Analytic study for the string theory landscapes via matrix models

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We demonstrate a first-principles analysis of the string theory landscapes in the framework of noncritical string/matrix models. In particular, we discuss nonperturbative instability, decay rate, and the true vacuum of perturbative string theories. As a simple example, we argue that the perturbative string vacuum of pure gravity *is* stable but that of Yang-Lee edge singularity is inescapably a false vacuum. Surprisingly, most perturbative minimal string vacua are unstable, and their true vacuum mostly does not suffer from nonperturbative ambiguity. Importantly, we observe that the instability of these tachyon-less closed string theories is caused by ghost D-instantons (or ghost ZZ-branes), the existence of which is determined only by nonperturbative completion of string theory.

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I. ANALYTIC ASPECTS OF THE STRING THEORY LANDSCAPE

The string theory landscape is a space of vacua in string theory, which hopefully includes the standard model in four dimensions. Despite of its importance, its progress is mainly in its statistical aspects [1], and little is known about analytic structures of the space. By "analytic structures" we mean general interrelationship among distinct perturbative string-theory vacua. Therefore, appearance of vacua in the landscape, relative stability/decay rate of vacua, and identification of the true vacuum are included. However, these aspects are generally far from our understanding. Therefore, within solvable noncritical string theory [2], we here discuss the analytic structures of the string theory landscapes and try to extract underlying essential structures that would be shared with critical string theory. In particular, we demonstrate how these aspects can be extracted by the nonperturbative completion of string theory. This is the purpose of this short paper.

The free energy of perturbative string theory, $\mathcal{F}(g)$, is an asymptotic series and is calculated from world-sheet conformal field theory [3],

$$\mathcal{F}(g)\simeq\sum_{n=0}^{\infty}g^{2n-2}\mathcal{F}_n+\sum_I\theta_I g^{\gamma_I}\exp\left[\frac{1}{g}\sum_{n=0}^{\infty}g^n\mathcal{F}_n^{(I)}\right]+\mathcal{O}(\theta^2). \tag{1}$$

The nonperturbative corrections are usually provided by D-instantons, i.e., their leading contributions, $\mathcal{F}_0^{(I)}$, are identified as D-instanton action $\mathcal{S}_I=-\frac{1}{g}\mathcal{F}_0^{(I)}$ [4,5]. The overall coefficient θ_I for each instanton is called D-instanton fugacity [6–8], which has no corresponding world-sheet observable. Usually, we assume that the D-instanton action is positive, $\mathcal{S}_I>0$. However, a negative-action partner of the instanton, $\mathcal{S}_{I_{\rm gh}}=-\mathcal{S}_I<0$, has also been observed [9] in noncritical string theory. They are then generally defined as ghost D-branes (or ghost D-instantons) in (non) critical string theory [10]. However, such a D-brane was not seriously taken into account, since it contradicts perturbation theory. Existence of the D-branes is discussed very recently, mainly in resurgent analysis [11–13], and it was found that these branes must be generally encoded in nonperturbative completion of string theory.[1] In this paper, we shall see how these ghost D-instantons play a role in formulating "analytic structures" of the string theory landscape.

Since the actions of ghost D-instantons are negative (their masses are negative), they are no longer "corrections" to perturbation theory; they are rather indications of the nonperturbative instability of the perturbative vacuum [12]. However, this is a cause of confusion because "in principle, the ghost partner is defined for every D-brane, but it does not necessarily mean that the string theory is unstable." In fact, it is nontrivial to know which ghost D-instantons are allowed (or not allowed) to appear in the spectrum. Naively, this information is given by the physics of the D-instanton fugacity $\{\theta_I\}_I$. However, it is subtle to directly deal with $\{\theta_I\}_I$, since they are coefficients of exponentially small corrections which are supposed to be negligible in asymptotic expansions (see, e.g., Ref. [15]). Therefore, we should first grasp all the information on D-instanton fugacity. In the following, we shall see that

[1]It is shown that "multi instanton-ghost-instanton sectors" have discrepancy with world-sheet predictions in the sense of $\mathcal{F}^{(n|m)}\neq\mathcal{F}^{(n-m|0)}$ [12,13], and this is a main objection to identifying them as "ghost D-branes." However, we insist on using the terminology because, according to the free-fermion analysis [7,14], multi-ghost-instanton sectors $\mathcal{F}^{(0|m)}$ are simply obtained by flipping the sign of the ZZ-brane boundary state operators in the multi-instanton sectors $\mathcal{F}^{(m|0)}$ in all-order perturbation theory.

once one can control the information of D-instanton fugacity, one can quantitatively extract most of the analytic aspects of the string theory landscapes, including metastability, decay rates, and the true vacuum.

The completion and fugacity. It is known that perturbative amplitudes, including instanton corrections, in various solvable string theories are obtained by the information of spectral curves, especially with topological recursions [16,17]. In particular, an all-order asymptotic expansion of Eq. (1) is explicitly shown in Ref. [17], with fugacity remaining free parameters. Then, for completion of the nonperturbative information in the asymptotic expansion, there are two main ways studied to control fugacity: one is resurgent analysis (e.g., Refs. [11–13,15]) and the other is isomonodromy analysis (for mathematical developments on isomonodromy theory, see Refs. [18–20] and with matrix models, see Refs. [21–25]). The former is based on the connection formula (or Stokes phenomena) for analytic continuation of g, and the latter is based on Stokes phenomena of the Baker-Akhiezer (BA) functions on the spectral curves. Here we explore analytic aspects of the landscape from the latter approach.

