

Casimir force due to condensed vortices in a planeJ. F. de Medeiros Neto,^{1,*} Rudnei O. Ramos,^{2,†} and Carlos Rafael M. Santos^{1,‡}¹*Faculdade de Física, Universidade Federal do Pará, Belém, Pará 66075-110, Brazil*²*Departamento de Física Teórica, Universidade do Estado do Rio de Janeiro, Rio de Janeiro, Rio de Janeiro 20550-013, Brazil*

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The Casimir force between parallel lines in a theory describing condensed vortices in a plane is determined. We make use of the relation between a Chern-Simons-Higgs model and its dualized version, which is expressed in terms of a dual gauge field and a vortex field. The dual model can have a phase of condensed vortices, and, in this phase, there is a mapping to a model of two noninteracting massive scalar fields from which the Casimir force is readily obtained. We also discuss the details concerning the boundary conditions required for the scalar fields and their association with those for the vectorial field. We show that this association is subtle for the case of the transformations considered.

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I. INTRODUCTION

The Casimir effect is a manifestation of the quantum vacuum fluctuations that can be tested at mesoscopic scales. This quantum phenomenon has been of interest to fundamental physics since its prediction by Casimir in 1948 [1], and it has been studied extensively both theoretically and experimentally since then (for some recent reviews on both theory and experiments, see, e.g., Refs. [2,3]). In particular, many experiments have been measuring the Casimir force with increasing precision. It is then of increasing interest to look for possible novel situations where the theoretically computed Casimir force can be confronted to experiments and where related quantum phenomena, associated with the quantum vacuum, can then be probed and tested in the laboratory.

In this work, we want to study how a vacuum state made of topological excitations, more precisely, a vacuum constituted of condensed vortices, will affect the Casimir force between perfectly conducting parallel lines in a plane. Let us recall that stable vortex configurations can appear in important condensed matter systems, like in high-temperature superconductors and superfluids (for a detailed presentation, see, e.g., Ref. [4] and references therein). There has been an increasing use of superconducting materials to study the Casimir effect (see, for example, Ref. [5]). It has also been pointed out in Ref. [6] that unusual behaviors of superconductors may be found when the sizes of the samples shrink. But we note that this is precisely the case, at the nanometer scales, that we expect that the Casimir force to become more appreciable. In the case of vortex-based superconducting detectors [7,8], for instance, it can be expected that the Casimir effect can possibly alter the microscopic parameters of the detector, analogously to the case reported in Ref. [9].

Since superconductors can naturally form condensed phases of vortices, it becomes a matter of interest to investigate how a vacuum state constituted of a condensate of vortex excitations would affect the Casimir force. A vortex condensed phase constitutes a particular example of a nontrivial vacuum state. The Casimir effect being a manifestation of the quantum vacuum, it is then a fundamental problem to investigate how a vacuum state with topological excitations can affect the Casimir force.

The simplest and, in our opinion, the most direct way for studying a vortex condensate state is through the use of dual transformations involving the field variables of the original Lagrangian density. By following this procedure, we can make explicit the system's topological excitations content. In the problem that we study in this work, these topological excitations will be vortex ones. The duality transformations are reminiscent of similar approaches first used in condensed matter studies performed on the lattice [10] and of routine use since then. Through a series of appropriate dual transformations involving the original fields in the functional action, an equivalent action is obtained, in which the vortex excitations are made explicit. By properly matching our dual action to a field theory model, it is then possible to write it in terms of a vortex field coupled to a vectorial field (for earlier implementations of this procedure, see, for example, the work done in Refs. [11–13], and references therein).

Here we investigate the Casimir force for a massive vectorial field in a Maxwell-Proca-Chern-Simons (MPCS) model. Following the work in Refs. [12,13], we show that this model can be seen as the dualized version of a Chern-Simons-Higgs (CSH) model, in which the vortex excitations of the CSH model are made explicit and considered in a vacuum state. Vortex condensation in Chern-Simons (CS)-type theories, particularly in self-dual models, have been shown possible for some critical value of the Chern-Simons parameter [14,15], with the determination of the condensation point explicitly obtained in Ref. [13]. The Casimir force for the dual MPCS type of

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model is studied here in this context, deep inside the vortex condensate phase.

Irrespective of the connection of the dual MPCS model with the CSH model, the study done in this work has an interest of its own, which is associated with the determination of the Casimir force for massive vectorial fields. Recall that the MPCS model represents, by itself, massive photons in $2 + 1$ dimensions, with the photon mass having contributions from the usual Proca and the CS terms. While the mass contribution coming from the Proca term can be seen as having been generated through a symmetry breaking scalar field term, the contribution from the Chern-Simons term is of purely topological origin [16]. The issue of the Casimir force for a vectorial field is closely related to important questions, from both experimental and theoretical points of views. For instance, in the case of $3 + 1$ dimensions, the authors of Ref. [17] have analyzed the existence of new expressions for the electromagnetic field between conducting plates, where the photon has a possibly non-null mass. They then study the dependence of the Casimir force with the photon mass. Later, in Ref. [18], the Proca equations were used to represent the photon mass. In Ref. [9], it was considered the mass acquired by the photon due to the spontaneous symmetry breaking that takes place when a superconducting detector passes from its normal (N) to the superconducting (S) state, as a consequence of the detection of an external photon. In that reference, it was argued that the Casimir effect can alter the S - N transition in a detectable way and to be able to alter the microscopic parameters of the detector. Also considered in Ref. [9] was the viability of describing the Casimir force when the corresponding Maxwell equations are replaced by the Proca ones for massive photons.

