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M5-branes in the ABJM theory and the Nahm equation

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We explicitly construct two classes of the BPS solutions in the Aharony-Bergman-Jafferis-Maldacena action—the funnel type solutions and the 't Hooft-Polyakov type solutions—and study their physical properties as the M2–M5 bound state. Furthermore, we give a one-to-one correspondence between the solutions of the BPS equation and the ones of an extended Nahm equation which includes the Nahm equation. This enables us to construct infinitely many conserved quantities from the Lax form of the Nahm equation.

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I. INTRODUCTION AND SUMMARY

After the ground-breaking work by Bagger and Lambert [1] and Gustavsson [2], the multiple M2-branes have been studied intensively and a three-dimensional $\mathcal{N} = 6$ supersymmetric Chern-Simons-matter conformal field theory with gauge group $U(N) \times U(N)$ was proposed as an action of the low energy limit of N M2-branes on $\mathbf{C}^4/\mathbf{Z}_k$ by Aharony *et al.* [3]. There has been significant progress in understanding the M2-branes by the Aharony-Bergman-Jafferis-Maldacena (ABJM) action. (See Ref. [4] for the recent review of this subject.)

On the other hand, the M5-branes are still poorly understood. The theory on the multiple M5-branes is highly mysterious. For instance, the gravity dual analysis implies that there should be $\mathcal{O}(N^3)$ degrees of freedom at large N, which cannot be understood from gauge theory[1] In order to study the M5-branes, the ABJM action will be useful because the bound state of the M5-branes and the M2-branes can be described by the M2-brane action, where the M5-branes will be the "solitons" of the action. Indeed, the BPS solution corresponding to the funnel type bound state of these was found in Refs. [10,11], which can be regarded as a variant of the famous solution in the Bagger-Lambert-Gustavsson (BLG) action by Basu and Havey [12], and has been studied further [13–15].[2] This is the M-theory lift of the bound state of the D2-branes and the D4-branes, which are described as the solution of the Nahm equation from the D2-branes or the monopole from the D4-branes. The shape of the solution should be a fuzzy $S^3/\mathbf{Z}_k$ at a point in the world volume of the M2-branes.

For the D2–D4 bound state (which is essentially the same as the D1–D3 bound state [18]), we can use the Nahm construction [19] to construct the monopole solution from the Nahm data. For the M2–M5 bound state, we expect that there will be such correspondence between the BPS solution in the multiple M5-branes and the ones in the ABJM action.[3] This may give us some important clues for understanding the M5-branes.[4] For this project, we obviously need the details of the solutions of the BPS equation of the ABJM action. However, the solutions of the BPS equations [10] are less known and the properties of the solutions, for example what is the moduli space, have not been studied. Even the positions of the M2-branes far from the M5-branes are unclear and ambiguous, as we see in Sec. II.

In this paper, we construct two classes of the BPS solutions explicitly and study their physical properties.[5] Solutions in one class include the one found in Refs. [10,11], but the positions of the M2-branes are more general. Solutions in the other class behave like the Nahm data of the 't Hooft-Polyakov monopole and represent the bound state of two M5-branes. Furthermore, we give a one-to-one correspondence between the solutions of the BPS equation and the ones of an extended Nahm equation which includes the Nahm equation. This enables us to construct infinitely many conserved quantities from the Lax form of the Nahm equation. We also

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[1]Recently, it was claimed [5,6] that the $\mathcal{O}(N^3)$ behavior is reproduced from the localization computation of the 5D SUSY gauge theory on S^5 [7–9].

[2]Moreover, an M-theory lift of the D4-branes with a constant magnetic field in type IIA string theory should be an M2–M5 bound state and it was also constructed in the ABJM action [16,17]. This system is also useful for understanding the M5-branes.

[3]For the BLG action, Gustavsson studied this Ref. [20]. In Refs. [21,22], the Nahm construction for the BLG and the ABJM actions were considered, but they assume rotational invariance of the solutions.

[4]We cannot use the BLG theory instead of the ABJM theory. Since scalar fields in the BLG theory live in $\mathcal{A}_4$, there is no natural way to get from the BPS solution the information of the positions of the M2-branes which is necessary to discuss how that bound state should be expressed on the M5-branes.

[5]One can obtain further BPS solutions by taking the direct sum of these BPS solutions. One can even construct the bound state of the M5-branes (each of which extends in different directions in space-time) and the M2-branes [23]. Though we do not consider in this paper, this direction would also be interesting as a future work.

investigate the space-time profiles of the solutions using the correspondence.

The organization of this paper is as follows. In Sec. II we construct the BPS solutions representing the M2-branes ending on the M5-branes and investigate their profiles in space-time. In Sec. III we show the one-to-one correspondence between the BPS solutions and the extended Nahm data, and using this, construct conserved quantities of the BPS solution. We also comment on the relation to the reduction from the M2-branes to the D2-branes discussed by Mukhi and Papageorgakis [24].