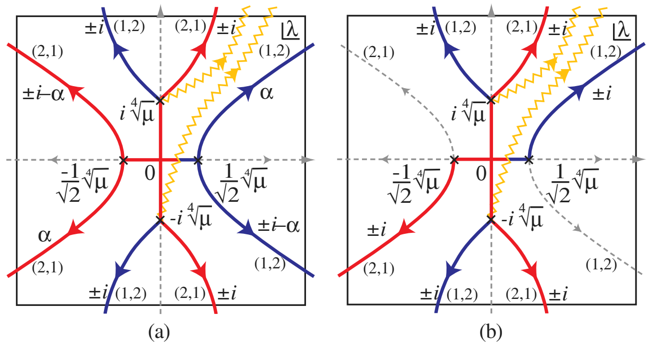


FIG. 1 (color online). The Deift-Zhou network for pure gravity. (a) General solutions to the nonperturbative completion with two-cut boundary condition. (b) A solution with the single-line condition. There is also another solution obtained by the reflection with respect to real axes.

II. RIEMANN-HILBERT PROBLEM FOR THE BAKER-AKHIEZER FUNCTIONS

For a given spectral curve $F(P, Q) = 0$ with a symplectic coordinate (P, Q), we define the BA function as follows:

1) We define a function $\varphi(\zeta)$ (called the string background) as

$$\varphi(\zeta) = \underset{1\le j\le k}{\mathrm{diag}}(\varphi^{(j)}(\zeta)), \qquad \varphi^{(j)}(\zeta) = \int^{\zeta} dP Q^{(j)}(P), \tag{2}$$

where $\{Q^{(j)}(P)\}_{j=1}^{k}$ are branches of the algebraic equation, the number of which is an integer, k. The function $\varphi(\zeta)$ is a rational function on the curve and, without loss of generality, it may have poles at $\zeta = \infty$ and $\zeta = \zeta_a$ ($a = 1, 2, \ldots, M-1$) in the following sense:

$$\varphi(\zeta) \sim \sum_{n=1}^{r_0} \varphi_{-n}\lambda^n + \mathcal{O}\Big(\frac{1}{\lambda}\Big), \qquad \zeta = \lambda^{\hat p_0} \to \infty,$$
$$\varphi(\zeta) \sim \sum_{n=1}^{r_a} \frac{\varphi_{-n}(\zeta_a)}{\lambda^n} + \mathcal{O}(\lambda), \qquad \zeta = \zeta_a + \lambda^{\hat p_a} \to \zeta_a, \tag{3}$$

with $a = 1, 2, \ldots, M-1$. Here $\{\hat p_a\}_{a=0}^{M-1}$ are proper integers and $\{r_a\}_{a=0}^{M-1}$ are the Poincaré indices. The BA function $\Psi(\zeta)$ is then a $k \times k$ matrix-valued sectional holomorphic function of $\zeta \in \mathbb{C}^* \backslash \mathcal{K}$ as

$$\Psi(\zeta) = Z(\zeta) e^{\varphi(\zeta)} \prod_{a=0}^{M-1} (\zeta - \zeta_a)^{\nu_a/\hat p_a} \equiv Z(\zeta) e^{\bar\varphi(\zeta)}, \tag{4}$$

where $\mathcal{K}$ is a collection of connected line elements, $\mathcal{K} = \bigcup_m \mathcal{K}_m$, equipped with a direction (e.g., Figs. 1 and 2), and $\zeta_0 = 0$. We often put $\hat p_0 = \hat p$, $r_0 = r$, and $\nu_0 = -\nu$.

Note that the line elements of the graph $\mathcal{K}$ flow from the poles of $\varphi(\zeta)$ (i.e., essential singularities of the BA function) and are drawn along anti-Stokes lines, $\mathrm{Re}[(\varphi^{(j)}(\zeta) - \varphi^{(l)}(\zeta))e^{-i\theta}] = 0$, with a proper θ so that the graph $\mathcal{K}$ attaches to saddle points, $\partial_\zeta(\varphi^{(j)}(\zeta) - \varphi^{(l)}(\zeta)) = 0$.

2) For each segment of the graph, $\mathcal{K}_m$, a $k \times k$ matrix S_m is assigned (which is called a Stokes matrix) and then the BA function has discontinuity along the segment,

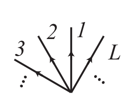
$$\Psi(\zeta + \epsilon) = \Psi(\zeta - \epsilon) S_m, \qquad \zeta \in \mathcal{K}_m, \tag{5}$$

where ϵ directs to the left-hand side of the segment $\mathcal{K}_m$. In particular, at the poles of $\varphi(\zeta)$, there are a number of lines [as in Eq. (7)] and the Stokes matrices are defined so that the BA function has standard asymptotic expansion around them,

$$\Psi(\zeta) \underset{\text{asym}}{\simeq} \Big[I_k + \sum_{n=1}^{\infty} \frac{Z_n}{\lambda^n}\Big] e^{\bar\varphi(\zeta)} \quad (\zeta = \lambda^{\hat p} \to \infty),$$
$$\Psi(\zeta) \underset{\text{asym}}{\simeq} \Big[I_k + \sum_{n=1}^{\infty} \lambda^n Z_n(\zeta_a)\Big] e^{\bar\varphi(\zeta)} E_a \quad (\zeta = \zeta_a + \lambda^{\hat p_a} \to \zeta_a), \tag{6}$$

with $\det E_a \neq 0$ and $a = 1, 2, \ldots, M-1$. Note that the expansion (6) does not depend on the direction of $\zeta \to \zeta_0$, and therefore this requires the standard form for the Stokes matrices (5).