The Casimir force between perfectly conducting parallel lines in a plane for a MCS model has been determined previously in Refs. [19–22]. In particular, it has already been shown in Ref. [19] that the Casimir force obtained in the MCS model is identical to that derived from a massive noninteracting scalar field in $2 + 1$ dimensions [23]. This result can be understood from the fact that in both theories the field satisfies the Klein-Gordon equation of motion and both have only 1 degree of freedom. Note that, in this case, the Chern-Simons term provides a mass term for the gauge field, but this is a topological mass that still maintains the field with only 1 (transverse) polarization degree of freedom. Besides, the boundary conditions (BCs) in both models can be matched, making the Casimir force in both models agree. On the other hand, in a symmetry-broken case, a Proca mass term is generated for the gauge field, which acquires a longitudinal polarization degree of freedom, in addition to the transverse polarization. For the case of the MPCS theory, the gauge field now has 2 polarization degrees of freedom. Similarly to the case of the MCS theory, we now expect that the respective Casimir force could be related to *two* massive noninteracting scalar

fields. In fact, it is well known that in this case the quantum mechanical analogue of the MPCS theory is equivalent to two noninteracting harmonic oscillators with distinct frequencies [16]. At the quantum field theory level, this fact must then correspond to the case of two noninteracting massive scalar fields. This has been shown explicitly in Ref. [24], where a mapping between the two theories was constructed. However, in order to associate the corresponding Casimir forces for both theories, a careful consideration of the boundary conditions must be accounted for. This issue was earlier discussed in Refs. [23,25]. Here we will give a detailed account for the issue of the boundary conditions when mapping our dual MPCS theory with vortices with a model corresponding to two noninteracting massive scalar fields. This will allow us then to readily obtain the Casimir force for the model we are studying here.

The remainder of this work is organized as follows. In Sec. II, we introduce the MPCS model as a particular limit of the vortex model considered in Ref. [13] and summarize the relevant equations and relations that will be of relevance for this work. In Sec. III, we discuss the mapping that leads from the initial MPCS theory to a model of two massive and noninteracting scalar fields. We also analyze the respective mapping between the boundary conditions needed for those two models. In Sec. IV, we then derive the Casimir force related to a vacuum state of condensed vortex excitations from the dual MPCS theory considered and contrast the result with the case where vortex excitations are absent. In Sec. V, we give our concluding remarks and discuss possible extensions of our work. Finally, in the Appendix, we give some technical details.

II. THE MPCS THEORY AS A DUAL MODEL FOR VORTICES IN A PLANE

Let us initially consider the CSH model in $2 + 1$ dimensions, written in terms of a complex scalar field and an Abelian gauge field, which here we will represent them by η and h_μ , respectively. The quantum partition function and the action of the model have the forms (in Euclidean space-time and with indices running from 1 to 3)

$$Z = \int \mathcal{D}h_\mu \mathcal{D}\eta \mathcal{D}\eta^* \exp\{-S_E[h_\mu, \eta, \eta^*]\}, \quad (2.1)$$

$$S_E[h_\mu, \eta, \eta^*] = \int d^3x \left[-i \frac{\Theta}{4} \epsilon_{\mu\nu\gamma} h_\mu H_{\nu\gamma} + |D_\mu \eta|^2 + V(|\eta|) \right], \quad (2.2)$$

with $H_{\mu\nu} = \partial_\mu h_\nu - \partial_\nu h_\mu$, $D_\mu \equiv \partial_\mu + ie h_\mu$, and Θ is the CS parameter. $V(|\eta|)$ is a symmetry breaking polynomial potential, independent of the phase of the complex scalar field, and has a non-null vacuum expectation value (VEV) $|\langle \eta \rangle| = \nu \neq 0$. As examples of $V(|\eta|)$, we can cite the

usual quartic order potential $V(|\eta|) = \lambda(|\eta|^2 - \nu^2)^2/4$ and the sixth-order self-dual potential [26]: $V(|\eta|) = e^4(|\eta|^2 - \nu^2)^2|\eta|^2/\Theta^2$. By writing η in a polar form $\eta = (\rho/\sqrt{2})\exp(i\chi)$, the VEV for η becomes $\nu = \rho_0/\sqrt{2}$.

The field equations associated with h_μ and η are known to have a nontrivial solution associated with a vortex field configuration [26]. When expressed in polar coordinates (r, χ) , the nontrivial solution can be put in the generic form that represents charged vortices:

$$\eta_{\text{vortex}} = \xi(r)\exp(in\chi), \quad (2.3)$$

$$h_{\mu,\text{vortex}} = \frac{n}{e}h(r)\partial_\mu\chi, \quad (2.4)$$

where n is an integer that can be interpreted as the vortex topological charge and $(\xi(r), h(r))$ are obtained by numerically solving the classical field differential equations, subjected to the BCs:

$$\lim_{r \rightarrow 0} \xi(r) = 0, \quad \lim_{r \rightarrow \infty} \xi(r) = \nu, \quad (2.5)$$

$$\lim_{r \rightarrow 0} h(r) = 0, \quad \lim_{r \rightarrow \infty} h(r) = 1. \quad (2.6)$$

A vortex represented by Eqs. (2.3) and (2.4) can be seen as carrying an ‘‘electric’’ charge Q (the spatial integral of the 0 component of the density current j_μ) attached to a magnetic flux Φ given by

$$\Phi \equiv \int d^2x H_{12} = \frac{Q}{\Theta}. \quad (2.7)$$

This fact is a direct consequence of the presence of the Chern-Simons term and also implies in an anyonic behavior of the charge-flux composite, which has spin $s = Q\Phi/(4\pi)$ [27]. It can also be demonstrated that, when r approaches infinity (or when it is sufficiently far from the vortex core), the flux Φ becomes quantized. Φ in this case is given by an integer multiple of flux quantum [27]: $\Phi = 2\pi n/e$.

The vortex degrees of freedom present in the original theory Eq. (2.2) can be made explicit through a series of duality transformations [12,13]. The final result is a theory of the form of a Maxwell-Chern-Simons-Higgs (MCSH) model, where the vortex solutions, represented by Eqs. (2.3) and (2.4), are associated with particles represented by a complex scalar field ψ that is coupled to a dual vector field A_μ . The original fields and the dual fields, at the classical level, are related to each other, e.g., by $\rho^2(\partial_\mu\chi + eh_\mu) = (\sigma/e)\epsilon_{\mu\nu\gamma}\partial_\mu A_\nu/(2\pi)$, where σ is an arbitrary parameter with mass dimension. The resulting dual Euclidean action becomes equivalent to a MCSH theory of the form [12,13]

$$S_{\text{dual}} = \int d^3x \left[\frac{\sigma^2}{16\pi^2 e^2 \rho_0^2} F_{\mu\nu}^2 + i \frac{\sigma^2}{8\pi^2 \Theta} \epsilon_{\mu\nu\gamma} A_\mu \partial_\nu A_\gamma + \left| \partial_\mu \psi + i \frac{2\sigma}{e} A_\mu \psi \right|^2 + V_{\text{vortex}}(|\psi|) + \mathcal{L}_G \right], \quad (2.8)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $V(|\psi|)$ is the effective potential term for the vortex field, and \mathcal{L}_G is a gauge fixing term. Note that in the dual model, Eq. (2.8), the new CS coefficient appears inversely proportional to the initial one in Eq. (2.2), $\Theta \rightarrow -1/(4\pi^2\Theta)$. This dualization of the CS coefficient is a consequence of the transformations used (see also Refs. [28,29]).

