$

to be sourced by monopoles of magnetic charge $\tilde{\mathcal{Q}} = \pm 2\pi$ at $x = 0$ and $x = y$

$$\partial_\mu\beta_\mu = 2\pi\delta^3(x) - 2\pi\delta^3(x - y). \tag{2.8}$$

The monopole charge of 2π is required by the Dirac quantization condition. Note that the monopole so defined has a subtle difference from those considered earlier [33,37,38], which were monopoles in a dynamical gauge flux. Here we have a *background* gauge flux, coupling to the field theory by gauging a global symmetry. Such monopole background fields were also discussed recently by Kapustin and Willett [40]. The monopole correlation function is

$$G_m(y) = \frac{\int \mathcal{D}\phi \exp(-\int d^3x \mathcal{L}_{XY}[\alpha])}{\int \mathcal{D}\phi \exp(-\int d^3x \mathcal{L}_{XY})}. \tag{2.9}$$

This correlation function has a power-law decay for the CFT3 [33], and this defines the scaling dimension of the monopole operator. This monopole scaling dimension was computed in Refs. [33,37] in the large N limit of a theory with N copies of the ϕ field; as we just noted, these

computations also include a fluctuating gauge field, but this plays no role in the leading large N limit.

We can also consider multipoint correlators of the operators defined above. A convenient probe of the structure of the theory is provided by 3-point correlators of the current J_μ with the matter fields. Thus we have the correlator with the electric operators

$$\langle J_\mu(p)\phi^*(k_1)\phi(k_2) \rangle_{\mathcal{L}_{XY}}, \tag{2.10}$$

(the arguments of the fields are momenta), and also with the monopole operators

$$\frac{\int \mathcal{D}\phi J_\mu(w)\exp(-\int d^3x \mathcal{L}_{XY}[\alpha])}{\int \mathcal{D}\phi \exp(-\int d^3x \mathcal{L}_{XY})}, \tag{2.11}$$

(the arguments of the fields are spacetime co-ordinates). Both these correlators are computed in Appendix B, which also provides a comparison with results from holography.

It is interesting to determine the conformal transformations of the monopole operator as defined above. For this, we need to specify the background field β_μ more completely. Being a "magnetic" field, it is natural to have β_μ transform as a vector of scaling dimension 2 (more properly, it is a 2-form field); it can then be checked that the divergence condition in (2.8) is indeed conformally invariant. We also need zero curl conditions: unlike (2.8), the conformally invariant form of these conditions depends upon the spacetime metric. With the conformally flat metric $ds^2 = \Omega^{-2}(x)dx_\mu dx_\mu$, the zero curl conditions are

$$\epsilon_{\mu\nu\lambda}\partial_\nu(\Omega(x)\beta_\nu(x)) = 0. \tag{2.12}$$

The Eqs. (2.12) and (2.8) define β_μ in a conformally invariant manner. In this way we obtain a definition of the monopole operator which transforms like an ordinary scalar under all conformal transformations. Note that our definition is intrinsically nonlocal as it involves a determination of a nonzero $\beta_\mu(x)$ at all spacetime points; the issue of the nonlocality of the monopole operator will appear again when we discuss holography in Sec. IV.

1. S duality

As reviewed in Appendix A, application of S duality to the XY model yields the abelian Higgs model [48,49]

$$\mathcal{L}_{XY}^S = |(\partial_\mu - ia_\mu)\psi|^2 + s|\psi|^2 + u|\psi|^4 + \frac{1}{2e^2}(\epsilon_{\mu\nu\lambda}\partial_\nu a_\lambda)^2, \tag{2.13}$$

which provides an alternative description of the same CFT3. The conserved U(1) current in (2.3) can now be written as

$$J_\mu = \frac{i}{2\pi}\epsilon_{\mu\nu\lambda}\partial_\nu a_\lambda. \tag{2.14}$$

The factor of i is a consequence of working in the Euclidean signature, and the exchange of electric and magnetic degrees of freedom under S duality. A gauge-invariant two-point correlator of the field ψ yields the same G_m as in (2.9), as shown in Appendix A:

$$G_m(y) = \left\langle \psi^*(y) \exp\left(-\frac{i}{2\pi}\int d^3x\, a_\mu \beta_\mu\right)\psi(0)\right\rangle_{\mathcal{L}^S_{XY}};$$
(2.15)

it is easy to verify that the above correlation function is gauge invariant after using (2.8). From this correlator, we can identify the gauge-invariant monopole operator as

$$\psi \mathcal{M}_a; \qquad \tilde{Q} = 2\pi,$$
(2.16)

where the operator $\mathcal{M}_a$ is defined from (2.15) as an insertion which couples the gauge field a_μ to the magnetic flux of a monopole via a Chern-Simons term; such an insertion also appeared in the analysis of supersymmetric CFT3s by Kapustin and Willett [40]. Explicitly we have the various representations

$$\mathcal{M}_a(y)\mathcal{M}_a^\dagger(0) = \exp\left(\frac{i}{2\pi}\int d^3x\, a_\mu(x)\epsilon_{\mu\nu\lambda}\partial_\nu \alpha_\lambda(x)\right)$$

$$= \exp\left(\frac{i}{2\pi}\int d^3x\, a_\mu(x)\beta_\mu(x)\right);$$

$$\mathcal{M}_a(y) = \exp\left(\frac{i}{2\pi}\int d^3x\, \frac{a_\mu(x)(x_\mu - y_\mu)}{2|x-y|^3}\right). \quad (2.17)$$

Note the nonlocal structure in the definition. In the Chern-Simons formulation, the boundary term obtained after the gauge transformation of the Chern-Simons term reduces to contributions on the surfaces of small spheres surrounding $x = y$ and $x = 0$, and this cancels the gauge transformation of $\psi(y)\psi^*(0)$. See also similar constructions in Refs. [50,51].

It is useful to interpret the operator (2.16) by the state-operator correspondence of CFT3s [37]. For this, we radially quantize the CFT3 on surface of a sphere S^2. Then we see that ψ creates a single quantum carrying a unit a_μ electric charge, delocalized over the spherical surface. While $\mathcal{M}_a$ is a background charge density, with a term $i\int(d\Omega/(4\pi))a_\tau$ (where $d\Omega$ is the spherical solid angle); the integral over a_τ merely projects the Hilbert space to states with unit electric charge. Thus from the point of view of the scalar QED theory in (2.13), $\psi\mathcal{M}_a$ is a gauge invariant "electric" operator, whereas it is a magnetic operator from the perspective of the direct XY model; our notation will always reflect the perspective of the direct theory. In previous work [52,53], Wilson line operators have been used to obtain gauge-invariant correlators of matter fields like ψ, but these are path-dependent, don't have simple conformal transformation properties, don't define pointlike operators which can be used in the

state-operator correspondence, and don't appear in our duality analysis.

Finally, to close the circle of dualities, we have a representation of the electric correlator in (2.2) in terms of a monopole background for $\mathcal{L}^S_{XY}$ [48], as described in Appendix A:

$$G_e(y) = \frac{\int \mathcal{D}\psi\,\mathcal{D}a_\mu \exp(-\int d^3x\,\mathcal{L}^S_{XY}[\alpha])}{\int \mathcal{D}\psi\,\mathcal{D}a_\mu \exp(-\int d^3x\,\mathcal{L}^S_{XY})}, \quad (2.18)$$

where

$$\mathcal{L}^S_{XY}[\alpha] = |(\partial_\mu - ia_\mu - i\alpha_\mu)\psi|^2 + s|\psi|^2 + u|\psi|^4.$$
(2.19)

This is actually a traditional local "monopole" correlator for the scalar QED in (2.13), which maps to the electric correlator of (2.1). The associated scaling dimension was computed using the above monopole insertion method in Refs. [33,37] in the large N limit of a theory with N copies of the ψ field; the fluctuating gauge field a_μ was included in these computations, but this will modify the scaling dimension only at order $1/N$.

Similar S duality mappings also apply to the 3-point correlators of the current J_μ and the matter fields, such as those in (2.10) and (2.11), and are discussed in Appendix B.