Note that if one flips the direction of a line $\mathcal{K}_m$, then the matrix is replaced by its inverse, S_m^{-1}. At a junction of lines, they satisfy a conservation equation,



$$\Leftrightarrow \qquad S_1 S_2 \cdots S_L = I_k. \tag{7}$$

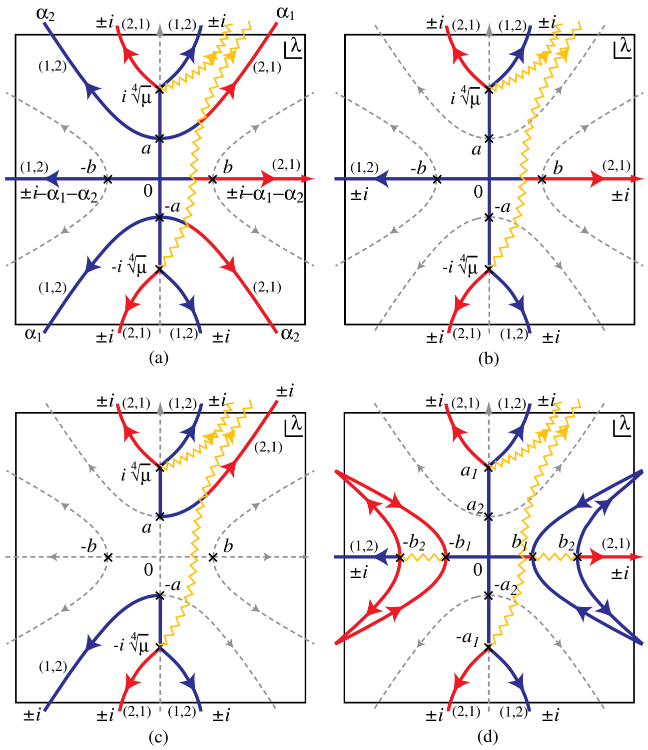


FIG. 2 (color online). The Deift-Zhou network for the Yang-Lee edge. Here there are two saddle points in the string background $\varphi(\lambda)$: $a = i\sqrt{\sqrt{\mu}(\sqrt{5}-1)/4}$ and $b = \sqrt{\sqrt{\mu}(\sqrt{5}+1)/4}$. (a) General solutions to the completion with two-cut boundary condition. (b) The solution with the single-line condition and Hermiticity condition. (c) Deformation of network around the essential singularity. This deformation changes the theory but helps us to obtain decay rates of perturbative string theory. (d) Deformation of spectral curve keeping the same Stokes data as (b). This gives the true vacuum if there is no large instanton along the network.

This is the basic algebra of the Stokes matrices. Obtaining explicit solutions to the algebra is a first nontrivial preparation for the Riemann-Hilbert (RH) calculus, and some recent progress for general k and r can be found in Refs. [24,25].

Importantly, the Stokes matrices S_m in Eq. (5) are independent from ζ of $\mathcal{K}_m$, which means that the graph $\mathcal{K}$ is topological and one can deform it continuously. In addition, we assume that the Stokes matrices are independent from the deformations of the leading Laurent coefficients in Eq. (3), i.e., $\{\varphi_{-n}\}_{n=1}^{r}$ and $\{\varphi_{-n}(\zeta_a)\}_{n=1}^{r_aM-1}{}_{a=1}$. They are the isomonodromy deformations that guarantee the integrable hierarchy (e.g., KP/Toda hierarchy according to the spectral curves [26]) and string equations of the system, which means that perturbative results coincide with that of the topological recursions. The partition function of matrix models is then given by the τ function of the integrable hierarchy (see, e.g., Ref. [27]).

3) The sectional holomorphic function $Z(\zeta)$ is uniquely fixed by giving the Stokes matrices of jump relations (5). In fact, $Z(\zeta)$ is calculable by solving the Riemann-Hilbert integral equation (see, e.g., Ref. [20]) around $\zeta = \lambda^{\hat{p}} \to \infty$,

$$\tilde{Z}(\lambda) = I_k + \int_{\mathcal{K}} \frac{d\xi}{2\pi i} \frac{\tilde{Z}(\xi - \epsilon)(G(\xi) - I_k)}{\xi - \lambda}, \tag{8}$$

along $\mathcal{K}$. Here $\tilde{Z}(\lambda) \equiv Z(\lambda^{\hat{p}})$ and $G(\lambda)$ is a sectional holomorphic function along $\mathcal{K}$, defined by $G(\lambda) = e^{\bar{\varphi}(\lambda^{\hat{p}})} S_m e^{-\bar{\varphi}(\lambda^{\hat{p}})}$ $(\lambda \in \mathcal{K}_m;\ m = 0, 1, \ldots)$.