As argued in Refs. [14,15], there is a critical value for the CS coefficient in the CSH theory, below which vortices are expected to be energetically favorable to condense. In terms of the dual action (2.8), this can be expressed in terms of an existence condition for a VEV for the dual vortex field, given in terms of the first derivative of the potential with respect to the vortex field, $V'_{\text{vortex}}(|\psi| = \psi_0/\sqrt{2}) = 0$, or, analogously, that the quadratic mass term in the vortex potential be negative below some critical Θ_c , with Θ_c determined by the condition on the second derivative of the effective vortex potential with respect to the vortex field, $V''_{\text{vortex}}(\Theta = \Theta_c) = 0$. In Ref. [13], this critical value has been obtained as given by $\Theta_c \simeq (e^2/\pi)\ln 6$ and shown to be robust against quantum corrections, changing by no more than about 17%. In this work, we are interested in deriving the Casimir force starting from the dual action (2.8) considering the case in which vortex condensation is favorable, i.e., for the region of parameters where $\Theta < \Theta_c$.

Since the Casimir force is related to quantum vacuum fluctuation of fields, if we want to determine an expression for that force in the case of the MCSH theory of the form of Eq. (2.8), we can, as an approximation, consider only small variations of the vortex field around its nontrivial constant VEV ψ_0 . In other words, if we are deep inside the vortex condensed phase, fluctuations of the vortex field can be neglected, much like in the London approximation in condensed matter problems [4]. This approximation can then be seen as a limiting case of Eq. (2.8), in which the term $|\partial_\mu \psi + 2i\sigma A_\mu \psi/e|^2$ in Eq. (2.8) gives rise to a Proca-like term (i.e., we are in the vortex symmetry-broken phase in the dual action), which will be written as $m^2 A^\mu A_\mu$. We can also make use of the arbitrariness of σ to rewrite Eq. (2.8) in the form of a MPCS model (see also the Appendix). Considering $\sigma \equiv 2\pi e \rho_0$ and going back to Minkowski space-time, the corresponding MPCS Lagrangian density then becomes

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}m^2 A^\mu A_\mu + \frac{\mu}{4}\epsilon^{\mu\nu\lambda}A_\mu \partial_\nu A_\lambda, \quad (2.9)$$

where in Eq. (2.9), for convenience, we have redefined the parameters as

$$\mu \equiv 2e^2 \rho_0^2 / \Theta, \quad m \equiv 4\pi\rho_0\psi_0. \quad (2.10)$$

The association of a covariant derivative of a field with a mass term for a boson, in the broken vacuum state (for the dual theory in our case), is a well known result. This is very similar to the mechanism of mass generation for photons inside superconductors, which can be explained in terms of a symmetry breaking in the Landau-Ginzburg model for superconductivity [30].

In the next section, the Casimir force related to the theory described by Eq. (2.9) is determined by noticing that it can be mapped to an equivalent model of two non-interacting massive scalar fields, as mentioned in the introduction, and by choosing the appropriate boundary conditions for the corresponding scalar fields and the dual vector field. In the association between the two theories, the two initial massive degrees of freedom of the MPCs model are transposed to 2 degrees of freedom represented by the scalar fields, as should be expected [16].

III. THE EQUIVALENT MODEL AND THE MAPPING BETWEEN THE BOUNDARY CONDITIONS

The MPCs theory given by Eq. (2.9) can be mapped, after a sequence of mathematical transformations, in a model of two noninteracting massive real scalar fields (ϕ and φ) in $2 + 1$ dimensions [24]. Next we will explain the main steps needed for this mapping and that will be useful for setting the respective BCs needed in the calculation of the Casimir force.

A. The MPCs theory equivalence to two noninteracting scalar fields

From Eq. (2.9), the Euler-Lagrange equation for the dual gauge field A_μ is

$$\partial_\mu F^{\mu\alpha} + m^2 A^\alpha + \frac{\mu}{2} \epsilon^{\alpha\rho\beta} F_{\rho\beta} = 0, \quad (3.1)$$

while the canonical momenta are

$$\pi^0 = \frac{\partial \mathcal{L}}{\partial \dot{A}^0} = 0, \quad \pi^i = \frac{\partial \mathcal{L}}{\partial \dot{A}^i} = F^{i0} + \frac{\mu}{2} \epsilon^{ij} A_j, \quad (3.2)$$

where the indexes i and j vary from 1 to 2. The relation $\pi^0 = 0$ is a primary constraint of the model, which also shows a secondary one, given by

$$\partial_i \pi^i + \frac{\mu}{4} \epsilon^{ij} F_{ij} + m^2 A_0 \approx 0. \quad (3.3)$$

The primary and the secondary constraints are directly related to the reduction of the number of degrees of freedom of the system (from 3 to 2). We also note that the secondary constraint permits one to write A_0 in terms of the components A_j . This possibility can be seen as a direct consequence of the fact that the vectorial field mass m is non-null. As a consequence of the constraints, the physical

degrees of freedom of the system are represented by A_i and π_i . The quantum partition function can now be written in the form

$$Z = \int \mathcal{D}A_i \mathcal{D}\pi^i \exp\left[i \int d^3x (\pi^i \dot{A}_i - \mathcal{H}) \right], \quad (3.4)$$

where \mathcal{H} is the physical Hamiltonian density

$$\mathcal{H} = \frac{1}{2} \pi^i K_{ij} \pi^j + \pi^i Q_{ij} A^j + A_i S^{ij} A_j, \quad (3.5)$$

where K_{ij} , Q_{ij} , and S^{ij} are defined, respectively, by

$$K_{ij} = g_{ij} + \frac{\partial i \partial j}{m^2}, \quad (3.6)$$

$$Q_{ij} = \frac{\mu}{2} \left(\epsilon_{ij} + \frac{1}{m^2} \partial_i \tilde{\partial}_j \right), \quad \tilde{\partial}_i = \epsilon_{ij} \partial^j, \quad (3.7)$$

$$S^{ij} = \frac{1}{2} \left(1 + \frac{\mu^2}{4m^2} \right) [(\nabla^2 - m^2) g^{ij} + \partial^i \partial^j]. \quad (3.8)$$