II. THE BPS SOLUTIONS REPRESENTING THE M2-BRANES ENDING ON THE M5-BRANES

In this paper, we study the half BPS solutions of the ABJM action which represent the M2-branes ending on the M5-branes. We assume that the M2-branes extend in (x^0, x^5, x^6) directions and that the fields on the M2-branes depend only on x^6, which we will denote as s. The bosonic fields of the ABJM action are the gauge fields and the $N \times N$ matrix valued complex scalar fields Y^1, Y^2, Y^3, Y^4 representing the positions in the transverse directions of M2-branes. Here N is the number of the M2-branes. We also assume that Y^3, Y^4 and the gauge fields vanish in the BPS solution. This implies that the M5-branes are extending along $(x^0, x^1, x^2, x^3, x^4, x^5)$ where we identify the directions of (Y^1, Y^2) with (x^1, x^2, x^3, x^4).

Thus the BPS equation [10,11] is

$$\dot{Y}^a = \frac{2\pi}{k}(Y^a Y^{b\dagger} Y^b - Y^b Y^{b\dagger} Y^a), \tag{2.1}$$

where k is the level of the Chern-Simons action, Y^a ($a = 1$, 2) is the $N \times N$ matrix valued scalar field representing the positions in the transverse directions of M2-branes and $\dot{Y}^a = \frac{dY^a}{ds}$. In the type IIA limit, the M2–M5 bound state reduces to the D2–D4 bound state, which is described by the Nahm equation from the D2-brane viewpoint and by the monopole equation from the D4-brane viewpoint. Therefore, the BPS equation (2.1) is an analogue of the Nahm equation in the M-theory. Note that the rhs of (2.1) can be written by the Lie 3-algebra [25] and then Eq. (2.1) can be regarded as a generalization of the Basu-Harvey equation [12] to the ABJM action.

Note that (2.1) is covariant under the $U(N) \times U(N)$ gauge transformation $Y^a \to U Y^a V^\dagger$, and the $SU(2)$ global transformation $Y^a \to \Lambda^{ab} Y^b$ of the ABJM action, where U, V should be constant because of our assumption $A_\mu = \tilde{A}_\mu = 0$.

A. The funnel type solutions

Here we will explicitly construct the solutions of the BPS equation (2.1) which represent N M2-branes ending on an M5-brane at $s = s_0$ and extending to $s = \infty$. We take the following ansatz for the solutions:

$$Y^a(s) = \sqrt{\frac{k}{4\pi}} f_a(s) G^a, \tag{2.2}$$

where $f_a(s)$ is a function of s, and G^a is the constant $N \times N$ matrix defined by [10,11]

$$-G^a = G^a G^{b\dagger} G^b - G^b G^{b\dagger} G^a. \tag{2.3}$$

Using the $U(N) \times U(N)$ gauge symmetry of the ABJM action, we can express them as

$$G^1_{mn} = \delta_{m,n-1}\sqrt{m}, \qquad G^2_{mn} = \delta_{m,n}\sqrt{N-n}, \tag{2.4}$$

where $m, n = 1, \ldots, N$. Then the BPS equation reduces to[6]

$$\dot{f}_a = -\frac{1}{2} f_a |f_b|^2 (b \neq a). \tag{2.5}$$

By the symmetry of the BPS equation, we can replace

$$Y^a \to \Lambda^{ab} U Y^b V^\dagger. \tag{2.6}$$

This can make f_a satisfy $f_2 \geq f_1 \geq 0$. Then, we can write down the solution explicitly

$$\begin{aligned} Y^1(s) &= \sqrt{\frac{k}{4\pi}} G^1 \cdot \frac{C \exp[-C^2(s-s_0)/2]}{\sqrt{1-\exp[-C^2(s-s_0)]}}, \\ Y^2(s) &= \sqrt{\frac{k}{4\pi}} G^2 \cdot \frac{C}{\sqrt{1-\exp[-C^2(s-s_0)]}}, \end{aligned} \tag{2.7}$$

where

$$C^2 = (f_2)^2 - (f_1)^2 \tag{2.8}$$

is a constant.

Now we will study the physical interpretation of this solution. First, s_0 is the position of the M5-brane because Y^a diverges at $s = s_0$ as in the solution obtained in Refs. [10,11]. We will shift the coordinate s such that $s_0 = 0$. If $Y^{1,2}(s)$ are equivalent to diagonal matrices by $U(N) \times U(N)$, we expect that the ith eigenvalues of Y^a represent the position of the ith M2-brane. Here defining $z^1 \equiv x^1 + ix^2$, $z^2 \equiv x^3 + ix^4$, we identify the eigenvalues of Y^a with z^a.[7] Then, at $s = \infty$, the position of the ith M2-brane ($i = 1, \ldots, N$) is

$$\begin{bmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ C\sqrt{\frac{k(N-i)}{4\pi}} \\ 0 \end{bmatrix}, \tag{2.9}$$

which is shown in Fig. 1. This clearly shows that the solution is not spherically symmetric. Thus, the symmetry

[6] This solution was considered in Ref. [10] and also in a recent work [26] independently.