Let us also note that we can define a U(1) current associated with the ψ matter field

$$\tilde{J}_\mu = 2\pi i(\psi^*(\partial_\mu - ia_\mu)\psi - \psi(\partial_\mu + ia_\mu)\psi^*). \quad (2.20)$$

In the scaling limit, where $e^2 \to \infty$ in (2.13), the equation of motion of a_μ imposes the constraint $\tilde{J}_\mu = 0$. Clearly, this implies that the two-point correlator of $\tilde{J}_\mu$ must also vanish. It is then easy to show diagrammatically that the *irreducible* $\tilde{J}_\mu$ correlator (with respect to a a_μ propagator) is nonzero, and equal to the inverse [30,54,55] of the correlator of J_μ in (2.14)

$$\langle \tilde{J}_\mu(k)\tilde{J}_\nu(-k)\rangle_{\mathcal{L}^S_{XY}}|_{\mathrm{irr}} = -g^2|k|\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right). \quad (2.21)$$

To summarize, the XY CFT3 has a global U(1) symmetry, an electric operator ϕ, and a magnetic operator $\psi\mathcal{M}_a$ [as we noted, this terminology reflects the perspective of the XY model in (2.1), and not its S dual in (2.13)]. Gauge-invariant correlators for these two operators can be written in both the direct and S dual formulations of the field theory. Such operators also obey nontrivial monodromy properties [56], but we will not explore this aspect here.

B. Abelian $\mathbb{CP}^{N-1}$ model

The XY CFT3 will not be rich enough to display all the possible phases of CFT3s in the presence of a chemical potential. So we consider the "easy-plane" or abelian $\mathbb{CP}^{N-1}$ model for which explicit S duality transformations can be performed (without supersymmetry) [57,58]. This is

also the simplest CFT3 which is both a gauge theory and has global U(1) symmetries. It should be noted that the existence of a conformally-invariant fixed point for this specific field theory has not been conclusively demonstrated for the simplest $N = 2$ case [59]. However, it is clear that such a CFT3 does exist in a large N limit [43,57], and all of our analysis can be extended to general N [58]. However, in the interests of simplicity, we will restrict ourselves to the simplest $\mathbb{CP}^1$ case, and work with the assumption that this CFT3 does exist.

In the direct formulation, the degrees of freedom are two complex scalars, z_1 and z_2, and a noncompact U(1) gauge field b_μ with Lagrangian

$$\mathcal{L}_{\mathbb{CP}} = |(\partial_\mu - ib_\mu)z_1|^2 + |(\partial_\mu - ib_\mu)z_2|^2$$
$$+ \frac{1}{2e^2}(\epsilon_{\mu\nu\lambda}\partial_\nu b_\lambda)^2 + \mathcal{L}_{z,\text{loc}}$$

$$\mathcal{L}_{z,\text{loc}} = s(|z_1|^2 + |z_2|^2) + u(|z_1|^2 + |z_2|^2)^2 + v|z_1|^2|z_2|^2,$$

(2.22)

with $u > 0$ and $-4u < v < 0$. For these negative values of v, the phase for s sufficiently negative has $|\langle z_1 \rangle| = |\langle z_2 \rangle| \neq 0$. We assume that the one relevant perturbation at the critical point has been tuned to obtain a CFT3.

This theory actually has two global U(1) symmetries, and associated conserved currents. The first is the ordinary global symmetry

$$\mathcal{Q}_1: z_1 \rightarrow z_1 e^{i\theta}, \qquad z_2 \rightarrow z_2 e^{-i\theta}. \quad (2.23)$$

The conserved current is

$$J_{1\mu} = i(z_1^*(\partial_\mu - ib_\mu)z_1 - z_1(\partial_\mu + ib_\mu)z_1^*)$$
$$- i(z_2^*(\partial_\mu - ib_\mu)z_2 - z_2(\partial_\mu + ib_\mu)z_2^*), \quad (2.24)$$

and

$$\mathcal{Q}_1 = \int d^2x J_{1\tau}. \quad (2.25)$$

However, there is also a conserved "topological" U(1) current

$$J_{2\mu} = \frac{i}{\pi}\epsilon_{\mu\nu\lambda}\partial_\nu b_\lambda \quad (2.26)$$

and a corresponding topological charge

$$\mathcal{Q}_2 = \int d^2x J_{2\tau}. \quad (2.27)$$

Another significant symmetry of $\mathcal{L}_{\mathbb{CP}}$ is the $\mathbb{Z}_2$ symmetry under which $z_1 \leftrightarrow z_2$. Note that $J_{1\mu}$ is odd under this symmetry, while $J_{2\mu}$ is even. This prohibits a bilinear coupling between these current operators.

As reviewed in Appendix C, we can perform the S dual mapping on *both* U(1) symmetries. This yields a theory with the same Lagrangian (unlike the situation for the XY model), but now expressed in terms of complex scalars w_1

and w_2, and a U(1) gauge field a_μ, which will have different physical interpretations. So the abelian $\mathbb{CP}^1$ model is self-dual [57] and has the Lagrangian

$$\mathcal{L}_{\mathbb{CP}}^S = |(\partial_\mu - ia_\mu)w_1|^2 + |(\partial_\mu - ia_\mu)w_2|^2$$
$$+ \frac{1}{2e^2}(\epsilon_{\mu\nu\lambda}\partial_\nu a_\lambda)^2 + \mathcal{L}_{w,\text{loc}}. \quad (2.28)$$

The values of the nonuniversal couplings in $\mathcal{L}_{w,\text{loc}}$ have been modified from before. The key feature of this dual representation is that the roles of the global and topological U(1) currents have been interchanged. Thus the currents in (2.24) and (2.26) now have the representation

$$J_{2\mu} = i(w_1^*(\partial_\mu - ia_\mu)w_1 - w_1(\partial_\mu + ia_\mu)w_1^*)$$
$$- i(w_2^*(\partial_\mu - ia_\mu)w_2 - w_2(\partial_\mu + ia_\mu)w_2^*), \quad (2.29)$$

and

$$J_{1\mu} = \frac{i}{\pi}\epsilon_{\mu\nu\lambda}\partial_\nu a_\lambda. \quad (2.30)$$

Let us now turn to an identification of the electric and magnetic operators associated with these U(1) symmetries. By analogy with the XY model, the simplest choices for electric operators are gauge invariant combinations of the matter fields which carry global charges

$$z_2^*z_1; \qquad \mathcal{Q}_1 = 2, \mathcal{Q}_2 = 0$$
$$w_2^*w_1; \qquad \mathcal{Q}_1 = 0, \mathcal{Q}_2 = 2. \quad (2.31)$$

Similarly, following the analysis for the XY model, we can write down gauge-invariant magnetic operators which are electrically neutral but carry magnetic charges (see Appendix C)

$$w_1w_2\mathcal{M}_a^2; \qquad \tilde{\mathcal{Q}}_1 = 2\pi, \qquad \tilde{\mathcal{Q}}_2 = 0$$
$$z_1z_2\mathcal{M}_b^2; \qquad \tilde{\mathcal{Q}}_1 = 0, \qquad \tilde{\mathcal{Q}}_2 = 2\pi. \quad (2.32)$$

However, examination of the operators in (2.31) and (2.32) shows that all of these can be written as composites of simpler *dyonic* operators that carry electrical charges of one U(1) symmetry and magnetic charges of the other U(1) symmetry, i.e., the operator product expansion of pairs of dyonic operators will produce operators in (2.31) and (2.32). We denote the magnetic charges by $\tilde{\mathcal{Q}}_1$ and $\tilde{\mathcal{Q}}_2$, and then the primary dyonic operators are

$$z_1\mathcal{M}_b; \quad \mathcal{Q}_1 = 1, \quad \mathcal{Q}_2 = 0, \quad \tilde{\mathcal{Q}}_1 = 0, \quad \tilde{\mathcal{Q}}_2 = \pi$$
$$z_2\mathcal{M}_b; \quad \mathcal{Q}_1 = -1, \quad \mathcal{Q}_2 = 0, \quad \tilde{\mathcal{Q}}_1 = 0, \quad \tilde{\mathcal{Q}}_2 = \pi$$
$$w_1\mathcal{M}_a; \quad \mathcal{Q}_1 = 0, \quad \mathcal{Q}_2 = 1, \quad \tilde{\mathcal{Q}}_1 = \pi, \quad \tilde{\mathcal{Q}}_2 = 0$$
$$w_2\mathcal{M}_a; \quad \mathcal{Q}_1 = 0, \quad \mathcal{Q}_2 = -1, \quad \tilde{\mathcal{Q}}_1 = \pi, \quad \tilde{\mathcal{Q}}_2 = 0.$$