We should note that by this procedure, one observes that fugacity is given by Stokes multipliers of $\{S_m\}_m$ mostly related linearly (see e.g., Ref. [20]). Importantly, this allows us to obtain the connection rules for analytic continuation of integrable flows (including string coupling g). In this sense, the information in the graph and matrices $\hat{\mathcal{K}} \equiv \bigcup_m (\mathcal{K}_m, S_m)$ has all the information of the D-instanton fugacity of Eq. (1). We shall refer to $\hat{\mathcal{K}}$ as the Deift-Zhou (DZ) network [28], following the recent naming fashion [29]. This is how we control the fugacity.

4) In the RH approach, if one fixes the integrable flows $\{\varphi_{-n}(\zeta_a)\}_{n=1}^{r_aM}{}_{a=1}$ and Stokes matrices $\{S_m\}_m$, every piece of information is determined. In particular, the BA function $\Psi(\zeta)$ is not changed by any deformations of the spectral curve, $F(P, Q) = 0 \to \tilde{F}(P, Q) = 0$ (i.e., $\varphi(\zeta) \to \tilde{\varphi}(\zeta)$), such that the resulting string-background $\tilde{\varphi}(\zeta)$ of Eq. (2) does not change the singular structure (3). In other words, this is just a matter of how to divide the BA function $\Psi(\zeta)$ into $Z(\zeta)$ and $\bar{\varphi}(\zeta)$. In this sense, the RH approach can be interpreted as an (off-shell) background independent formulation of string theory [24].

By this fact, we define the string theory landscape $\mathfrak{L}_{\rm str}$ by the moduli space of spectral curves $F(P, Q) = 0$ which preserves the pole structure (i.e., integrable flows) of $\varphi(\zeta)$ in Eq. (3). Schematically, we define it as a set of string-background $\varphi(\zeta)$,

$$\mathfrak{L}_{\rm str} = \{\varphi(\zeta);\ \text{keeping Eq.(3)}\}, \tag{9}$$

and the potential of the landscape is determined by the RH integral equation (8).

Note that the background independence of noncritical string theory was first explicitly shown in the topological recursions [17], i.e., within perturbation theory; however, for the potential picture of the landscape we need to know the nonperturbative completion of the string theory. In the following, as a first nontrivial example for the analytic aspects of the landscape, we shall show how metastability, decay rates, and the true vacuum are obtained with the information on the fugacity/network which are controlled as above.

III. CASES OF MINIMAL STRING THEORY

We now consider minimal string theory [2] as an example described by matrix models/spectral curves [30]. The spectral curve of (p, q) minimal string theory is given by

$$F(\zeta, Q) = T_p(Q/\beta\mu^{q/2p}) - T_q(\zeta/\sqrt{\mu}) = 0, \tag{10}$$

with Chebyshev polynomials of the first kind $T_n(\cos\theta) = \cos n\theta$ [9,31]. Therefore, $\varphi(\zeta)(\equiv \varphi_{\rm mstr}(\zeta))$ in this background is given as $\varphi^{(j)}_{\rm mstr}(\zeta)=\varphi^{(1)}_{\rm mstr}(e^{-2\pi i\frac{j-1}{p}}\zeta)$ with $\varphi^{(1)}_{\rm mstr}(\zeta) = \beta\mu^{\frac{q+2}{4}}\int^{\zeta/\sqrt{\mu}} dx T_{q/p}(x)$. That is, (p, q) minimal string theory is $p \times p$ isomonodromy systems (i.e., $k = p$) with only one essential singularity at $\zeta = \infty$ of Poincaré index $r = p + q$ [we put $\zeta = \lambda^p$ in Eq. (3)]. We put the monodromy ν_0 as $\nu_0(= -\nu) = -\frac{p-1}{2}$, and this background also preserves $\mathbb{Z}_p$ symmetry in the sense of Chan *et al.* [24]. Here, following the discussion in Ref. [24], we use the same notation.

Around the singularity $\zeta \to \infty$, there are $2rp$ Stokes matrices $\{S_n\}_{n=0}^{2rp-1}$ of $p \times p$, and their algebraic relations [20,24] are expressed as

- $\mathbb{Z}_p$-symmetry condition: $S_{n+2r} = \Gamma^{-1} S_n \Gamma$ $(n = 0, 1, \ldots, 2rp - 1)$,
- Monodromy condition: $S_0 S_1 \ldots S_{2rp-1} = e^{\pi i(p-1)} I_p$,
- Hermiticity condition: $S_n^* = \Delta\Gamma S^{-1}_{(2r-1)p-n}\Gamma^{-1}\Delta$ $(n = 0, 1, \ldots, 2rp - 1)$,

with $\Gamma = (\Gamma_{ij})_{1\le i,j\le p} = (\delta_{j,i+1} + \delta_{i,p}\delta_{j,1})_{1\le i,j\le p}$ and $\Delta = (\Delta_{ij})_{1\le i,j\le p} = (\delta_{i+j,p+1})_{1\le i,j\le p}$. For components of the matrices $\{S_n\}_{n=0}^{2pr-1}$, one should consult [24].