[7] This diagonalization has $U(1)^N$ ambiguity, i.e., $z^a \to e^{i\theta} z^a$ for each diagonal component. However, we expect them physically inequivalent due to Chern-Simons term in the same way to the analysis of vacuum moduli space in Ref. [3].

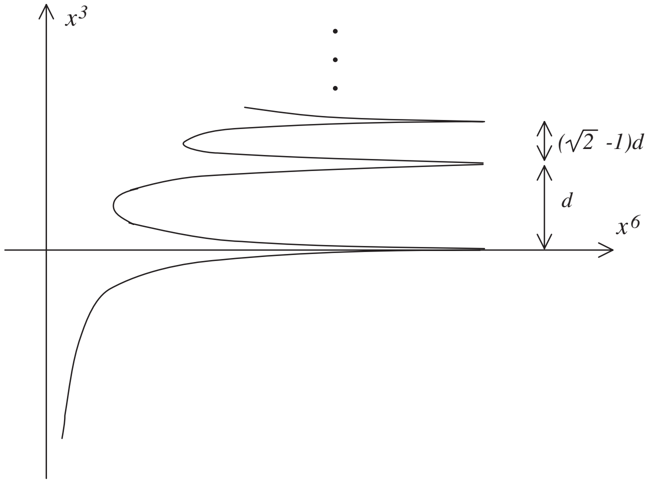

FIG. 1. The expected profile of the solution, where $d = C\sqrt{\frac{k}{4\pi}}$.

rotations give different solutions. For example, using the global $SU(2)$ and a choice of C, we can set an M2-brane at an arbitrary point in (z^1, z^2). Furthermore, because the scalars are diagonalized at $s = \infty$, we can make a $U(1)^N$ transformation, which is the diagonal part of $U(N)$, to move each of the M2-branes at z_i^a to $e^{i\phi_i} z_i^a$.

Note that for the funnel type solution obtained in [10,11],

$$Y^a(s) = \sqrt{\frac{k}{4\pi}} G^a \frac{1}{\sqrt{s - s_0}}, \tag{2.10}$$

all the M2-branes are at $z^1 = z^2 = 0$ at $s = \infty$.

The divergent behavior of this solution represents the existence of an M5-brane. Indeed, the energy of this configuration in ABJM theory is calculated as

$$E \sim 2 \int \mathrm{Tr} D_\mu Y_a^\dagger D^\mu Y^a = 2 \int d^2x ds \frac{k}{16\pi} \frac{1}{s^3} \mathrm{Tr} G^a G^{a\dagger} = \int d^2x ds \frac{kN(N-1)}{32\pi s^3}. \tag{2.11}$$

Since the radius of the fuzzy three-sphere is approximated as

$$r = \sqrt{\frac{1}{N} \mathrm{Tr} Y^a Y^{a\dagger}} = \sqrt{\frac{k(N-1)}{4\pi s}}, \tag{2.12}$$

this energy can be written as

$$E \sim \frac{1}{k} \frac{N}{N-1} \int d^2x dr r^3. \tag{2.13}$$

This represents the energy of a $(1+5)$-dimensional object expanding in $\mathbf{R}^{1,1} \times \mathbf{C}^4/\mathbf{Z}_k$. The tension is constant and independent of k and, in large N limit, N. Moreover, the unbroken supersymmetry (SUSY) generators for the configuration are the same as the ones for the M2–M5 bound state. Therefore we can interpret the BPS solution as an M2–M5 bound state.

This interpretation is based on the classical analysis. In order to know its M-theoretical aspects more extensively we need nonperturbative quantum analysis, although we do not do it in this paper.

Our solution contains this as the $C \to 0$ limit. Moreover, we can easily see that if Y^a diverges at a point $s = s_0$, the solution should be approximated by a diagonal sum of the solutions (2.10) with a symmetry transformation (2.6). This means there is an M5-brane at $s = s_0$ because of the interpretation of the solution (2.10). We can check that our solution behaves like this near $s = s_0$.

B. ’t Hooft-Polyakov type solutions

In this subsection, we will consider the $N = 2$ case only and construct a solution of the BPS equation corresponding to the Nahm data of the ’t Hooft-Polyakov monopole. We take the following ansatz:

$$\begin{aligned} Y^1(s) &= \sqrt{\frac{k}{4\pi}} (f_1(s)\sigma^1 + i f_2(s)\sigma^2), \\ Y^2(s) &= \sqrt{\frac{k}{4\pi}} (f_3(s)\sigma^3 + f_4(s)\sigma^4), \end{aligned} \tag{2.14}$$

where $f_i(s)$ is real, σ^1, σ^2, σ^3 are Pauli matrices and σ^4 is the unit matrix.[8]