(2.33)

We emphasize that these operators are all gauge-invariant. Expressions for the correlators of all four operators can be

obtained in both the direct and S dual representations. As an example, let us consider the two-point correlator of $z_1 \mathcal{M}_b$: from its definition we have as in (2.15)

$$G_{z_1}(y) = \left\langle z_1^*(y) \exp\left(-\frac{i}{2\pi} \int d^3x\, b_\mu \beta_\mu\right) z_1(0) \right\rangle_{\mathcal{L}_{\mathbb{CP}}}.$$
(2.34)

We perform the S duality mapping of this correlator in Appendix C and find

$$G_{z_1}(y) = \frac{\int \mathcal{D}w_1 \mathcal{D}w_2 \mathcal{D}a_\mu \exp(-\int d^3x\, \mathcal{L}^S_{\mathbb{CP}}[\alpha])}{\int \mathcal{D}w_1 \mathcal{D}w_2 \mathcal{D}a_\mu \exp(-\int d^3x\, \mathcal{L}^S_{\mathbb{CP}})}, \quad (2.35)$$

where a monopole background has been minimally coupled to w_1 via

$$\mathcal{L}^S_{\mathbb{CP}}[\alpha] = |(\partial_\mu - ia_\mu - i\alpha_\mu)w_1|^2 + |(\partial_\mu - ia_\mu)w_2|^2$$
$$+ \frac{1}{2e^2}(\epsilon_{\mu\nu\lambda}\partial_\nu a_\lambda)^2 + \mathcal{L}_{w,\text{loc}}. \quad (2.36)$$

Similar expressions can be obtained for the remaining operators in (2.33).

III. DOPED CONFORMAL FIELD THEORIES

This section will describe step A of Fig. 1, applied to the CFT3s described in Sec. II.

A. *XY* model

We apply a chemical potential, μ, to the *XY* model of (2.1)

$$\mathcal{L}_{XY}[\mu] = [(\partial_\tau + \mu)\phi^*][(\partial_\tau - \mu)\phi] + |\partial_i \phi|^2$$
$$+ s|\phi|^2 + u|\phi|^4. \quad (3.1)$$

In the S dual formulation of (3.1), this chemical potential also couples to J_τ:

$$\mathcal{L}^S_{XY}[\mu] = \mathcal{L}^S_{XY} - \frac{\mu}{2\pi}\epsilon_{ij}\partial_i a_j. \quad (3.2)$$

The following subsections describe the superfluid and solid phases that can appear in such a CFT3 at a nonzero μ. A Bose metal phase is not possible in such a CFT3 without a gauge field in the direct formulation, and the reason for this will become clear in the next subsection.

1. Superfluid

Notice from (3.1) that μ induces a negative mass term $-\mu^2|\phi|^2$. So the most likely consequence is that we obtain a superfluid phase with a ϕ condensate.

In the S dual formulation, we see that μ induces a net magnetic flux $\langle\epsilon_{ij}\partial_i a_j\rangle$. Assuming there is no Higgs phase with a ψ condensate, one consequence is that the spectrum of ψ quanta has the form of gapped Landau levels. And so the superfluid phase is characterized by

$$\langle\phi\rangle \neq 0, \qquad \langle\psi\rangle = 0. \quad (3.3)$$

The broken U(1) symmetry due to the ϕ condensate implies that there is a gapless Goldstone boson. In the S dual formulation, this gapless mode is the a_μ photon.

2. Solid

If quantum fluctuations are sufficiently strong, it is possible that for certain CFTs a solid phase is obtained. For the *XY* model, note that μ lowers the energy of the ϕ particle, while raising the energy of its antiparticle so we can consider a low-energy theory of the particle alone. One possible nonzero density ground state is a crystal of these particles. Clearly such a phase preserves the global U(1) symmetry, while breaking translational symmetry.

In the S dual formulation, the solid is obtained by treating $\mathcal{L}^S_{XY}[\mu]$ in the classical limit, and allowing for a ψ condensate in a Higgs phase. Indeed, this Lagrangian is precisely that for an Abrikosov flux lattice in the Landau-Ginzburg theory [60]. So we obtain a spatially modulated solution for $\langle\psi\rangle$ in the form of a triangular lattice. Abrikosov's argument also determines the size of the unit cell of this lattice. To keep the argument general, let us imagine that the field ϕ has electric charge $\mathcal{Q} = q_e$ (the present model has $q_e = 1$), and that the field ψ has magnetic charge $\tilde{\mathcal{Q}} = q_m$ (present model has $q_m = 2\pi$), and the total area of the system is L^2. The average flux density $\langle\epsilon_{ij}\partial_i a_j\rangle$ equals $q_e\langle\mathcal{Q}\rangle/(2\pi L^2)$ via the S duality relation (2.14), and so Abrikosov's condition of a flux quantum per unit cell implies [61]

$$q_e q_m \frac{\langle\mathcal{Q}\rangle}{L^2} A = 2\pi, \quad (3.4)$$

where A is the area of the unit cell. This corresponds to a U(1) charge $\mathcal{Q} = 1$ per unit cell.

So we see that the solid phase is characterized by

$$\langle\phi\rangle = 0, \qquad \langle\psi\rangle \neq 0. \quad (3.5)$$

and, as claimed, the monopole operator ψ is the solid order parameter.

B. Abelian $\mathbb{CP}^1$ model

We will apply a chemical potential, μ, to the $\mathcal{Q}_1$ charge only. Then

$$\mathcal{L}_{\mathbb{CP}}[\mu] = [(\partial_\tau + ib_\tau + \mu)z_1^*][(\partial_\tau - ib_\tau - \mu)z_1]$$
$$+ [(\partial_\tau + ib_\tau - \mu)z_2^*][(\partial_\tau - ib_\tau + \mu)z_2]$$
$$+ |(\partial_i - ib_i)z_1|^2 + |(\partial_i - ib_i)z_2|^2$$
$$+ \frac{1}{2e^2}(\epsilon_{\mu\nu\lambda}\partial_\nu b_\lambda)^2 + \mathcal{L}_{z,\text{loc}}, \quad (3.6)$$

and in the S dual theory

$$\mathcal{L}^S_{\mathbb{CP}}[\mu] = \mathcal{L}^S_{\mathbb{CP}} - \frac{\mu}{\pi}\epsilon_{ij}\partial_i a_j. \quad (3.7)$$

The superfluid and solid phases of $\mathcal{L}_{\mathbb{CP}}[\mu]$ have a structure similar to that of the *XY* model, and so our discussion of these will be brief.

The superfluid phase has a condensate of z_1 and z_2, and hence a gauge-invariant condensate of ϕ_1. The b_μ gauge field is Higgsed. There is a gapless Goldstone mode associated with the broken $\mathcal{Q}_1$ symmetry, and this is S dual to the a_μ photon. And the w_1 and w_2 quanta are gapped.