It is important to note that, in order to write the Hamiltonian density in the form Eq. (3.5), the surface terms generated by the integrals of $\partial_i (\pi^i \partial_j \pi^j)$, $\partial_i (\pi^i \epsilon^{jk} F_{jk})$, $\partial^i (A^j \partial_i A_j)$, and $\partial^i (A^j \partial_j A_i)$ are neglected. As we will show below, this can be shown to be indeed the case for the boundary conditions considered here.

Next, we introduce two new variables \tilde{A}_i and $\tilde{\pi}^i$ ($i = 1, 2$), defined by the relations [24]

$$A_1 = (\hat{O}_1^{-1} \tilde{A}_1 - \hat{O}_2^{-1} \tilde{A}_2) / (2\theta), \quad (3.9)$$

$$A_2 = \hat{O}_1 \tilde{\pi}^1 + \hat{O}_2 \tilde{\pi}^2, \quad (3.10)$$

$$\pi^1 = \theta \hat{O}_1 \tilde{\pi}^1 - \theta \hat{O}_2 \tilde{\pi}^2, \quad (3.11)$$

$$\pi^2 = -(\hat{O}_1^{-1} \tilde{A}_1 + \hat{O}_2^{-1} \tilde{A}_2) / 2, \quad (3.12)$$

where

$$\theta = \sqrt{m^2 + \frac{\mu^2}{4}} \quad (3.13)$$

and \hat{O}_1 and \hat{O}_2 are operators whose squares are given, respectively, by

$$\hat{O}_1^2 = \left(-\frac{1}{2} \theta^2 K_{11} - \theta Q_{12} + S^{22} \right)^{-1}, \quad (3.14)$$

$$\hat{O}_2^2 = \left(-\frac{1}{2} \theta^2 K_{11} + \theta Q_{12} + S^{22} \right)^{-1}. \quad (3.15)$$

We note from the above equations that, when acting the operators \hat{O}_1 and \hat{O}_2 on some function [e.g., $\phi(x)$], they cannot be simply written in terms of the derivatives of the function. In Eqs. (3.9) and (3.12), $\tilde{\pi}^i$ and \tilde{A}_i can be seen as intermediate variables, related to the fields $\{\phi, \varphi\}$ and their respective momenta $\{\pi_\phi, \pi_\varphi\}$, as

$$\tilde{\pi}_1 = \frac{1}{\sqrt{2}}\pi_\phi - \sqrt{2}\left(\frac{S^{12}}{\theta} + \frac{Q_{22}}{2}\right)\phi, \quad (3.16)$$

$$\tilde{\pi}_2 = \frac{1}{\sqrt{2}}\pi_\varphi - \sqrt{2}\left(\frac{-S^{12}}{\theta} + \frac{Q_{22}}{2}\right)\varphi, \quad (3.17)$$

$$\tilde{A}_1 = \sqrt{2}\phi, \quad (3.18)$$

$$\tilde{A}_2 = \sqrt{2}\varphi. \quad (3.19)$$

The set of mathematical transformations shown above makes it possible to rewrite the Hamiltonian of the MPCs model as a sum of two separated and independent Hamiltonians associated with two noninteracting scalar fields φ and ϕ , i.e.,

$$\mathcal{H} = \frac{1}{2}[\pi_\phi^2 + \phi(m_1^2 - \nabla^2)\phi] + \frac{1}{2}[\pi_\varphi^2 + \varphi(m_2^2 - \nabla^2)\varphi], \quad (3.20)$$

where

$$m_1 = \theta - \frac{\mu}{2}, \quad m_2 = \theta + \frac{\mu}{2}. \quad (3.21)$$

The relation between the model described by \mathcal{H} , Eq. (3.20), and the MPCs theory can now be used to obtain the Casimir force for the dual model Eq. (2.9), describing a condensed vortex in the dual formalism. Since the Casimir force for a massive scalar field in $2 + 1$ dimensions is well known [23], provided well defined BCs are considered, we now turn our attention to this issue of setting the BCs for the mapped theory.

B. The BC mapping between the gauge field and the scalar fields

The method that we use here for determining the Casimir force for the MPCs model is to associate it with a model of scalar fields, as explained in the previous subsection. The involved mathematical form of the mapping between the vectorial field and the two scalar fields, however, makes the problem of fixing the BCs in this case a nontrivial one. Below, we will elaborate on this problem of mapping the required BCs. As we will show next, some usual BCs considered for scalar fields in Casimir problems cannot be directly written in terms of the vectorial field A_μ (at least in a simple form). This is an important issue, since it is well known that the Casimir force (for both its modulus and orientation) depends significantly on the BCs considered.

Our aim is to obtain the Casimir force for the vectorial field by equating it to a sum of two previously known expressions of Casimir forces for two scalar fields that have well-posed BCs. To be able to make this association between the two models and to use the corresponding Casimir force result known for massive scalar fields, the BCs for the scalar fields have to be related to well-posed

and physically acceptable BCs for the vectorial field. As an illustration, we could wonder whether the condition for the fields ϕ and φ to vanish at the boundaries, which is a well studied BC for scalar fields in Casimir problems, would or not imply in perfect conductor BCs (for instance) for the vectorial field and vice versa. To answer this question requires having a clear map from $\{\phi, \varphi\}$ (and/or the derivatives of those fields) into $\{A_0, A_1, A_2\}$ (and/or the derivatives of those fields components), at least at the boundaries. Hence, we need to invert the relations $\{A_0, A_1, A_2\} \rightarrow \{\phi, \varphi\}$ given in the previous subsection. With this aim, we first use the expressions for \tilde{A}_1 and \tilde{A}_2 , given by Eqs. (3.18) and (3.19), and substitute them in Eq. (3.9). From this, we obtain

$$A_1 = [\hat{\mathcal{O}}_1^{-1}\phi - \hat{\mathcal{O}}_2^{-1}\varphi]/(\sqrt{2}\theta). \quad (3.22)$$

Since the physical BCs are specified in configuration space, we need to further elaborate on the meaning of the terms $\hat{\mathcal{O}}_1^{-1}\phi$ and $\hat{\mathcal{O}}_2^{-1}\varphi$ appearing in Eq. (3.22), in particular at the boundaries.