By the symmetry transformation (2.6), we can make $|f_1| \le f_{2,3,4}$ at a given point in s, say $s = s_0$. Then, the BPS equation becomes

$$\dot{f}_i = -2 f_j f_k f_l, \tag{2.15}$$

where $\epsilon_{ijkl} \neq 0$. This implies that there are the following independent conserved quantities:

$$\alpha^2 \equiv f_2^2 - f_1^2, \qquad \beta^2 \equiv f_3^2 - f_1^2, \qquad \gamma^2 \equiv f_4^2 - f_1^2. \tag{2.16}$$

The remaining equation is

$$\dot{f}_1 = -2\sqrt{(f_1^2 + \alpha^2)(f_1^2 + \beta^2)(f_1^2 + \gamma^2)}, \tag{2.17}$$

which is solved as

$$2s = \int_{f_1(s)}^{\infty} \frac{df}{\sqrt{(f^2 + \alpha^2)(f^2 + \beta^2)(f^2 + \gamma^2)}}, \tag{2.18}$$

[8]A solution of the similar form for the Basu-Harvey equation was found in Ref. [27]. The BLG action for the A_4 algebra is equivalent to the $SU(2) \times SU(2)$ ABJM action which is equivalent to the $N = 2$ ABJM action if we forget the gauge fields. Thus our solution and the one in Ref. [27] are essentially the same by an appropriate map. In BLG theory, however, we did not know the relation between the parameters and the positions of the M2-branes.

$$f_2(s) = \sqrt{\alpha^2 + f_1^2}, \qquad f_3(s) = \sqrt{\beta^2 + f_1^2}, \qquad f_4(s) = \sqrt{\gamma^2 + f_1^2}, \tag{2.19}$$

where we have chosen the integration constant such that $f_1(s) = \infty$ at $s = 0$. We can see that f_1 also diverges at $s = s_*$ where

$$s_* = \frac{1}{2}\int_{-\infty}^{\infty} \frac{df}{\sqrt{(f^2+\alpha^2)(f^2+\beta^2)(f^2+\gamma^2)}}. \tag{2.20}$$

Thus, this solution represents the two M2-branes stretching between the two M5-branes at $s = 0$ and $s = s_*$. Using the first kind of the elliptic integral

$$\mathcal{F}(\phi, k) \equiv \int_0^{\phi} \frac{d\theta}{\sqrt{1 - k^2\sin^2\theta}}, \tag{2.21}$$

the integration (2.18) is written as

$$\begin{aligned} \left|\frac{s_*}{2} - s\right| &= \frac{1}{2\sqrt{\alpha_2^2(\alpha_1^2 - \alpha_3^2)}}\mathcal{F}\left(\arcsin\sqrt{\frac{\alpha_1^2 - \alpha_3^2}{\alpha_1^2}\frac{f_1^2}{f_1^2 + \alpha_3^2}}, \sqrt{\frac{\alpha_1^2(\alpha_2^2 - \alpha_3^2)}{\alpha_2^2(\alpha_1^2 - \alpha_3^2)}}\right) \\ &= \frac{1}{2\sqrt{\alpha_2^2(\alpha_1^2 - \alpha_3^2)}}\,\mathrm{sn}^{-1}\left(\sqrt{\frac{\alpha_1^2 - \alpha_3^2}{\alpha_1^2}\frac{f_1^2}{f_1^2 + \alpha_3^2}}, \sqrt{\frac{\alpha_1^2(\alpha_2^2 - \alpha_3^2)}{\alpha_2^2(\alpha_1^2 - \alpha_3^2)}}\right), \end{aligned} \tag{2.22}$$

where

$$s_* = \frac{1}{\sqrt{\alpha_2^2(\alpha_1^2 - \alpha_3^2)}}\mathcal{F}\left(\arcsin\sqrt{\frac{\alpha_1^2 - \alpha_3^2}{\alpha_1^2}}, \sqrt{\frac{\alpha_1^2(\alpha_2^2 - \alpha_3^2)}{\alpha_2^2(\alpha_1^2 - \alpha_3^2)}}\right) \tag{2.23}$$

and $(\alpha_1, \alpha_2, \alpha_3)$ is (α, β, γ) or another permutation such that $\alpha_1^2 \geq \alpha_2^2 \geq \alpha_3^2$. The schematic profiles of f_1 and f_2 are shown in Fig. 2.