As in the *XY* model, if quantum fluctuations are sufficiently strong, it is possible that a solid phase is obtained. For the abelian $\mathbb{CP}^1$ model, note that μ lowers the energy of the z_1 particles and the z_2 antiparticles, while raising the energy of z_1 anti-particles and the z_2 particles; so we can consider a low-energy theory of the z_1 (z_2) particles (anti-particles) alone. These excitations carry opposite charges under the b_μ gauge field. So one possible ground state with a nonzero $\mathcal{Q}_1$ density is a crystalline arrangement of these charges. Note that while the z_1 and z_2 excitations carry opposite b_μ charges, they carry the same $\mathcal{Q}_1$ charge, and this prevents them from annihilating each other. In the S dual formulation, the solid appears as an Abrikosov flux lattice, but in a theory with 2 "superconducting" order parameters [62]; there is flux $\langle \epsilon_{ij}\partial_i a_j \rangle$ of π per unit cell, and this corresponds to a charge of $\mathcal{Q}_1 = 2$, one each for the z_1 particles and z_2 antiparticles.

1. Bose metal

However, the most interesting feature is the possibility of a compressible phase which is neither a solid nor a superfluid, and which breaks no symmetries. As we noted above, μ prefers particles of z_1 and antiparticles z_2, and so let us write $\mathcal{L}_{\mathbb{CP}}[\mu]$ in a nonrelativistic approximation by integrating out the antiparticles of z_1 and the particles of z_2:

$$\mathcal{L}_{\mathbb{CP}}^{\mathrm{nr}} = z_1^*(\partial_\tau - ib_\tau - \mu)z_1 + z_2(\partial_\tau + ib_\tau - \mu)z_2^*$$
$$- \frac{1}{2m} z_1^*(\partial_i - ib_i)^2 z_1 - \frac{1}{2m} z_2(\partial_i + ib_i)^2 z_2^* + \cdots$$
$$+ \frac{1}{2e^2}(\epsilon_{\mu\nu\lambda}\partial_\nu b_\lambda)^2 + \mathcal{L}_{z,\mathrm{loc}}, \tag{3.8}$$

where the ellipses represent higher order terms in the bosons kinetic energy, and we expect by scaling that the boson effective mass $m \sim \mu$. Now we can apply an *exact* transformation which fermionizes the z_1 and z_2 quanta by attaching 2π gauge flux tube of another U(1) gauge field c_μ [63,64]. We write this transformation as

$$f_1 = \mathcal{F}_c z_1 \qquad f_2 = \mathcal{F}_c z_2^*, \tag{3.9}$$

where $\mathcal{F}_c$ is flux tube attachment operator [65]. Note the formal analogy to the relativistic monopole flux operator $\mathcal{M}_b$ which was attached to $z_{1,2}$ in (2.33). In the present nonrelativistic context, we attach a flux tube, and this converts nonrelativistic bosons to nonrelativistic fermions. The Lagrangian for the fermions is

$$\mathcal{L}_{\mathbb{CP}}^{\mathrm{nr}} = f_1^\dagger(\partial_\tau - ib_\tau - ic_\tau - \mu)f_1 + f_2^\dagger(\partial_\tau + ib_\tau + ic_\tau - \mu)f_2$$
$$- \frac{1}{2m} f_1^\dagger(\partial_i - ib_i - ic_i)^2 f_1 - \frac{1}{2m} f_2^\dagger(\partial_i + ib_i + ic_i)^2 f_2$$
$$+ \cdots + \frac{i}{4\pi}\epsilon_{\mu\nu\lambda}c_\mu\partial_\nu c_\lambda + \frac{1}{2e^2}(\epsilon_{\mu\nu\lambda}\partial_\nu b_\lambda)^2 + \mathcal{L}_{f,\mathrm{loc}}, \tag{3.10}$$

and (3.8) and (3.10) are exactly equivalent. Note the Chern-Simons term in the c_μ gauge field.

A key feature of (3.10) is the equation of motion of c_τ:

$$f_1^\dagger f_1 - f_2^\dagger f_2 = \frac{1}{2\pi}\epsilon_{ij}\partial_i c_j. \tag{3.11}$$

This implies that in a state with $\langle f_1^\dagger f_1 \rangle = \langle f_2^\dagger f_2 \rangle$, which corresponds to the compressible phases we are interested in, the net c_μ flux will be zero, and the f_1 and f_2 fermions move in a net zero magnetic field. This is a key feature which allows the Bose metal phase here. And it is this step that fails when we apply the fermionization transformation to the *XY* model.

To proceed, we follow [66]: we map $b_\mu \to b_\mu - c_\mu$, and then integrate out the Gaussian c_μ fluctuations. Then, dropping higher derivative terms, we obtain a theory without a Chern-Simons term

$$\mathcal{L}_{\mathbb{CP}}^{\mathrm{nr}} = f_1^\dagger(\partial_\tau - ib_\tau - \mu)f_1 + f_2^\dagger(\partial_\tau + ib_\tau - \mu)f_2$$
$$- \frac{1}{2m} f_1^\dagger(\partial_i - ib_i)^2 f_1 - \frac{1}{2m} f_2^\dagger(\partial_i + ib_i)^2 f_2$$
$$+ \cdots + \mathcal{L}_{f,\mathrm{loc}}. \tag{3.12}$$

This describes a compressible Bose metal [44]. Of course, there is the possibility that there is a pairing instability with a $f_1 f_2$ condensate. This will lead to a superfluid state, but this is not identical to the superfluid discussed earlier in this section; the present superfluid has a gapless b_μ photon mode, which was not present earlier. We assume that this superfluid instability is somehow suppressed to a low energy scale. In Ref. [44] this is accomplished by endowing the f_1 and f_2 fermions with different Fermi surface shapes, and some analog of this may be possible here.

IV. HOLOGRAPHY

We begin with a discussion of step B in Fig. 1.

First, let us consider the holographic representation of the *XY* CFT3 on AdS$_4$. We propose that the bulk theory should have fields corresponding to the conserved current, and to the electric and magnetic operators:

$$J_\mu \to A_a, \qquad \phi \to \Phi, \qquad \psi \mathcal{M}_a \to \Psi. \tag{4.1}$$

Also we note Witten's observation [29] that S duality on the boundary theory corresponds to electromagnetic duality in the bulk theory. This suggests that $\psi \mathcal{M}_a$ couples to the electromagnetic dual of A_a, which we denote $\tilde{A}_a$: so we have the following minimal structure of the action

$$\mathcal{S}_{XY} = \int d^4x \sqrt{g} \Bigg[\frac{1}{4g^2} F_{ab}^2 + |(\partial_a - iA_a)\Phi|^2 + m_e^2 |\Phi|^2$$

$$+ |(\partial_a - i2\pi \tilde{A}_a)\Psi|^2 + m_m^2 |\Psi|^2 \Bigg], \qquad (4.2)$$

where

$$F_{ab} = \partial_a A_b - \partial_b A_a, \qquad \tilde{F}_{ab} = \partial_a \tilde{A}_b - \partial_b \tilde{A}_a,$$

$$\tilde{F}_{ab} = \frac{i}{2} \epsilon_{abcd} F^{cd}. \qquad (4.3)$$

The factor of i is needed in the last expression as in (2.14); it is also connected to $S^2 = -1$ [29], and will be important in Appendix B. The mass m_e is determined by the scaling dimension of the electric operator, the mass m_m by the scaling dimension of the magnetic operator, and g^2 is the universal number in (2.5). We have not written out the gravitational sector of the action, along with other possible neutral scalars. Indeed (4.2) should be considered a minimal theory with bulk fields which correspond to the simplest primary operators of the XY model, and so describes models similar to the XY model. We are not attempting to obtain the full holographic equivalent of the XY CFT3.

Note that there is a nonlocal relationship between the vector potentials A_a and $\tilde{A}_a$ in (4.3). Here A_a is related to the local observable J_μ in (4.1). So, clearly, the nonlocality of $\tilde{A}_a$ is linked to the nonlocality in the definition of the monopole operator $\psi \mathcal{M}_a$ which was noted in Sec. II A. An important check of the coupling between $\tilde{A}_a$ and the monopole field Ψ in (4.2) is provided by its predictions for the 3-point correlator of A_a with Ψ. We compute this in Appendix B, and show that the holographic result has the same form as the corresponding CFT3 correlator of J_μ and $\psi \mathcal{M}_a$.