In addition, we consider the cases related to matrix models. The corresponding conditions for the Stokes matrices are known as the multicut boundary condition [24]. In particular, in the case of (p, q) minimal string theory, the constraint is the same as p-cut critical points of the multicut matrix models [32–35].[2] A major difference from the previous cases [24,25] is, however, that the Poincaré index is greater than the number of cuts,

$$r(= p + q) > k(= p), \tag{11}$$

which greatly simplifies the quantum integrable structure of the condition [25]. Therefore, for simplicity, we consider below the cases of $p = 2$, i.e., one-matrix models. Then we can completely solve the conditions, and the Stokes matrices are given with $(L = 1, 2, \ldots)$ as

(1) $r = q + 2 = 4L + 1$ *cases* $(m = 1, 2, \ldots, L)$

$$S_{4m-5} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad S_{4m-3} = \begin{pmatrix} 1 & \alpha_m \\ 0 & 1 \end{pmatrix}$$
$$S_{4L-1} = \begin{pmatrix} 1 & 0 \\ \pm i & 1 \end{pmatrix}, \qquad S_{4L+1} = \begin{pmatrix} 1 & \pm i \\ 0 & 1 \end{pmatrix}$$
$$S_{4(2L-m)+3} = \begin{pmatrix} 1 & 0 \\ \alpha_{-m} & 1 \end{pmatrix}, \qquad S_{4(2L-m)+5} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

with $\sum_{m=1}^{L}(\alpha_m + \alpha_{-m}) = \pm i$.

(2) $r = q + 2 = 4L + 3$ *cases* $(m = 0, 1, \ldots, L)$

$$S_{4m-3} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad S_{4m-1} = \begin{pmatrix} 1 & 0 \\ \alpha_m & 1 \end{pmatrix}$$
$$S_{4L+1} = \begin{pmatrix} 1 & \pm i \\ 0 & 1 \end{pmatrix}, \qquad S_{4L+3} = \begin{pmatrix} 1 & 0 \\ \pm i & 1 \end{pmatrix}$$
$$S_{4(2L-m)+5} = \begin{pmatrix} 1 & \alpha_{-m} \\ 0 & 1 \end{pmatrix},$$
$$S_{4(2L-m)+7} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

with $\alpha_0 + \sum_{m=1}^{L}(\alpha_m + \alpha_{-m}) = \pm i$.

Note that $S_{2m} = I_2$ $(m \in \mathbb{Z})$ and the remaining Stokes matrices are obtained by the $\mathbb{Z}_p$-symmetric condition. In addition, the Hermiticity condition is given as $\alpha_m = -\alpha^*_{-m}$. In the following, we show the Deift-Zhou networks and results of the RH calculus in the $p = 2$ cases.

A. Metastability and decay rates of string theory

1. $(p, q) = (2, 3)$ *minimal strings: Pure gravity*

Nonperturbative completion in the pure-gravity case is studied from various points of view [6,11,22]. The DZ network of the spectral curve is drawn in Fig. 1(a). Note that the Hermiticity condition, $\alpha - \alpha^* = \pm i$, requires that the free energy is a real function. On the other hand, it is known that nonperturbative solutions of the matrix models should break Hermiticity [6]. This situation is recovered if one discards the Hermiticity condition and insists that contours in the DZ network attached to saddle points (i.e., D-instantons) should be a single line with a uniform Stokes multiplier [see Fig. 1(b)]. We here refer to this condition as the single-line condition, which corresponds to the standard solutions from matrix models. In fact, the result is the following $(g \to 0,\ \mu > 0)$:

$$\mathcal{F} \simeq_{\rm asym} \left[-\frac{4}{15}\frac{\mu^{\frac{5}{2}}}{g^2} + \mathcal{O}(g^0)\right] + \frac{\mp i}{2}\left[\frac{\sqrt{g}(1+\mathcal{O}(g))}{8\sqrt{3^{\frac{3}{2}}\pi}\mu^{\frac{5}{8}}}\right]e^{-\frac{8\sqrt{3}}{5g}\mu^{\frac{5}{4}}} \tag{12}$$

and gives one half of the fugacity obtained in Ref. [8]. This value is natural for the same reason as Ref. [36] and has also been argued in Ref. [37]. Here $u = g^2\mathcal{F}''(t)$, $g^2u'' + 6(u^2 + t) = 0$ and $\mu = -t$.

This result suggests an important implication: (Anti-Stokes) lines of the DZ networks correspond to "the mean field path-integral of the many-eigenvalue system" (discussed in Ref. [6]). Therefore, the DZ networks are a remnant of the path-integral in string theory. If this is so, with the standard prescription in QM/QFT systems [36], one expects that metastability and decay rates can be discussed in string theory. For further discussions, we

[2]The constraint requires "p cuts" (not one cut) around $\lambda \to \infty$ in the resolvent function of this $p \times p$ isomonodromy system because the spectral parameter $\lambda(= \zeta^{1/p})$ creates p copies of the physical cuts in a $\mathbb{Z}_p$-symmetric way.

consider the minimal string of the Yang-Lee edge, $(p, q) = (2, 5)$.