For $\alpha = 0$, we find

$$s_* = \int_0^{\infty} \frac{\mathrm{d}f}{f\sqrt{(f^2+\beta^2)(f^2+\gamma^2)}} = \infty, \tag{2.24}$$

which means there is only one M5-brane and the solution is in a funnel shape. In this case, because $f_1(\infty) = f_2(\infty) = 0$, $f_3(\infty) = \beta$ and $f_4(\infty) = \gamma$, the positions of the two M2-branes are

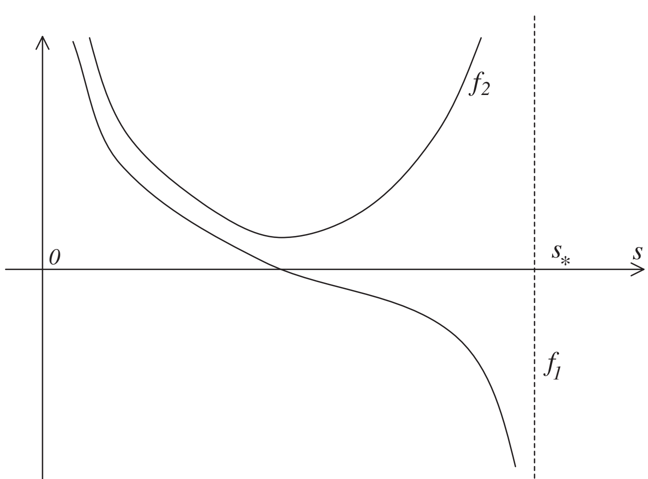

FIG. 2. Profiles of f_1 and f_2.

$$\begin{bmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \sqrt{\frac{k}{4\pi}}(\beta + \gamma) \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \sqrt{\frac{k}{4\pi}}(-\beta + \gamma) \\ 0 \end{bmatrix}. \tag{2.25}$$

By the choice of β, γ and the symmetry transformation (2.6) with

$$\beta = \sqrt{\frac{\pi}{k}}(R_1 + R_2), \qquad \gamma = \sqrt{\frac{\pi}{k}}(R_1 - R_2), \tag{2.26}$$

$$\Lambda = \begin{pmatrix} \sin\theta_1 e^{i(\phi_1 - \psi_1)} & \cos\theta_1 \\ -\cos\theta_1 & \sin\theta_1 e^{i(\psi_1 - \phi_1)} \end{pmatrix}, \qquad U = \begin{pmatrix} e^{i\phi_1} & 0 \\ 0 & e^{i\phi_2} \end{pmatrix}, \qquad V = 1, \tag{2.27}$$

we can have the solution representing the two M2-branes at

$$\begin{bmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \end{bmatrix} = \begin{bmatrix} R_1\cos\theta_1\cos\phi_1 \\ R_1\cos\theta_1\sin\phi_1 \\ R_1\sin\theta_1\cos\psi_1 \\ R_1\sin\theta_1\sin\psi_1 \end{bmatrix}, \qquad \begin{bmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \end{bmatrix} = \begin{bmatrix} R_2\cos\theta_1\cos\phi_2 \\ R_2\cos\theta_1\sin\phi_2 \\ R_2\sin\theta_1\cos(\psi_1 + \phi_2 - \phi_1) \\ R_2\sin\theta_1\sin(\psi_1 + \phi_2 - \phi_1) \end{bmatrix}. \tag{2.28}$$

If we further put $\beta = \gamma$, the solution reduces to the funnel type solution (2.7) obtained in the previous subsection for $N = 2$.

Finally, we will consider the limit of the reduction from the M2-branes to the D2-branes discussed by Mukhi and Papageorgakis for the solution (2.14). We take the parameters as

$$\alpha = \frac{\alpha_0}{\sqrt{4\pi k}}, \qquad \beta = \frac{\beta_0}{\sqrt{4\pi k}}, \qquad \gamma = \gamma_0\sqrt{\frac{k}{4\pi}}, \tag{2.29}$$

and take the $k \to \infty$ limit with α_0, β_0 and γ_0 fixed. We assume $\beta_0 > \alpha_0$ for simplicity. In this case, by (2.23) [here $(\alpha_1, \alpha_2, \alpha_3) = (\gamma, \beta, \alpha)$],

$$s_* = \frac{4\pi}{\sqrt{\beta_0^2(\gamma_0^2 - \frac{\alpha_0^2}{k^2})}}\mathcal{F}\left(\arcsin\sqrt{1 - \frac{1}{k^2}\cdot\frac{\alpha_0^2}{\gamma_0^2}}, \sqrt{\frac{\beta_0^2 - \alpha_0^2}{\beta_0^2(1 - \frac{1}{k^2}\cdot\frac{\alpha_0^2}{\gamma_0^2})}}\right) \underset{k\to\infty}{\longrightarrow} \frac{4\pi}{\beta_0\gamma_0}\mathcal{F}\left(\arcsin(1), \sqrt{1 - \frac{\alpha_0^2}{\beta_0^2}}\right) \tag{2.30}$$

is finite. Also, around $s = s_*/2$, since $f_1 \approx 0$, $f_2 \approx \alpha$, $f_3 \approx \beta$ and $f_4 \approx \gamma$,

$$Y^1 = \mathcal{O}(1), \tag{2.31}$$

$$Y^2 = \frac{k}{4\pi}\gamma_0 + \mathcal{O}(1). \tag{2.32}$$

The region around $s_*/2$ where these are true is of finite range. Indeed, the integration (2.20) is dominated by the contribution from $f = \mathcal{O}(1/\sqrt{k})$. Therefore, by (2.18), if $|s - s_*| = \mathcal{O}(1)$, then $f_1(s) = \mathcal{O}(1/\sqrt{k})$. With $\gamma_0 = 4\pi v/k$, this behavior of Y^a is just as assumed in the reduction from the M2-branes to the D2-branes (3.22) discussed in Sec. III.