For the abelian $\mathbb{CP}^1$ model, the analogous proposal has two copies of this structure, and the boundary $\rightarrow$ bulk correspondence leads to dyonic operators

$$J_{1\mu} \rightarrow A_a, \qquad z_1 \mathcal{M}_b \rightarrow Z_1, \qquad z_2 \mathcal{M}_b \rightarrow Z_2$$

$$J_{2\mu} \rightarrow B_a, \qquad w_1 \mathcal{M}_a \rightarrow W_1, \qquad w_2 \mathcal{M}_a \rightarrow W_2, \qquad (4.4)$$

with the minimal action

$$\mathcal{S}_{\mathbb{CP}} = \int d^4x \sqrt{g} \Bigg[\frac{1}{4g^2} F_{ab}^2 + \frac{1}{4g^2} G_{ab}^2$$

$$+ |(\partial_a - iA_a - i\pi \tilde{B}_a)Z_1|^2$$

$$+ m^2 |Z_1|^2 + |(\partial_a + iA_a - i\pi \tilde{B}_a)Z_2|^2 + m^2 |Z_2|^2$$

$$+ |(\partial_a - iB_a - i\pi \tilde{A}_a)W_1|^2 + m^2 |W_1|^2$$

$$+ |(\partial_a + iB_a - i\pi \tilde{A}_a)W_2|^2 + m^2 |W_2|^2 \Bigg], \qquad (4.5)$$

where as in (4.3)

$$G_{ab} = \partial_a B_b - \partial_b B_a, \qquad \tilde{G}_{ab} = \partial_a \tilde{B}_b - \partial_b \tilde{B}_a,$$

$$\tilde{G}_{ab} = \frac{i}{2} \epsilon_{abcd} G^{cd}. \qquad (4.6)$$

Note that to this order, there is no direct coupling between the $\mathcal{Q}_1$ and $\mathcal{Q}_2$ sectors in $\mathcal{S}_{\mathbb{CP}}$ apart from their common coupling to gravitation. The simplest terms are prohibited by the $\mathbb{Z}_2$ symmetry under which $z_1 \leftrightarrow z_2$, which was mentioned earlier. Dyonic operators also appeared [67,68] in holographic studies of the quantum Hall effect, but in that case they carried electric and magnetic charges of the same gauge field; in our case, the dyons carry electric charges under one gauge field, and magnetic charges of a second gauge field.

We are now ready to turn to step C of Fig. 1.

The generalization to the nonzero μ case is now immediate. We simply apply the chemical potential as a boundary condition to A_τ [6]. We can also add various dilaton fields and potentials, as appropriate for the infrared metric [10,20–22].

Let us now discuss the possible phases of $\mathcal{S}_{XY}$. A state with superfluid order has a Φ condensate [5]. With the monopole Ψ in hand, here we can also obtain a phase with crystalline order, in a manner similar to that for the boundary theory in Sec. III A 2. Notice that with an applied chemical potential, there is an electric field in $\langle F_{ab} \rangle$, which translates into a magnetic field in $\langle \tilde{F}_{ab} \rangle$. So the bulk theory for Ψ is the same as that of an electrically charged scalar moving in a background magnetic field. The condensation of Ψ in such a situation has been considered in supergravity theories [27], and leads to a vortex lattice [27,69], the bulk analog of the Abrikosov flux lattice. In terms of the original direct variables, this clearly corresponds to a boundary state with crystalline order. However, as we noted in Sec. I, these superfluid and solid phases are, strictly speaking, *not* the conventional superfluids or solids of Sec. III A. They contain a horizon in the infrared, and so correspond to "fractionalized" phases with additional deconfined excitations.

Indeed, it is best to think of the symmetry-broken phases above as descending from, and retaining many of the features of, the symmetric phase with no condensate or broken symmetry. So let us turn to a characterization of such a possible symmetric state in which none of the fields Φ, Ψ, Z_1, Z_2, W_1, W_2 condense, as may be arranged by making their masses very large. The most natural conclusion from our previous analysis of the boundary theory for the abelian $\mathbb{CP}^1$ model is that such a symmetric phase is a Bose metal, or related non-Fermi liquid. It is useful to define the Bose metal in gauge-invariant terms, to help identify it in the holographic theory: the Bose metal is a compressible phase with gapless excitations at all momenta, accompanied by signatures of a gauge-charged Fermi surface, which include (i) Friedel oscillations in the density correlations at the extremal wave vector, $2k_F$, of the gauge-charged Fermi surface; (ii) logarithmic violation

of the area law of the entanglement entropy, with a coefficient fixed by the charge density [10,22]. A strong form of the conjecture of Refs. [9,10,70] is that all compressible phases which do not break any symmetry are ultimately Bose metals, Fermi liquids, or allied phases with visible and/or gauge-charged Fermi surfaces.

So can theories like those in (4.2) and (4.5) describe Bose metals? We leave the answer of this question to future work, and just make some general remarks here. As discussed earlier [10,22], the entanglement entropy for certain hyperscaling violating backgrounds has numerous features consistent with a gauge-charged Fermi surface. So a key question is whether the holographic theories (4.2) and (4.5), or their extensions, contain Friedel oscillations. It is clear that information on the oscillatory structure is already present: after all, condensates of the monopole fields Ψ, W_1, W_2, leads to crystalline order with precisely the right period of an integer number of particles per unit cell. So we need to make Ψ, W_1, W_2 "almost" condensed to obtain Friedel oscillations. Furthermore, the theories (4.2) and (4.5) are similar to theories of vortex liquids in classical superconductors in an applied magnetic field: the latter systems have been studied using Feynman graph expansions [71], density-functional theories [72,73] and numerical simulations [74], and show clear oscillatory structure in the vortex-vortex correlation functions. From these studies, we can expect that the bulk $\tilde{F}_{ab}$ correlations will have a structure factor with a maximum at a nonzero wave vector, and the boundary limit of this structure factor is the density-density correlator of the doped CFT3.

V. CONCLUSIONS

We have presented an analysis of possible phases of doped CFT3s. The S duality properties of the parent CFT3, and its electric and magnetic operators were important in our analysis, and for our proposed bulk theory on AdS$_4$. We found that the doped CFT3s had phases with superfluid and solid order, and a Bose metal phase which broke no symmetries. The magnetic operators of the parent CFT3 served as order parameters for the solid, and also determined the size of its unit cell.

We checked the structure of the bulk theory on AdS$_4$ by both bulk and boundary computations of 3-point correlators in Appendix B. We exhibited connections between the Bose metal phases of doped CFT3s and the holographic compressible phase with no broken symmetries on asymptotically AdS$_4$ spacetimes, and these were summarized in Sec. IV. The magnetic operators of the CFT3 translated into new terms in the holographic theory which are sensitive to the quantization of particle number, and produce associated periodic correlations in the density. We also noted that holographic states with broken symmetries are best understood as Bose metals upon which a broken symmetry has been superimposed: thus holographic superfluids and solids have broken particle number and

translational symmetries, respectively, concomitant with the excitations of a deconfined gauge theory.

The key step in obtaining a Bose metal in our doped CFT3 was the flux *tube* attachment operator $\mathcal{F}_c$ [65] in (3.9) which converted the nonrelativistic bosons z_1 and z_2^* in (3.8) to nonrelativistic fermions f_1 and f_2 in (3.10). This operation has a formal similarity to a relativistic analog in our discussion of the CFT3 in (2.33), where the *monopole* flux operator $\mathcal{M}_b$ [defined in (2.17)] was attached to the relativistic fields z_1 and z_2 to obtain gauge-invariant primary fields of the CFT3. The relativistic $\mathcal{M}_b$ operator has proposed counterparts in bulk monopole/dyon fields on AdS$_4$, as discussed in Sec. IV. We now need to understand the holographic extension of $\mathcal{F}_c$ better, by finding additional signatures of the gauge-charged fermions in the Bose metal.

A. Generalizations

The explicit analyses of this paper have been for CFT3s with a global U(1) symmetry which are also Abelian gauge theories, and are expressible using only bosonic degrees of freedom. We conclude by briefly noting the applicability of our results to other CFT3s with a global U(1) symmetry.