2. $(p, q) = (2, 5)$ *minimal strings: Yang-Lee edge*

The DZ network of the spectral curve is drawn in Fig. 2(a). In this case, the Hermiticity condition and the single-line condition are consistent with each other [Fig. 2(b), as expected from matrix models]. However, the RH integral picks up an exponentially large instanton contribution from the second saddle point ($\lambda = \pm b$ in Fig. 2), $\mathcal{F} \underset{\text{asym}}{\cong} \mathcal{F}_{\text{pert}}(g;\mu) + \mathcal{F}_{\text{nonpert}}(g;\mu)$,

$$\mathcal{F}_{\text{nonpert}}(g;\mu) = \mp \frac{\sqrt{g/2}\exp\Big[+\frac{10}{21g}\sqrt{2(5-\sqrt{5})}\mu^{\frac{7}{4}}\Big]}{\sqrt{5\pi(2\sqrt{5})^{\frac{3}{2}}(\sqrt{5}+1)^{\frac{5}{2}}\mu^{\frac{7}{8}}}} + \cdots, \tag{13}$$

with $u = g^2\partial_t^2\mathcal{F}(t;\mu)$, $0 = 2t + 2(g/2)^4u'''' - 5\mu u + 5(gu'/2)^2 + 10(g/2)^2uu'' + 5u^3$, and $t \to 0$. This is the standard contribution from the (1, 2) ghost ZZ-brane (see also Refs. [14,38]).[3] Note that $\mathcal{F}(g)$ is a real function and here is shown as the leading one-instanton contribution, and the multi-instanton contributions have stronger exponential behavior. By this, we conclude that the Yang-Lee edge perturbative string vacuum is unstable. More precisely, since there is no tachyon in its perturbative spectrum, this string theory vacuum is metastable.

Generally metastable vacua in quantum systems have an important characteristic by decay rate.[4] Here, with use of the network, i.e., the path-integral degree of freedom in string theory, we calculate the decay rate by applying the prescription of Coleman [36]. The way is to deform the path/network to avoid the instability [now by discarding the Hermiticity condition while keeping the single-line condition as in Fig. 2(c)],

$$\mathcal{F}(g;\mu) \overset{\text{deform}}{\to} \mathcal{F}^{(\text{def})}(g;\mu) \underset{\text{asym}}{\cong} \mathcal{F}_{\text{pert}}(g;\mu) + \mathcal{F}^{(\text{def})}_{\text{nonpert}}. \tag{14}$$

Then its imaginary part is the decay rate of the Yang-Lee edge string vacuum (g, $\mu > 0$),

$$\mathrm{Im}\mathcal{F}^{(\text{def})}_{\text{nonpert}} = \frac{\mp 1}{2}\frac{\sqrt{g/2}\exp\Big[-\frac{10}{21g}\sqrt{2(5+\sqrt{5})}\mu^{\frac{7}{4}}\Big]}{\sqrt{5\pi(2\sqrt{5})^{\frac{3}{2}}(\sqrt{5}-1)^{\frac{5}{2}}\mu^{\frac{7}{8}}}} + \cdots. \tag{15}$$

This is one half of the standard (1, 1) ZZ-brane contribution (see also Ref. [38]).

Generally, one can see that the $(2, q)$ minimal string theory is metastable: The string theory with Hermiticity has ghost $(1, 2m)$ ZZ-branes $(m = 1, 2, \ldots, (q-3)/2)$, which render the vacuum unstable, and the decay rate is given by half of the (1, 1) ZZ-brane. String theory of pure gravity, (2, 3), is an exception, since there is no ghost ZZ-brane in its background and therefore it is a stable vacuum. In this way, we have shown that the physics of the landscape is given by the networks/fugacity which control the spectrum of (ghost) D-instantons. Importantly, this example shows that different networks [Figs. 2(b) and 2(c)], i.e., different fugacities, may represent different physical situations of the same theory.

B. The true vacuum and landscape of string theory

Since the Yang-Lee edge string theory is metastable, there is the true vacuum, into which the string theory decays. For that purpose, we choose the spectral curve in the landscape,

$$\varphi_{\text{tv}}(\zeta) \in \mathfrak{L}_{\text{str}} \qquad (\varphi_{\text{mstr}}(\zeta) \in \mathfrak{L}_{\text{str}}), \tag{16}$$

in such a way that there are large instantons along the Deift-Zhou network of the RH integral (8). Roughly speaking, the instanton actions on the saddle points λ_* $(\partial_\lambda[\varphi^{(j)}(\lambda_*^p) - \varphi^{(l)}(\lambda_*^p)] = 0)$ should be positive real,

$$\mathrm{Re}[\varphi^{(j)}(\zeta_*) - \varphi^{(l)}(\zeta_*)] < 0, \tag{17}$$

if the corresponding lines of the network $\mathcal{K}_m$ (with nonzero Stokes multiplier $s_{m,j,l} \neq 0$) are attached to the saddle point λ^*. In the current case $p = 2$ is eventually equivalent to the vanishing condition around the B-cycle on the network $\mathcal{K}$,

$$\oint_B d\zeta\,\partial_\zeta[\varphi^{(1)}_{\text{tv}}(\zeta) - \varphi^{(2)}_{\text{tv}}(\zeta)] = 0 \qquad B \subset \mathcal{K}; \tag{18}$$

this is known as the Boutroux equations in the RH context [20]. This type of condition has also been discussed in previous literature on matrix models [6]. This simply means that the eigenvalues should fill up to the same Fermi level of the effective potential along the DZ network. Here we simply show the result,

$$\zeta = \sqrt{\mu}(\wp(z) + c), \qquad \partial_\zeta\varphi^{(1)}_{\text{tv}}(\zeta^{1/2}) = \sqrt{2\mu^{\frac{5}{2}}}(\wp(z) - \alpha)\wp'(z). \tag{19}$$

Here the Weierstrass $\wp$ function is given by $(\wp'(z))^2 = 4(\wp(z))^3 - g_2\wp(z) - g_3$. The normalization of the system is now fixed as $\alpha = \frac{5}{2}c$, $g_2 = 5(1–3c^2)$, $g_3 = 5c(2–7c^2)$ so that the corresponding string-background $\varphi_{\text{tr}}(\zeta)$ belongs to the landscape $\mathfrak{L}_{\text{str}}$. Therefore, the parameter c is a coordinate of the string theory landscape of the Yang-Lee edge,

[3] For more about ZZ-branes [39], see also Ref. [9].