III. NAHM DATA AND ABJM

In this section, we will show that any solution of the BPS equation (2.1) of the ABJM action gives two sets of Nahm data of N monopoles which satisfy the Nahm equation. This correspondence allow us to construct the conserved quantities.

In the ABJM action, there is the $U(N) \times U(N)$ gauge symmetry. By taking the product of Y^a and $Y^{a\dagger}$ we have an adjoint representation of a $U(N)$ gauge symmetry which is a singlet under the other $U(N)$. We will define

$$T^M(s) = \frac{2\pi}{k}\sigma^M_{ab}Y^bY^{a\dagger} \tag{3.1}$$

and

$$\tilde{T}^M(s) = \frac{2\pi}{k}\bar{\sigma}^M_{ab}Y^{a\dagger}Y^b, \tag{3.2}$$

where

$$\sigma^M = (\sigma^I, 1), \qquad \bar{\sigma}^M = (\sigma^I, -1), \tag{3.3}$$

$M = 1$ and $\ldots, 4, I = 1, 2, 3$. Under the $U(N) \times U(N)$ transformation ($Y^a \to UY^aV^\dagger$), they transform as

$$T^M \to UT^MU^\dagger, \qquad \tilde{T}^M \to V\tilde{T}^MV^\dagger. \tag{3.4}$$

Then, from the BPS equation of the ABJM theory (2.1) we can show that T^M and $\tilde{T}^M$ satisfy the following differential equations:

$$\dot{T}^I = i\epsilon_{IJK}T^JT^K, \tag{3.5}$$

$$\dot{T}^4 = (T^1)^2 + (T^2)^2 + (T^3)^2 - (T^4)^2, \tag{3.6}$$

and

$$\dot{\tilde{T}}^I = i\epsilon_{IJK}\tilde{T}^J\tilde{T}^K, \tag{3.7}$$

$$\dot{\tilde{T}}^4 = (\tilde{T}^1)^2 + (\tilde{T}^2)^2 + (\tilde{T}^3)^2 - (\tilde{T}^4)^2. \tag{3.8}$$

Therefore, T^I ($\tilde{T}^I$) is the solution of the Nahm equations (3.5) [(3.7)], which is called Nahm data with appropriate boundary conditions.[9] Here, we have four matrices which are the Nahm data T^I [$\tilde{T}^I$] and an additional one T^4 [$\tilde{T}^4$]. We will call them extended Nahm data and the differential equations (3.5) and (3.6) [(3.7) and (3.8)] the extended Nahm equation.

Now we will show the one-to-one correspondence

$$\{Y^a(s)\}/V \leftrightarrow \{T^M(s)|T(s_0) = AA^\dagger \quad \text{for some} \quad A \in \mathrm{Mat}_{2N\times N}(\mathbf{C})\}, \tag{3.9}$$

where we defined

$$T \equiv \sigma^M \otimes T^M = \begin{bmatrix} T^4 + T^3 & T^1 - iT^2 \\ T^1 + iT^2 & T^4 - T^3 \end{bmatrix}. \tag{3.10}$$

s_0 is a fixed constant and we consider the BPS solutions and the extended Nahm data which are finite in the neighborhood of s_0.

First, the quotient on the left-hand side is necessary since (Y^1, Y^2) and (Y^1V, Y^2V) give the same T^M by (3.1). This V must be a constant matrix since a local transformation is forbidden by the gauge fixing in (2.1). Second, since the extended Nahm equation is the set of the first order differential equations, its solution is uniquely determined by the initial condition. Thus the condition on the rhs of (3.9) is equivalent with that T^M given by (3.1) with Y^a satisfying

[9]This can be considered as a generalization of the result for the BLG action in Ref. [20]. A similar phenomena for the monopole in the ABJM action was observed in Ref. [28].

$$\begin{bmatrix} Y^1(s_0) \\ Y^2(s_0) \end{bmatrix} = \sqrt{\frac{k}{2\pi}} AV. \tag{3.11}$$

Such Y^a is unique for A up to V since the BPS equation is also the set of the first order differential equations. Therefore, (3.1) gives the one-to-one correspondence (3.9).

If $T^4(s_0) + T^3(s_0)$ is invertible, the condition $T(s_0) = AA^\dagger$ can be written explicitly with $T^M(s_0)$ only:[10]

$$\begin{cases} T^4(s_0) + T^3(s_0) \text{ is positive definite,} \\ T^4(s_0) - T^3(s_0) = (T^1(s_0) + iT^2(s_0))(T^4(s_0) + T^3(s_0))^{-1}(T^1(s_0) - iT^2(s_0)) \end{cases}. \tag{3.13}$$

The second condition is obtained by writing

$$A = \begin{bmatrix} A_1 \\ BA_1 \end{bmatrix}, \tag{3.14}$$

where A_1 ,$B \in \mathrm{Mat}_{N\times N}(\mathbf{C})$ and eliminating B from $T(s_0) = AA^\dagger$. Of course, we can have essentially the same correspondence between Y^a and $\tilde{T}^M$, instead of T^M.