The analyses defining the monopole operator in direct representation of the *XY* model in Sec. II A generalize to any CFT3 with a global U(1). We can always gauge the global U(1) as in (2.6), and insert monopole sources in the background gauge field. Consequently, we expect there to be an analog of the bulk field Ψ in (4.2) for all such CFT3s. The global U(1) current J_μ will be holographically dual to a bulk U(1) gauge field A_a, and Ψ will be electrically coupled to the S dual gauge field $\tilde{A}_a$, just as in (4.2). It is Ψ that carries the information on periodic density modulations in the compressible phases, and so these are ubiquitous, as expected. However, the field Φ in (4.2) is not as ubiquitous: its existence requires the presence of a gauge-invariant CFT3 operator carrying the global U(1) charge, and these need not be present, or could carry large enough dimensions to be irrelevant.

When we restrict attentions to CFT3s with a global U(1) which are also Abelian gauge theories, then further applications of our result are possible, even in cases where explicit S dual mappings are not known. As an example, we can consider a CFT3 with Dirac fermions, such as those in Ref. [9], obtained by replacing z_1, z_2 in $\mathcal{L}_{\mathbb{CP}}$ [Eq. (2.22)] by two-component Dirac fermions, q_1, q_2. In this case also, as in (2.33), we will have gauge-invariant operators $q_1 \mathcal{M}_b$, $q_2 \mathcal{M}_b$ carrying electric charge $\mathcal{Q}_1$ and magnetic charge $\tilde{\mathcal{Q}}_2$, even though their S dual counterparts are not evident. And there should be fermionic bulk operators, Q_1, Q_2 with the same quantum numbers which could reveal the "hidden" Fermi surfaces of Bose metal-like phases in these Abelian gauge theories. However, this construction of gauge-invariant operators carrying the fundamental global electric charge does not appear to generalize to non-Abelian gauge theories.

APPENDIX A: *S* DUALITY OF THE *XY* MODEL

We review the duality mapping of the *XY* model [48,49] in (2.6). We begin by writing $\phi \sim e^{i\theta}$, and expressing the action in the Villain form on a cubic lattice of sites i:

$$\mathcal{L}_{XY}[\alpha] = \frac{K}{2}(\Delta_\mu \theta_i - \alpha_{i\mu} - 2\pi n_{i\mu})^2, \quad \text{(A1)}$$

where Δ_μ is a discrete lattice derivative, $n_{i\mu}$ are integers on the links of the cubic lattice, and $\alpha_{i\mu}$ is the monopole background field. We perform a Fourier transform and write

$$\mathcal{L}_{XY}[\alpha] = \frac{1}{2K}J_{i\mu}^2 + iJ_{i\mu}(\Delta_\mu \theta_i - \alpha_{i\mu}), \quad \text{(A2)}$$

where $J_{i\mu}$ are another set of integers on the links of the cubic lattice. Integrating over θ_i, we obtain the zero divergence condition $\Delta_\mu J_{i\mu} = 0$. We solve this condition by writing

$$J_{i\mu} = \frac{1}{2\pi}\epsilon_{\mu\nu\lambda}\Delta_\nu a_{J\lambda}, \quad \text{(A3)}$$

where $a_{J\mu}$ resides on the links of the dual cubic lattice with sites J, and takes values which are integer multiples of 2π. Then we have

$$\mathcal{L}_{XY}[\alpha] = \frac{1}{8\pi^2 K}(\epsilon_{\mu\nu\lambda}\Delta_\nu a_{J\lambda})^2 - \frac{i}{2\pi}a_{J\mu}\beta_{J\mu}, \quad \text{(A4)}$$

where $\beta_{J\mu} = \epsilon_{\mu\nu\lambda}\Delta_\nu\alpha_\lambda$ is the magnetic flux associated with the monopole insertion. At this point, we can drop the "Dirac string" contribution to $\beta_{J\mu}$ because it only changes the Lagrangian by integer multiples of $2\pi i$.

So far, everything has been an exact rewriting of (A1). Now we promote $a_{J\mu}$ to a continuous real field by writing

$$\mathcal{L}_{XY}^S[\alpha] = \frac{1}{8\pi^2 K}(\epsilon_{\mu\nu\lambda}\Delta_\nu a_{J\lambda})^2 - y\cos(a_{J\mu}) - \frac{i}{2\pi}a_{J\mu}\beta_{J\mu}. \quad \text{(A5)}$$

Note that (A5) is exactly equivalent to (A4) in the limit $y \to \infty$. But, as argued in Ref. [49], the physics at finite y is the same as that as $y \to \infty$, and so will work with the S dual Lagrangian $\mathcal{L}_{XY}[\alpha]$. We can make the Lagrangian have the structure of a gauge theory by the shift $a_{J\mu} \to a_{J\mu} - \Delta_\mu\vartheta_J$, where ϑ_J is a dual angular variable. The integral

over ϑ_J only introduces a redundancy on configuration space, and the original expression merely corresponds to the gauge choice $\vartheta_j = 0$. Finally, taking the continuum limit with $\psi \sim e^{i\vartheta}$, we obtain the S dual Lagrangian (2.13), and also the correlator (2.15) after using (2.8).

Conversely, let us begin with a lattice version of the S dual theory in the presence of a monopole background, $\mathcal{L}_{XY}^S[\alpha]$ in (2.19):

$$\mathcal{L}_{XY}^S[\alpha] = \frac{y}{2}(\Delta_\mu \vartheta_J - a_{J\mu} - \alpha_{J\mu} - 2\pi n_{J\mu})^2$$
$$+ \frac{1}{8\pi^2 K}(\epsilon_{\mu\nu\lambda}\Delta_\nu a_{J\lambda})^2, \quad \text{(A6)}$$

After similar steps, this maps exactly to

$$\mathcal{L}_{XY}^S[\alpha] = \frac{1}{8\pi^2 y}(\epsilon_{\mu\nu\lambda}\Delta_\nu b_{i\lambda})^2 + \frac{K}{2}(\Delta_\mu\gamma_i - b_{i\mu})^2$$
$$- \frac{i}{2\pi}b_{i\mu}\beta_{i\mu}, \quad \text{(A7)}$$

where $b_{i\mu}$ takes values which are integer multiples of 2π on the links of the direct lattice, and γ_i is a real variable on the direct lattice. Then promoting $b_{i\mu}$ to a real variable, and shifting $b_{i\mu} \to b_{i\mu} - \Delta_\mu\theta_i$, $\gamma_i \to \gamma_i - \theta_i$, as below (A4), we obtain

$$\mathcal{L}_{XY}[\alpha] = \frac{1}{8\pi^2 y}(\epsilon_{\mu\nu\lambda}\Delta_\nu b_{i\lambda})^2 + \frac{K}{2}(\Delta_\mu\gamma_i - b_{i\mu})^2$$
$$- \frac{i}{2\pi}(b_{i\mu} - \Delta_\mu\theta_i)\beta_{i\mu} - y\cos(\Delta_\mu\theta_i - b_{i\mu}). \quad \text{(A8)}$$

This expression shows that the b_μ gauge field has been Higgsed by $e^{i\gamma}$, and so ignoring the massive Higgs mode we can set $b_{i\mu} = 0$ in the gauge $\gamma_i = 0$. The resulting theory is just a lattice version of the XY model of (2.1) with $\phi \sim e^{i\theta}$. Upon using (2.8), the effect of the monopole insertion $\alpha_{J\mu}$ is to yield the electric correlator in (2.2).

Other relationships between the correlators of the direct and S dual theories can be obtained in a similar manner, by inserting appropriate sources in the starting Lagrangian.