Let us consider initially the first term in Eq. (3.22), $\hat{\mathcal{O}}_1^{-1}\phi$. Using the explicit forms of the operators K_{11} , Q_{12} , and S^{22} , given in Eqs. (3.6), (3.7), and (3.8), we can write that

$$\hat{\mathcal{O}}_1^2\phi = (A - B\partial_1^2)^{-1}\phi, \quad (3.23)$$

where two new constants A and B have been introduced in the above equation and they are given, respectively, by

$$A \equiv \frac{\theta^2}{2} - \frac{\theta\mu}{2} + \frac{m^2}{2}\left(1 + \frac{\mu^2}{4m^2}\right), \quad (3.24)$$

$$B \equiv -\frac{\theta^2}{2m^2} - \frac{\theta\mu}{2m^2} - \frac{1}{2} - \frac{\mu^2}{4m^2}.$$

From Eq. (3.23), we see that $\hat{\mathcal{O}}_1^{-1}\phi$ can be written as $(A - B\partial_1^2)^{1/2}\phi$. Let us now evaluate this expression at the boundaries. Our physical system is constrained in an infinite strip, with boundaries at $x = 0$ and $x = a$. By also considering that the field ϕ obeys the Neumann BC, with $\partial_1\phi(x = 0) = \partial_1\phi(x = a) = 0$, thus, *at the boundaries*, we can write

$$(A - B\partial_1^2)\phi = (\sqrt{A} + i\sqrt{B}\partial_1)^2\phi. \quad (3.25)$$

From Eqs. (3.23) and (3.25), we can now write

$$\hat{\mathcal{O}}_1^{-2}\phi = (A - B\partial_1^2)\phi = (\sqrt{A} + i\sqrt{B}\partial_1)^2\phi. \quad (3.26)$$

Hence, at the boundaries $x = 0$ and $x = a$, we determine that

$$\hat{\mathcal{O}}_1^{-1}\phi = (\sqrt{A} + i\sqrt{B}\partial_1)\phi = \sqrt{A}\phi. \quad (3.27)$$

With these results, it is now easy to write the first term of the left-hand side of Eq. (3.22) in the configuration (coordinate) space and at the boundaries. We note that, in

Eq. (3.27), \sqrt{A} does not represent an eigenvalue of $\hat{\mathcal{O}}_1^{-1}$ but the mathematical expression of that operator itself (at the boundaries).

We can use analogous considerations also for $\hat{\mathcal{O}}_2^{-1}\varphi$, the second term in the left-hand side of Eq. (3.22). From similar arguments as those used for $\hat{\mathcal{O}}_1^{-1}\phi$ and considering the Neumann BC for φ , we can write, at the boundaries, that

$$\hat{\mathcal{O}}_2^{-1}\varphi \equiv \sqrt{C}\varphi, \quad (3.28)$$

where the constant C in the above equation is defined as

$$C \equiv \frac{\theta^2}{2} + \frac{\theta\mu}{2} + \frac{m^2}{2} \left(1 + \frac{\mu^2}{4m^2}\right). \quad (3.29)$$

From the above results, we can write Eq. (3.22), at the boundaries, as $A_1 = [\sqrt{A}\phi - \sqrt{C}\varphi]/(\sqrt{2}\theta)$. Hence we see that A_1 must also obey the Neumann BC:

$$\partial_1 A_1(x=0) = \partial_1 A_1(x=a) = 0. \quad (3.30)$$

Likewise, we can proceed analogously to obtain the required conditions for A_2 . By making use of Eqs. (3.10), (3.16), and (3.17), we obtain that

$$A_2 = \frac{(\hat{\mathcal{O}}_1\pi_\phi + \hat{\mathcal{O}}_2\pi_\varphi)}{\sqrt{2}} - \sqrt{2}\hat{\mathcal{O}}_1\left(\frac{S^{12}}{\theta} + \frac{Q_{22}}{2}\right)\phi - \sqrt{2}\hat{\mathcal{O}}_2\left(-\frac{S^{12}}{\theta} + \frac{Q_{22}}{2}\right)\varphi. \quad (3.31)$$

We can now use Eq. (3.31) to determine the behavior of A_2 at the boundaries. Using Eqs. (3.27) and (3.28), we can write (for $x=0$ and $x=a$) that $\hat{\mathcal{O}}_1\phi = \phi/\sqrt{A}$ and $\hat{\mathcal{O}}_2\varphi = \varphi/\sqrt{C}$. Noticing that we are considering the Neumann BC for ϕ and φ , we can use the Hamilton equations ($\pi_\phi = \partial_0\phi$ and $\pi_\varphi = \partial_0\varphi$) and the explicit forms of Q_{22} and S^{12} to rewrite Eq. (3.31) as

$$A_2 = \frac{\partial_0\phi}{\sqrt{2A}} + \frac{\partial_0\varphi}{\sqrt{2C}}. \quad (3.32)$$

Equation (3.32) implies that A_2 must also obey the Neumann BC (since ϕ and φ are subjected to the same type of BC).