[4] Here we define decay rates by the imaginary part of "energy" of the metastable states in the following sense: $e^{\mathcal{F}} = \langle\text{vac}|e^{-TH}|\text{vac}\rangle \sim e^{-TE_{\text{vac}}}(T \to \infty)$, $E_{\text{vac}} = E + i\Gamma_{\text{vac}}$. Since minimal string theory is a Euclidean theory with "compact Euclidean time," our decay rate is simply given by the imaginary part of the free energy of the metastable vacuum. Therefore, this definition/terminology can be easily generalized to the Lorentzian situations of string theory.

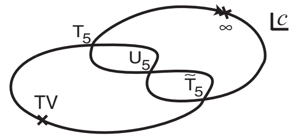


FIG. 3. One-parameter landscape of Yang-Lee edge string theory. The coordinate is given by c. T_5, $\tilde{T}_5$ and U_5 are backgrounds that are described by spectral curves with genus zero. The value of c for each vacuum is $c = \frac{1\pm\sqrt{5}}{6}$, $\frac{-1\pm\sqrt{5}}{6}$, $\frac{\pm\sqrt{5}}{3\sqrt{2}}$, respectively.

and the true-vacuum condition is expressed with the Weierstrass elliptic functions,

$$\left[\frac{4g_2}{5}\zeta_W(\omega_B) - \frac{6g_3\omega_B}{5}\right]\alpha = \frac{6g_3}{7}\zeta_W(\omega_B) - \frac{g_2^2\omega_B}{21}, \quad (20)$$

where ω_B is the Weierstrass half period along the B-cycle, and $\zeta_W(z)$ is the Weierstrass ζ function, $\zeta_W'(z) = -\wp(z)$. The numerical value of c is given as $c \simeq -0.184963725\ldots$. Then the perturbative amplitude around this true vacuum is obtained (by the RH approach with the network of Fig. 2(d) as

$$u(\mu) \simeq -\sqrt{\mu}(\wp(\omega_A) + \wp(\omega_B) - \wp(\omega_A + \omega_B) + c). \quad (21)$$

Here ω_A is the half period of the A-cycle.

The perturbative structure around the true vacuum does not receive any contributions from nonperturbative ambiguities, which is a result of universality. Note that since the expression includes elliptic functions, this vacuum represents a nonperturbative vacuum whose classical dynamics would not be stringy degrees of freedom, although quantum corrections still resemble the closed-string behavior g^{2n-2}. It would be worth drawing the string theory landscape with the parameter c (Fig. 3). The pinched points correspond to perturbative-string vacua (T_5, $\tilde{T}_5$, and U_5) in which the perturbative amplitudes have simply the power-scale behavior. T_5 is the original minimal-string vacuum (10). Note that not all the vacua have a simple interpretation by matrix models and, therefore, most of the vacua are an off-shell background of this nonperturbative string theory.

These analyses can clearly be generalized to many other systems. Further investigations including general (p, q) cases will be reported in a future paper [40].

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[1] M. R. Douglas, J. High Energy Phys. 05 (2003) 046 (for recent survey).
[2] A. M. Polyakov, Phys. Lett. **103B**, 207 (1981).
[3] M. B. Green, J. H. Schwarz, and E. Witten, *Superstring Theory* (Cambridge University Press, Cambridge, England, 1987), Vol. 1, p. 469 and Vol. 2, p. 596; J. Polchinski, *String Theory* (Cambridge University Press, Cambridge, England, 1998), Vol. 1, p. 402 and Vol. 2, p. 531.
[4] S. H. Shenker, in *Cargese 1990, Proceedings, Random Surfaces and Quantum Gravity* (Plenum, New York, 1990), p. 191; J. Polchinski, Phys. Rev. D **50**, R6041 (1994); Phys. Rev. Lett. **75**, 4724 (1995).
[5] J. McGreevy and H. L. Verlinde, J. High Energy Phys. 12 (2003) 054; E. J. Martinec, arXiv:hep-th/0305148; I. R. Klebanov, J. M. Maldacena, and N. Seiberg, J. High Energy Phys. 07 (2003) 045; S. Y. Alexandrov, V. A. Kazakov, and D. Kutasov, J. High Energy Phys. 09 (2003) 057; V. A. Kazakov and I. K. Kostov, arXiv:hep-th/0403152.
[6] F. David, Phys. Lett. B **302**, 403 (1993).
[7] M. Fukuma and S. Yahikozawa, Phys. Lett. B **396**, 97 (1997); Phys. Lett. B **393**, 316 (1997); Phys. Lett. B **460**, 71 (1999).
[8] M. Hanada, M. Hayakawa, N. Ishibashi, H. Kawai, T. Kuroki, Y. Matsuo, and T. Tada, Prog. Theor. Phys. **112**, 131 (2004).
[9] N. Seiberg and D. Shih, J. High Energy Phys. 02 (2004) 021.
[10] T. Okuda and T. Takayanagi, J. High Energy Phys. 03 (2006) 062.
[11] S. Garoufalidis, A. Its, A. Kapaev, and M. Marino, arXiv:1002.3634.
[12] A. Klemm, M. Marino, and M. Rauch, J. High Energy Phys. 10 (2010) 004.
[13] I. Aniceto, R. Schiappa, and M. Vonk, arXiv:1106.5922.
[14] M. Fukuma, H. Irie, and S. Seki, Nucl. Phys. **B728**, 67 (2005); M. Fukuma, H. Irie, and Y. Matsuo, J. High Energy Phys. 09 (2006) 075.
[15] M. Marino, in *Topological Strings, Modularity and Nonperturbative Physics* (Erwin Schrödinger International Institute for Mathematical Physics, Vienna, 2010).