APPENDIX B: THREE-POINT CORRELATORS OF THE *XY* MODEL

First, we compute the 3-point correlator between the conserved current J_μ and the electrically charged field ϕ of the *XY* model in (2.10) shown in Fig. 2. For the free CFT3, or in the large flavor number limit of the interacting CFT3, this is

$$K(p, k_1, k_2) = \varepsilon_\mu(p)\langle J_\mu(p)\phi^*(k_1)\phi(k_2)\rangle_{\mathcal{L}_{XY}}$$
$$= \frac{\varepsilon_\mu(p)(k_{1\mu} + k_{2\mu})}{k_1^2 k_2^2}, \quad \text{(B1)}$$

$$\phi \to \Phi$$
$$k_2$$
$$J_\mu \to A_a \quad p$$
$$k_1$$
$$\phi^* \to \Phi^*$$
$$(a)$$

$$\psi \, \mathcal{M}_a \to \Psi$$
$$k_2$$
$$J_\mu \to A_a \quad p$$
$$k_1$$
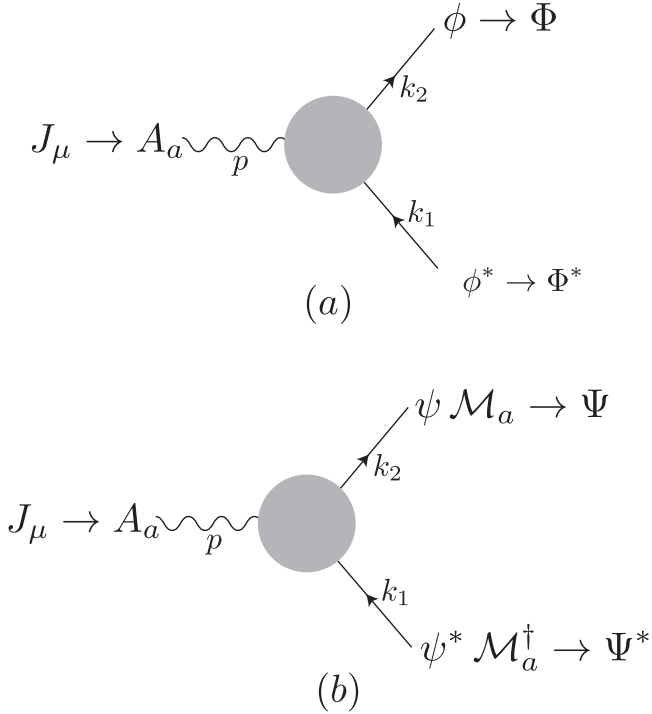$$\psi^* \, \mathcal{M}_a^\dagger \to \Psi^*$$
$$(b)$$

FIG. 2.   3-point correlators of the *XY* model: (a) the electric correlator K in (B1), (b) the magnetic correlator K_m in (B5). The labels are the boundary → bulk fields.

where $p_\mu \equiv k_{1\mu} - k_{2\mu}$, and $\varepsilon_\mu(p)$ is a polarization vector orthogonal to p_μ, $\varepsilon_\mu(p)p_\mu = 0$.

Let us now compare the result (B1) with that obtained by a tree-level holographic computation from the bulk action $\mathcal{S}_{XY}$ in (4.2). We will label the holographic direction z and the AdS$_4$ metric

$$ds^2 = \frac{dz^2 + dx_\mu^2}{z^2}. \tag{B2}$$

The correlator is given by the 3-point interaction in the action $\mathcal{S}_{XY}$, evaluated with the bulk fields taking values specified by the boundary-bulk propagators [4,75–77]; in the gauge $A_z = 0$, $\partial_\mu A_\mu = 0$ the bulk fields are

$$\Phi(k, z) = |k|^{\Delta_e - \frac{3}{2}} z^{3/2} K_{\Delta_e - \frac{3}{2}}(|k|z)$$
$$A_\mu(p, z) = \varepsilon_\mu(p) e^{-|p|z}, \tag{B3}$$

where Δ_e is the scaling dimension of the ϕ field. Then the 3-point correlator is

$$K(p, k_1, k_2) = \int_0^\infty \frac{dz}{z^2}(k_{1\mu} + k_{2\mu})\Phi^*(k_1, z)\Phi(k_2, z)A_\mu(p, z)$$
$$= \frac{\varepsilon_\mu(p)(k_{1\mu} + k_{2\mu})}{|k_1|^{5/2 - \Delta_e}|k_2|^{5/2 - \Delta_e}} \mathcal{F}\left(\frac{|k_1|}{|p|}, \frac{|k_2|}{|p|}\right), \tag{B4}$$

where $\mathcal{F}$ is a dimensionless function of its dimensionless arguments whose value can be deduced from the expressions above. Notice the similarity between the vector

structure of the expressions in (B1) and (B4). It is expected that the two results will match when the CFT3 computation is extended beyond the free field limit to nontrivial values of Δ_e.

We now extend these computations to the 3-point correlator between the current J_μ and the monopole operator of the *XY* model, specified in (2.11) and shown in Fig. 2. On the CFT3 side, it is difficult to work with (2.11), and this computation is more easily performed using the S dual representation of Sec. . Under this mapping, using the transformations of Appendix A, (2.11) becomes the gauge-invariant correlator

$$K_m(p, k_1, k_2)$$
$$= \int d^3y d^3w e^{ipw - ik_1y} \varepsilon_\mu(p)$$
$$\times \left\langle J_\mu(w)\psi^*(y) \exp\left(-\frac{i}{2\pi}\int d^3x a_\mu \beta_\mu\right)\psi(0)\right\rangle_{\mathcal{L}_{XY}^S}, \tag{B5}$$

where the current J_μ is now given by (2.14). This correlator is best computed in the large N_f limit of the CFT3s in which ψ has N_f flavors. Then in the leading large N_f limit the transverse gauge field propagator is [78]

$$\langle a_\mu(p)a_\nu(p)\rangle = \frac{16}{N_f|p|}\left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right). \tag{B6}$$

Evaluating (B5) to order $1/N_f$ with this propagator we obtain

$$K_m(p, k_1, k_2) = \left(\frac{16}{N_f \pi}\right)\frac{\epsilon_{\mu\nu\lambda}\varepsilon_\mu(p)k_{1\nu}k_{2\lambda}}{k_1^2 k_2^2 |p|}. \tag{B7}$$

We note that the exponential factor in (B5) has a vanishing contribution at this order, and (B7) arises only from the current vertex of ψ.

Finally, let us compare (B7) with the tree-level holographic computation from the bulk action $\mathcal{S}_{XY}$ in (4.2). Just as in (B3), we will now have the bulk fields

$$\Psi(k, z) = |k|^{\Delta_m - \frac{3}{2}} z^{3/2} K_{\Delta_m - \frac{3}{2}}(|k|z)$$
$$A_\mu(p, z) = \varepsilon_\mu(p) e^{-|p|z}, \tag{B8}$$

where Δ_m is the scaling dimension of $\psi \mathcal{M}_a$. We now need to convert the above result for A_μ in (B8) to an expression for $\tilde{A}_\mu$. Let the holographic indices $a, b \ldots$ extend over the directions z, x_1, x_2, x_3, and let us choose the momentum $p_\mu = (p, 0, 0)$ (with $p > 0$) and $\varepsilon_\mu = (0, 1, 0)$. Then, the Maxwell tensor is

$$F_{12} = ipe^{-pz}, \qquad F_{z2} = -pe^{-pz} \tag{B9}$$

and all other components are zero. So, from (4.3), we have the dual tensor

$$\tilde{F}_{3z} = -pe^{-pz}, \qquad \tilde{F}_{13} = -ipe^{-pz}, \tag{B10}$$

which corresponds to a dual vector potential

$$\tilde{A}_\mu(p, z) = -(0, 0, 1)e^{-pz}. \tag{B11}$$

From this we deduce the following result for general p_μ and ε_μ:

$$\tilde{A}_\mu(p, z) = \frac{\epsilon_{\mu\nu\lambda}p_\nu\varepsilon_\lambda(p)}{|p|}e^{-|p|z}, \tag{B12}$$

and $\tilde{A}_z = 0$. Note that the simple exponential form of the boundary-bulk correlator of the gauge field was crucial in the above analysis leading to the simple result for the dual gauge field in (B12). We can now obtain the 3-point correlator as in (B4)

$$\begin{aligned} K_m(p, k_1, k_2) &= \int_0^\infty \frac{dz}{z^2}(k_{1\mu} + k_{2\mu})\Psi^*(k_1, z)\Psi(k_2, z)\tilde{A}_\mu(p, z) \\ &= \frac{\epsilon_{\mu\nu\lambda}\varepsilon_\mu(p)k_{1\nu}k_{2\lambda}}{|k_1|^{5/2-\Delta_m}|k_2|^{5/2-\Delta_m}|p|}\tilde{\mathcal{F}}\left(\frac{|k_1|}{|p|}, \frac{|k_2|}{|p|}\right), \end{aligned} \tag{B13}$$

where again $\tilde{\mathcal{F}}$ is a dimensionless function of its dimensionless arguments whose value can be deduced from the expressions above. Now notice the remarkable match of the magnetic holographic result (B13) to the CFT3 computation in (B7), similar to that for the electric operator case between (B1) and (B4).