The BCs considered for A_1 and A_2 , together with the Euler-Lagrange equations and the definitions of the canonical momenta, define the components of the strength tensor at the boundaries. The behavior of those components should not be confused with a new BC imposed to the vectorial field but just direct implications of the Neumann BCs considered for A_1 and A_2 . For instance, from the definition of π^i given in Eq. (3.2), we get

$$F^{20} = \pi^2 + \mu A_1/2, \quad (3.33)$$

or yet, from Eqs. (3.9), (3.12), (3.18), and (3.19),

$$F^{20} = \frac{\mu}{2}A_1 - \frac{\sqrt{2}}{2}[\hat{\mathcal{O}}_1^{-1}\phi + \hat{\mathcal{O}}_2^{-1}\varphi]. \quad (3.34)$$

Thus, at the boundaries and using Eqs. (3.27) and (3.28), we obtain that

$$F^{20} = \frac{\mu}{2}A_1 - \frac{\sqrt{2}}{2}[\sqrt{A}\phi + \sqrt{C}\varphi]. \quad (3.35)$$

Since A_1 , ϕ , and φ are subjected to the Neumann BC, Eq. (3.35) implies that F^{20} is also subjected to the same BC. We can also write those BCs in terms of the dual tensor F^μ , defined by $F^\mu \equiv \epsilon^{\mu\nu\rho}\partial_\nu A_\rho$, to obtain

$$\partial_1 F^1(x=0) = \partial_1 F^1(x=a) = 0. \quad (3.36)$$

The result given by Eq. (3.36) can be seen as a BC for the vectorial field and a direct consequence of the Neumann BCs considered for A_1 and A_2 , which, in turn, are a direct consequence of the Neumann BCs considered for ϕ and φ . We can say that Eq. (3.36) is the analogue of the BC $F^1(x=0) = F^1(x=a) = 0$ considered in Ref. [19]. Also, we note that, in a similar manner to what occurred in Ref. [19], the BC given by Eq. (3.36) can be seen as a consequence of the Bianchi identity $\partial_\nu F^\nu = 0$, together with the statics requirement $\partial_0 F^0 = 0$, imposed to a perfect conductor. To better see this in a clearer manner, we can first evaluate Eq. (3.1) for $\alpha = 0$ and $\alpha = 1$. Using the BCs considered above, we can write (at the boundaries)

$$(\partial_1\partial^1 + m^2)A_0 - \mu\partial_2 A_1 + \partial_2 F^{20} = 0, \quad (3.37)$$

$$(\partial_0\partial^0 + \partial_2\partial^2 + m^2)A^1 - \partial_0\partial^1 A_0 + \mu F_{20} = 0. \quad (3.38)$$

From Eq. (3.38), it is easy to see (by taking the derivative with respect to x and using the BCs) that $\partial_0\partial_1\partial^1 A_0 = 0$. Using this result and representing A_0 in terms of its transverse Fourier transform (in y and t) [19],

$$A_0(x, y, t) = \int \frac{d\omega}{2\pi} e^{-i\omega t} \int \frac{dk}{2\pi} e^{iky} \tilde{A}_0(x, k, \omega), \quad (3.39)$$

we see that the condition $\partial_0\partial_1\partial^1 A_0 = 0$ (valid for any t and y) implies that $\partial_1\partial^1 \tilde{A}_0(x, k, \omega) = 0$ and, therefore, $\partial_1\partial^1 A_0 = 0$ (for $x=0$ and $x=a$). We can now use this result in Eq. (3.37) to obtain

$$m^2 A_0 - \mu\partial_2 A_1 + \partial_2 F^{20} = 0. \quad (3.40)$$

Since A_1 and F^{20} are subjected to the Neumann BC, Eq. (3.40) implies (by deriving with respect to x) that A_0 is also subjected to the same kind of BC as well: $\partial_1 A_0 = 0$ at $x=0$ and $x=a$. Hence, at the boundaries, we have

$$F^2 = F_{01} = \partial_0 A_1, \quad (3.41)$$

where we made use of the BC for A_0 .

By considering that F^0 is subjected to the statics requirement $\partial_0 F^0 = 0$ [19], we get (at the boundaries, where $\partial_1 A_2 = 0$)

$$\partial_0 F^0 = \partial_0 F_{12} = \partial_0 \partial_2 A_1 = 0. \quad (3.42)$$

We can now use Eq. (3.42) to establish the value of $\partial_2 F^2$ at the boundaries and show that it must vanish as well. This result will then be used below, together with the Bianchi identity, to obtain equally that $\partial_1 F^1 = 0$, which can be seen as a direct consequence of the Bianchi identity and the statics requirement. First, we note that, since $\partial_1 A_0 = 0$ at the boundaries, we can write for $x = 0$ or $x = a$ that

$$\partial_2 F^2 = \partial_2 F_{01} = \partial_2 \partial_0 A_1. \quad (3.43)$$

By comparing Eqs. (3.43) and (3.42), we see that the statics requirement implies that $\partial_2 F^2 = 0$ at the boundaries. Using this condition together with the statics requirement, we get likewise that $\partial_1 F^1 = 0$ at the boundaries. Thus, the BC $\partial_1 F^1 = 0$ can be seen as a consequence of the statics requirement and the Bianchi identity considered here and in Ref. [19].

By using the definitions of the canonical momenta and the considerations about the behavior of A_i at the boundaries, it is easy to prove that the surface terms generated by the integrals of $\partial_i(\pi^i \partial_j \pi^j)$, $\partial_i(\pi^i \epsilon^{jk} F_{jk})$, $\partial^i(A_j \partial_i A_j)$, and $\partial^i(A^j \partial_j A_i)$, that appear in the generating functional, will give no contributions. This justifies neglecting those contributions to the partition function, as we have assumed. Analogously, the BC considered here, written in terms of ϕ , φ , and their respective conjugate momenta, allow us to neglect the surface terms related to those fields in the process of obtaining the final Hamiltonian density Eq. (3.20).

IV. THE CASIMIR FORCE

By having the relevant BCs fixed, it becomes straightforward to find the Casimir force for the dual MPCS theory Eq. (2.9). This follows directly from the equivalence between the original theory Eq. (2.9) with the model represented by Eq. (3.20). The Casimir force for a massive scalar field subjected to the Neumann (or Dirichlet) BC in $2 + 1$ dimensions (which is also the same as the one computed for a MCS theory) is [19,23]

$$f_{\text{scalar}}(m_s, a) = -\frac{1}{16\pi a^3} \int_{2m_s, a}^{\infty} dy \frac{y^2}{e^y - 1}, \quad (4.1)$$

where m_s is the mass of the scalar field. The integral in Eq. (4.1) is a second Debye function [31],

$$\int_x^{\infty} dy \frac{y^2}{e^y - 1} = \sum_{k=1}^{\infty} e^{-kx} \left(\frac{x^2}{k} + 2\frac{x}{k^2} + 2\frac{1}{k^3} \right), \quad (4.2)$$

indicating that the Casimir force due to massive scalars exponentially decays with $m_s a$.