[16] B. Eynard and N. Orantin, arXiv:math-ph/0702045.
[17] B. Eynard and M. Marino, J. Geom. Phys. **61**, 1181 (2011).
[18] H. Flaschka and A. C. Newell, Commun. Math. Phys. **76**, 65 (1980); M. Jimbo, T. Miwa, and a. K. Ueno, Physica (Amsterdam) **2**, 407 (1981).
[19] A. S. Fokas and M. J. Ablowitz, Commun. Math. Phys. **91**, 381 (1983); A. S. Fokas and M. J. Ablowitz, Phys. Rev. Lett. **47**, 1096 (1981); A. S. Fokas and X. Zhou, Commun. Math. Phys. **144**, 601 (1992).
[20] A. S. Fokas, A. R. Its, A. A. Kapaev, and V. Y. Novokshenov, in *Painlevé Transcendents: The Riemann-Hilbert Approach*, Mathematical Surveys and Monographs Vol. 128 (American Mathematical Society, Providence, 2006), p. 553.A. R. Its and V. Y. Novokshenov, in *The Isomonodromic Deformation Method in the Theory of Painlevé Equations*, Lecture Notes in Mathematics Vol. 1191 (Springer-Verlag, Berlin Heidelberg, 1986), p. 313.
[21] G. W. Moore, Commun. Math. Phys. **133**, 261 (1990); Prog. Theor. Phys. Suppl. **102**, 255 (1990).
[22] A. S. Fokas, A. R. Its, and A. V. Kitaev, Commun. Math. Phys. **147**, 395 (1992); Commun. Math. Phys. **142**, 313 (1991).
[23] J. M. Maldacena, G. W. Moore, N. Seiberg, and D. Shih, J. High Energy Phys. 10 (2004) 020; N. Seiberg and D. Shih, J. High Energy Phys. 01 (2005) 055; C.R. Physique **6**, 165 (2005).
[24] C.-T. Chan, H. Irie and C.-H. Yeh, Nucl. Phys. **B854**, 67 (2012).
[25] C.-T. Chan, H. Irie and C.-H. Yeh, Nucl. Phys. **B855**, 46 (2012).
[26] M. R. Douglas, Phys. Lett. B **238**, 176 (1990); E. J. Martinec, Commun. Math. Phys. **138**, 437 (1991).
[27] M. Fukuma, H. Kawai, and R. Nakayama, Commun. Math. Phys. **143**, 371 (1992).
[28] P. A. Deift and X. Zhou, Ann. Math. **137**, 295 (1993); P. A. Deift and X. Zhou, Commun. Pure Appl. Math. **48**, 277 (1995).
[29] D. Gaiotto, G. W. Moore, and A. Neitzke, arXiv:1204.4824.
[30] E. Brezin and V. A. Kazakov, Phys. Lett. B **236**, 144 (1990); M. R. Douglas and S. H. Shenker, Nucl. Phys. **B335**, 635 (1990); D. J. Gross and A. A. Migdal, Phys. Rev. Lett. **64**, 127 (1990); V. A. Kazakov, Mod. Phys. Lett. A **04**, 2125 (1989).
[31] I. K. Kostov, in *Cargese 1990, Proceedings, Random Surfaces and Quantum Gravity* (Plenum, New York, 1990), p. 135.
[32] C. Crnkovic and G. W. Moore, Phys. Lett. B **257**, 322 (1991).
[33] H. Irie, Nucl. Phys. **B819**, 351 (2009).
[34] C.-T. Chan, H. Irie, S.-Y. D. Shih, and C.-H. Yeh, Nucl. Phys. **B828**, 536 (2010).
[35] C.-T. Chan, H. Irie and C.-H. Yeh, Nucl. Phys. **B838**, 75 (2010).
[36] S. R. Coleman, Subnuclear series **15**, 805 (1979).
[37] Y. Matsuo, Nucl. Phys. **B740**, 222 (2006).
[38] A. Sato and A. Tsuchiya, J. High Energy Phys. 02 (2005) 032.
[39] A. B. Zamolodchikov and Al. B. Zamolodchikov, arXiv: hep-th/0101152.
[40] C.-T. Chan, H. Irie, and C.-H. Yeh (unpublished).