APPENDIX C: S DUALITY OF THE ABELIAN $\mathbb{CP}^1$ MODEL

We proceed just as in Appendix A, following Ref. [57]. We start from the S dual action (2.28), write $w_{1,2} \sim e^{i\vartheta_{1,2}}$, and introduce the Villain action on the dual cubic lattice

$$\begin{aligned} \mathcal{L}_{\mathbb{CP}}^S[\alpha] &= \frac{K}{2}(\Delta_\mu\vartheta_{1J} - a_{J\mu} - \eta_1\alpha_{J\mu} - 2\pi n_{1J\mu})^2 \\ &+ \frac{K}{2}(\Delta_\mu\vartheta_{2J} - a_{J\mu} - \eta_2\alpha_{J\mu} - 2\pi n_{2J\mu})^2 \\ &+ \frac{1}{2e^2}(\epsilon_{\mu\nu\lambda}\partial_\nu a_{J\lambda})^2. \end{aligned} \tag{C1}$$

Here $\alpha_{J\mu}$ is a monopole background field defined by (2.7) and (2.8). The choices of $\eta_{1,2} = 0, \pm 1$ will give expressions for the different operator insertions. Thus, the choice $\eta_1 = 1$, $\eta_2 = 0$ yields (2.36).

We begin with a Fourier transform, as in (A2), to obtain

$$\begin{aligned} \mathcal{L}_{\mathbb{CP}}^S[\alpha] &= \frac{1}{2K}(J_{1J\mu}^2 + J_{2J\mu}^2) - ia_{J\mu}(J_{1J\mu} + J_{2J\mu}) \\ &- i\alpha_{J\mu}(\eta_1J_{1J\mu} + \eta_2J_{2J\mu}) + \frac{e^2}{2}f_{i\mu}^2 \\ &+ if_{i\mu}\epsilon_{\mu\nu\lambda}\Delta_\nu a_{J\lambda}, \end{aligned} \tag{C2}$$

where $J_{1i\mu}$ and $J_{2i\mu}$ are integer valued currents obeying $\Delta_\mu J_{1i\mu} = \Delta_\mu J_{2i\mu} = 0$, and $f_{J\mu}$ is a real-valued flux on the links of the dual lattice. Integrating over $a_{J\mu}$ we obtain the

additional constraint $J_{1i\mu} + J_{2i\mu} = \epsilon_{\mu\nu\lambda}\Delta_\nu f_{J\lambda}$. We solve these constraints by writing

$$\begin{aligned} J_{1J\mu} &= \frac{1}{2\pi}\epsilon_{\mu\nu\lambda}\Delta_\nu b_{1i\lambda} \\ J_{2J\mu} &= \frac{1}{2\pi}\epsilon_{\mu\nu\lambda}\Delta_\nu b_{2i\lambda} \\ f_{J\mu} &= \frac{1}{2\pi}(b_{1i\mu} + b_{2i\mu} - \Delta_\mu\gamma_i), \end{aligned} \tag{C3}$$

where $b_{1i\mu}$ and $b_{2i\mu}$ are integer multiples of 2π, and γ_i is real-valued. So the action is

$$\begin{aligned} \mathcal{L}_{\mathbb{CP}}^S[\alpha] &= \frac{1}{8\pi^2K}((\epsilon_{\mu\nu\lambda}\Delta_\nu b_{1i\lambda})^2 + (\epsilon_{\mu\nu\lambda}\Delta_\nu b_{2i\lambda})^2) \\ &- \frac{i}{2\pi}\beta_{i\mu}(\eta_1b_{1i\mu} + \eta_2b_{2i\mu}) \\ &+ \frac{e^2}{8\pi^2}(b_{1i\mu} + b_{2i\mu} - \Delta_\mu\gamma_i)^2. \end{aligned} \tag{C4}$$

As in Appendix A, we can now drop the Dirac string in $\beta_{i\mu}$ because it only changes the action by integer multiples of $2\pi i$. Also, up to this point, all transformations have been exact.

Now we promote $b_{1i\mu}$ and $b_{2i\mu}$ to continuous real fields, and shift $b_{1i\mu} \to b_{1i\mu} - \Delta_\mu\theta_{1j}$, $b_{2i\mu} \to b_{2i\mu} - \Delta_\mu\theta_{2j}$, and $\gamma_i \to \gamma_i - \theta_{1i} - \theta_{2i}$. Then we obtain the direct lattice theory

$$\begin{aligned} \mathcal{L}_{\mathbb{CP}}[\alpha] &= \frac{1}{8\pi^2K}((\epsilon_{\mu\nu\lambda}\Delta_\nu b_{1i\lambda})^2 + (\epsilon_{\mu\nu\lambda}\Delta_\nu b_{2i\lambda})^2) \\ &- \frac{i}{2\pi}\beta_{i\mu}[\eta_1(b_{1i\mu} - \Delta_\mu\theta_{1i}) \\ &+ \eta_2(b_{2i\mu} - \Delta_\mu\theta_{2i})] \\ &+ \frac{e^2}{8\pi^2}(b_{1i\mu} + b_{2i\mu} - \Delta_\mu\gamma_i)^2 \\ &- y(\cos(\Delta_\mu\theta_{1i\mu} - b_{1i\mu}) + \cos(\Delta_\mu\theta_{2i\mu} - b_{2i\mu})). \end{aligned} \tag{C5}$$

This action has the structure of a U(1) × U(1) gauge theory, in the presence of charged matter fields, $e^{i\theta_1}$ and $e^{i\theta_2}$. One of the diagonal U(1)s has been Higgsed by the term proportional to e^2, with $e^{i\gamma}$ acting as the Higgs field. So we can drop the massive excitations associated with this diagonal U(1) by setting $b_{1i\mu} = -b_{2i\mu} = b_{i\mu}$ in the gauge $\gamma_i = 0$. Then, in the continuum limit with $z_1 \sim e^{i\theta_1}$ and $z_2 \sim e^{-i\theta_2}$, and $\eta_1 = \eta_2 = 0$, we obtain the action of the Abelian $\mathbb{CP}^1$ model in (2.22). Other values of η_1, η_2 can now be used to establish the duality mappings of the operator insertions, and we note typical examples

  (i) $\eta_1 = 1$, $\eta_2 = 0$: This establishes the equality between the correlators in (2.34) and (2.35) upon applying (2.8).

  (ii) $\eta_1 = 1$, $\eta_2 = -1$: This monopole flux couples to the global $\mathcal{Q}_2$ charge of (2.28), and so corresponds

to operator insertions with $\tilde{\mathcal{Q}}_2 = 2\pi$, and all other electric and magnetic charges equal to zero. The above analysis shows that this is the two-point correlator of $z_1 z_2 \mathcal{M}_b^2$, corresponding to the operator identification in (2.32).

(iii) $\eta_1 = 1$, $\eta_2 = 1$: This is a monopole gauge flux in the a_μ gauge field, and so via (2.30) only carries $\mathcal{Q}_1 = 2$ electrical charge. Above we find the correlator $z_2^* z_1$, which carries the expected charge, as in (2.31).

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