Using the equivalence between Eqs. (2.9) and (3.20), we can then immediately write the corresponding Casimir force, in the presence of a vortex condensate, as

$$f_{\text{vortex}} = f_{\text{scalar}}(m_1, a) + f_{\text{scalar}}(m_2, a), \quad (4.3)$$

where m_1 and m_2 , using Eqs. (2.10), (3.13), and (3.21), are given by

$$m_{1(2)} = \frac{e^2 \rho_0^2}{|\Theta|} \left(\sqrt{1 + \frac{16\pi^2 \psi_0^2 \Theta^2}{e^4 \rho_0^2}} \mp 1 \right). \quad (4.4)$$

For small values of mass $ma \lesssim 1$, Eq. (4.1) can be expressed as

$$f_{\text{scalar}}(m_s, a) = -\frac{1}{8\pi a^3} \left[\zeta(3) - (am_s)^2 + \frac{2(am_s)^3}{3} - \frac{(am_s)^4}{6} + \mathcal{O}(a^5 m_s^5) \right], \quad (4.5)$$

where $\zeta(x)$ is the Riemann zeta function. Using Eq. (4.4) and keeping for simplicity up to the quadratic term in the mass in Eq. (4.5), we obtain for the Casimir force Eq. (4.3) the result

$$f_{\text{vortex}} \simeq -\frac{1}{4\pi a^3} \left[\zeta(3) - \left(\frac{e^2 \rho_0^2}{\Theta} \right)^2 a^2 \left(1 + \frac{8\pi^2 \psi_0^2 \Theta^2}{e^4 \rho_0^2} \right) \right]. \quad (4.6)$$

The result (4.3) allows us to immediately conclude that, in the presence of vortex matter ($\psi_0 \neq 0$), the Casimir force is always *smaller* in magnitude than in the absence of vortices.

There are two mass scales in our original model Eq. (2.2), which are the mass for the gauge field h_μ in the broken phase, m_h , and the mass for the scalar field η , m_η . These masses can be related to the relevant scales in the context of superconductivity. The two naturally occurring length scales in the theory of superconductivity are the penetration depth $\lambda = 1/m_h$, which describes the typical length into which a magnetic field can penetrate into a superconductor, and the coherence length $\xi = 1/m_\eta$, which describes the length scale at which the order parameter varies in space. The ratio between these two lengths is the Ginzburg-Landau parameter $\kappa = \lambda/\xi \equiv m_\eta/m_h$. Values of $\kappa > 1/\sqrt{2}$ characterize type-II superconductors. Type-II superconductors in the presence of a magnetic field can form a stable vortex state (the Shubnikov phase [32]). On the other hand, materials with $\kappa < 1/\sqrt{2}$ characterize type-I superconductors. In type-I superconductors, a magnetic field will destroy superconductivity without allowing the formation of a stable vortex state.

Using the parameters of the original CSH model Eq. (2.2) and taking as an example the self-dual potential for the scalar field [26], we have that $m_h = e\rho_0$ and $m_\eta = e^2 \rho_0^2 / \Theta$. The Ginzburg-Landau parameter becomes $\kappa = e\rho_0 / \Theta$. As shown in Ref. [13], vortices are energetically favored to condense for values of the CS parameter below a critical value $\Theta_c \approx (e^2/\pi) \ln 6 \approx 0.57e^2$, and for $\Theta < \Theta_c$ we have for the vortex condensate $\psi_0^2 \approx m_\eta \sqrt{6 - \exp(\pi\Theta/e^2)}$. By expressing Eq. (4.6) in

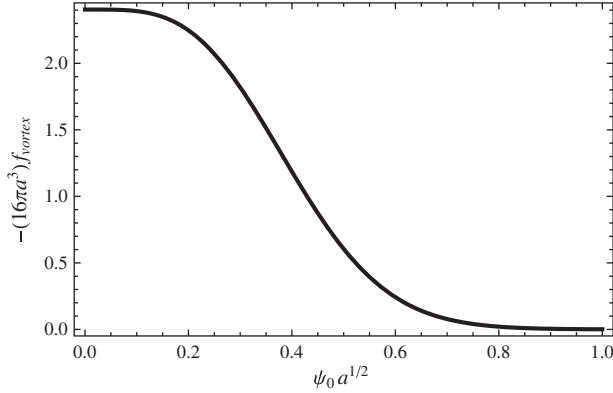


FIG. 1. The Casimir force as a function of the vortex condensate ψ_0 , for the choice of parameters $\Theta/e^2 = 0.1$ and $\rho_0 a^{1/2} = 1$.

terms of these values, we can write the fractional difference for the Casimir force without vortices, $f_{\text{vortex}}(\psi_0 = 0)$, and in the presence of vortices ($\psi_0 \neq 0$) as

$$\begin{aligned} \frac{\Delta f}{f} &\equiv \frac{f_{\text{vortex}}(\psi_0 = 0) - f_{\text{vortex}}(\psi_0)}{f_{\text{vortex}}(\psi_0 = 0)} \\ &\approx \frac{(m_\eta a)^2 8\pi^2 \frac{\Theta}{e^2} \sqrt{6 - \exp(\pi\Theta/e^2)}}{\zeta(3) - (m_\eta a)^2}. \end{aligned} \quad (4.7)$$

If we use representative values consistent with the above requirements of vortex condensation and in the regime of validity of Eq. (4.6), e.g., $\Theta/e^2 = 0.1$ and $m_\eta a = 0.1$, we obtain for the ratio Eq. (4.7) the result $\Delta f/f \simeq 0.14$, representing already a Casimir force that is 14% smaller due the presence of a vortex condensate. For larger values of $m_\eta a$, or equivalently for $m_{1(2)} a \gtrsim 1$, we need to solve numerically for the integral in Eq. (4.1), with the corresponding Casimir force decreasing exponentially due to the characteristic second Debye function displayed by the Casimir force for a massive scalar particle Eq. (4.1). In Fig. 1, we show the Casimir force Eq. (4.3) as a function of arbitrary values for the vortex condensate.