Below we will consider the relation between the t^M and z^a, which are T^M and Y^a for the $N = 1$ case and so usual coordinates. The relation between them is given by

$$t^M(z) = \frac{2\pi}{k}\sigma^M_{ab} z^b \bar{z}^a. \tag{3.15}$$

If we parametrize z^a by real coordinates (r, θ, ϕ, ψ) by

$$z^1 = r\cos\theta e^{i\phi}, \qquad z^2 = r\sin\theta e^{i\psi}, \tag{3.16}$$

where $0 \le r < \infty$, $0 \le \theta \le \pi/2$, $\phi \sim \phi + 2\pi$, $\psi \sim \psi + 2\pi$, we have

$$\begin{aligned} t^1 &= R\sin\Theta\cos\Phi, \qquad & t^2 &= R\sin\Theta\sin\Phi, \\ t^3 &= R\cos\Theta, & t^4 &= R, \end{aligned} \tag{3.17}$$

where

$$\begin{aligned} &0 \le R \equiv \frac{4r^2\pi}{k} < \infty, \qquad 0 \le \Theta \equiv 2\theta \le \pi, \\ &\qquad \Phi \equiv \psi - \phi \sim \Phi + 2\pi. \end{aligned} \tag{3.18}$$

Thus, $(t^4)^2 = (t^1)^2 + (t^2)^2 + (t^3)^2$ [which is surely consistent with (3.13)], and $\{t^I\}$ parametrize $\mathbf{C}^2/U(1) = \mathbf{R}_{\ge 0} \times S^2$, where $\mathbf{Z}_k$ of the ABJM action is in the $U(1)$.

The two sets of extended Nahm data would be related to the D3-NS5 (and D5) system in Ref. [3]. This system consists of N D3-branes winding around a compact direction and two NS5-branes which break the D3-branes into two intervals, and the strings connecting these two intervals give rise to the bifundamental hypermultiplets of ABJM theory. Therefore, it seems that the extended Nahm data (3.1) and (3.2) occur by the connection of a string extending from an interval to the other with one extending in the opposite direction. This interpretation explains why there are two different sets of Nahm data constructed from the BPS configuration. However, the space which T^M represents is not flat at least naively. By now, we should say that a physical meaning or a string theoretical meaning of this map is unclear; however, we expect that the correspondence to the (extended) Nahm data will be important for further understanding of the M2–M5 brane system.

Here we would like to comment on the relation between our map (3.1) and (3.2), and the reduction from the M2-branes to the D2-branes discussed by Mukhi and Papageorgakis. In Ref. [24] they obtained three-dimensional Yang-Mills theory from the action of M2-branes by expanding one of the scalars on M2-branes around its vacuum expectation value (VEV) v and taking the $v, k \to \infty$ limit with k/v fixed.[11] For example, if one gives the VEV in the x^3 direction, the effect of $\mathbf{Z}_k$ orbifolding is

$$\begin{bmatrix} x^1 + ix^2 \\ v + x^3 + ix^4 \end{bmatrix} \sim e^{i\frac{2\pi}{k}} \begin{bmatrix} x^1 + ix^2 \\ v + x^3 + ix^4 \end{bmatrix}, \tag{3.19}$$

$$\xrightarrow[k,v\to\infty]{} \begin{bmatrix} x^1 + ix^2 \\ v + x^3 + i(x^4 + \frac{2\pi v}{k}) \end{bmatrix}. \tag{3.20}$$

Therefore the fluctuation (x^1, x^2, x^3, x^4) lives in S^1 times flat $\mathbf{R}^3$. This S^1 becomes the so-called M-theory direction, and one obtains the D2–D4 bound state in flat spacetime.

Actually, writing

$$\begin{aligned} Y^1(s) &= \frac{k}{4\pi v}(T'^1(s) - iT'^2(s)), \\ Y^2(s) &= v + \frac{k}{4\pi v}(-T'^3(s) + iT'^4(s)), \end{aligned} \tag{3.21}$$

and assuming k, $v \gg |T'^I|$, one can obtain the Nahm equation for T'^I from the BPS equation (2.1). For these

[10] Even if $T^4(s_0) + T^3(s_0)$ is not invertible, there is a continuous deformation which makes $T^4(s_0) + T^3(s_0)$ invertible. Concretely, writing $T(s_0) = AA^\dagger$, the deformation of this $T(s_0)$ into $T(s_0) = A_\epsilon A_\epsilon^\dagger$, where

$$A_\epsilon = A + \epsilon \cdot \begin{bmatrix} 1_N \\ 0 \end{bmatrix} \tag{3.12}$$

with ϵ continuous parameter, is a continuous deformation and it makes $T^4(s_0) + T^3(s_0)$ invertible if ϵ is sufficiently small (but nonzero). Therefore the condition $T(s_0) = AA^\dagger$ can be rewritten also for such $T(s_0)$, such that there exists a continuous deformation allowed under (3.13) to reach that $T(s_0)$.