The overall decrease of the Casimir force when in the presence of a vacuum state with vortices can be interpreted as follows. Vortices are expected to repel each other, much like in the standard mean-field phenomenology for type-II superconductors when vortices can form [32], e.g., in the Shubnikov phase, where above some critical magnetic field vortices are present. The repelling vortices will exert an opposite, repulsive force on the external conducting lines that tend to counterbalance the attractive Casimir force, tending to make it smaller the larger the VEV of the vortex condensate is. The resulting Casimir force can then be made sufficiently small in the presence of vortex matter, though it will never be exactly zero or become repulsive, as it can be clear from the expression for the Casimir force and from Eq. (4.4), where of course $m_{1(2)} > 0$.

V. CONCLUSIONS

We have studied in this work how a nontrivial vacuum state, with condensed vortex excitations, affects the Casimir force between two conducting lines in a plane. By starting from a CSH model with field equations having vortex solutions, and using its dualized form, which results to be a MCSH model, vortex degrees of freedom are made explicit. In the vortex condensation regime of the dual model, it can be expressed simply as a MPCSH theory, which in turn can be mapped in a two noninteracting massive scalar field model. Using the known expression for the Casimir force for a massive scalar field, the corresponding Casimir forces for the case of vortex matter between the two lines have been computed.

We have shown that the Casimir force in the presence of vortex matter is smaller than in the absence of vortices. This result may have implications for Casimir effect experiments using, e.g., superconductors, like in the next generation of experiments [33], in the case that type-II superconductors could eventually be used. The results we have obtained are indicative that the presence of vortices in the superconducting materials can make the Casimir effect much smaller, making its detection through measurements more difficult. Earlier experiments on the Casimir effect performed by using superconducting materials, e.g., in Ref. [5], investigated the variation of the Casimir energy in the transition from the normal to the superconducting state. Though this variation can be very small, it can have a magnitude comparable to the condensation energy of a semiconducting film. It has been shown in Ref. [5] that this can cause a measurable increase in the value of the critical magnetic field required for the transition. However, these experiments were performed by using type-I superconductors, where a vortex state is absent. It is feasible to expect, based on the results we have obtained here, that in the case of type-II superconductors there should also be observed another variation of the Casimir energy in the transition from the superconducting state to the Shubnikov phase, where vortices are formed.

Another important issue that must be cited is the possibility of using our results to find the Casimir force, for the MPCSH theory, in the case of moving boundaries (i.e., the dynamical Casimir effect). As mentioned in the introduction, it is expected that the Casimir energy plays an important role in superconductors, especially at the nanometer scale. Recently, the first experimental observation of the dynamical Casimir effect in a superconductor circuit [34] has brought great attention to this matter. Some of the considerations that we have done here are also valid in the dynamical case. Of course, where we set the boundaries (e.g., $x = 0$ and $x = a$) is of decisive importance for determining the expression for the Casimir force. However, the mapping between the initial MPCSH theory and the model of the scalar fields makes use only of the values of the derivatives of the functions ϕ , φ , and A_i at the

boundaries. But the value of x itself at those boundaries is never actually needed there at any step. In other words, the mapping used here is expected also to be valid in the case of moving boundaries, as long as the BCs remain valid (e.g., a perfect conductor parallel to the y axis, in a movement in the x direction). Hence we conclude that we can use the same arguments used here to study the dynamical Casimir effect for the MPCS model. However, to find the Casimir force in that case, we must know the force for a massive scalar field between moving boundaries, which is an issue that we intend to treat in a future work.

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APPENDIX: THE ENERGY-MOMENTUM TENSOR AND THE CASIMIR FORCE

The Casimir force for the dual theory is expressed, as usual, in terms of the VEV of the T^{11} component of the symmetrized energy-momentum tensor [19,22]: force/length = $\langle 0|T^{11}|0\rangle$. Thus, we can first write

$$T^{\mu\nu} = F^\mu F^\nu + m^2 A^\mu A^\nu - \frac{1}{2} g^{\mu\nu} (F_\lambda F^\lambda + m^2 A_\alpha A^\alpha), \quad (\text{A1})$$

where, for the sake of simplicity, we made use of the definition of the dual tensor F^μ :

$$F^\mu \equiv \frac{1}{2} \epsilon^{\mu\nu\rho} F_{\nu\rho}. \quad (\text{A2})$$

Usually, the components F^μ are associated to the components of the “electric” and “magnetic” fields ($F^1 = -E_y$, $F^2 = E_x$, and the scalar $B = F^0$). In this work, E_x , E_y , and B may or may not (in the case of the dual gauge field) represent a physical massive electromagnetic field (we are just borrowing an usual nomenclature). Hence, $\langle 0|T^{11}|0\rangle$ at the boundaries can be written in terms of VEVs of products like $A^\mu A^\nu$ and derivatives of them, taken at $x = 0$ or $x = a$ (the explicit values of x at the boundaries will not be necessary for our purposes). Following Ref. [19], we write those VEVs, at $x = 0$, as

$$\langle 0|A^\mu(x)A^\nu(x)|0\rangle_{x_1=0} = \lim_{x_1 \rightarrow x'_1=0} \langle 0|A^\mu(x)A^\nu(x')|0\rangle, \quad (\text{A3})$$

where x and x' stand for points in the three-dimensional space-time. But the VEVs in the right-hand side of Eq. (A3) are the two-point functions of the model, which can be written in terms of the functional derivatives of the normalized generating functional $Z[J_\alpha]$, where J_α is a source:

$$\begin{aligned} \langle 0|A^\mu(x)A^\nu(x')|0\rangle &= - \frac{\delta^2 Z[J]}{\delta J_\mu(x) \delta J_\nu(x')} \Big|_{J_\alpha=0} \\ &= - \frac{\delta^2}{\delta J_\mu(x) \delta J_\nu(x')} \\ &\quad \times \left[\frac{\int \mathcal{D}A_\beta \exp(iS + i \int J_\alpha A^\alpha)}{\int \mathcal{D}A_\gamma \exp(iS)} \right]_{J_\alpha=0}. \end{aligned} \quad (\text{A4})$$

The derivatives appearing in Eq. (A4) are independent of the norm of the field. This also shows that the Casimir force should also be independent of the normalization of the fields. For instance, the arbitrary mass parameter σ appearing in Eq. (2.8) can be put as a global multiplicative constant σ^2 in all terms in Eq. (2.9) and be reabsorbed in a redefinition of the norm of A_μ .

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