[11] They used the BLG theory, but we can do the same thing also in the ABJM theory.

there is a clear physical interpretation of the system as a D2–D4 bound state [24]. However, since this procedure contains a limit, the information of the BPS solution is considerably lost. Moreover it can be used only for the BPS solution of the form (3.21) with $k, v \gg |T'^I|$. On the other hand, our equations for T^I are valid for arbitrary BPS solutions and, together with T^4 ($\tilde{T}^4$), it keeps all of the information of Y^a other than V. We also note that for the BPS solution of the form (3.21), our T^I and $\tilde{T}^I$ coincide with T'^I up to the translation:

$$T^{1,2} = \tilde{T}^{1,2} + \mathcal{O}\left(\frac{1}{k}\right) = T'^{1,2} + \mathcal{O}\left(\frac{1}{k}\right), \tag{3.22}$$

$$T^3 = \tilde{T}^3 + \mathcal{O}\left(\frac{1}{k}\right) = -\frac{2\pi v^2}{k} + T'^3 + \mathcal{O}\left(\frac{1}{k}\right). \tag{3.23}$$

A. Examples of the extended Nahm data

In this subsection, we will show the extended Nahm data T^M and $\tilde{T}^M$ explicitly for the BPS solutions obtained in Secs. II A and II B. We will compare the parameters of the solutions and the ones of the corresponding Nahm data. We will see that, in particular, the translation moduli which is trivially realized in the Nahm data is realized nontrivially in the solutions in the ABJM.

1. The funnel type solutions

From the funnel type solution (2.7), we compute

$$T^1(s) = F_1\tau^1, \qquad T^2(s) = F_2\tau^2,$$
$$T^3(s) = F_3\tau^3 + F_4\tau^4, \qquad T^4(s) = F_4\tau^3 + F_3\tau^4, \tag{3.24}$$

where

$$F_1(s) = F_2(s) = f_1 f_2 = C^2 \cdot \frac{\exp[-C^2 s/2]}{1-\exp[-C^2 s]},$$
$$F_3(s) = \frac{f_1^2 + f_2^2}{2} = \frac{C^2}{2}\frac{1+\exp[-C^2 s]}{1-\exp[-C^2 s]}, \tag{3.25}$$
$$F_4(s) = \frac{f_1^2 - f_2^2}{2} = -\frac{C^2}{2},$$

and

$$\tau^1 = \frac{G^1 G^{2\dagger} + G^2 G^{1\dagger}}{2}, \qquad \tau^2 = i\frac{G^1 G^{2\dagger} - G^2 G^{1\dagger}}{2},$$
$$\tau^3 = \frac{G^1 G^{1\dagger} - G^2 G^{2\dagger}}{2}, \qquad \tau^4 = \frac{G^1 G^{1\dagger} + G^2 G^{2\dagger}}{2}. \tag{3.26}$$

The matrices τ^M satisfy

$$[\tau^I, \tau^J] = i\epsilon_{IJK}\tau^K, \tag{3.27}$$

$$[\tau^I, \tau^4] = 0, \tag{3.28}$$

and then the τ^I is a generator of the $SU(2)$ as observed in Ref. [14]. Explicitly, they are given by

$$(\tau^1)_{mn} = \frac{1}{2}(\delta_{m,n-1}\sqrt{m(N-m-1)} + \delta_{n,m-1}\sqrt{(m-1)(N-m)}),$$
$$(\tau^2)_{mn} = \frac{i}{2}(\delta_{m,n-1}\sqrt{m(N-m-1)} - \delta_{n,m-1}\sqrt{(m-1)(N-m)}),$$
$$(\tau^3)_{mn} = \begin{cases} \frac{2m-N}{2}\delta_{mn} & (m,n<N) \\ 0 & (m,n=N) \end{cases},$$
$$(\tau^4)_{mn} = \begin{cases} \frac{N}{2}\delta_{mn} & (m,n<N) \\ 0 & (m,n=N) \end{cases}, \tag{3.29}$$

where we have used (2.4). Thus τ^I is the representation of $(\mathbf{N}-1)\oplus\mathbf{1}$. In the $s\to\infty$ limit, the location of the ith D1-brane[12] is

$$\begin{bmatrix} t_i^1 \\ t_i^2 \\ t_i^3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{C^2(i-N)}{2}, \end{bmatrix} \quad (i = 1, 2, \ldots, N). \tag{3.30}$$

One can obtain similar results for $\tilde{T}^M$. In that case, coefficient matrices $\tilde{\tau}^M$ corresponding to τ^M in T^M are the representation of $\mathbf{N}$ and

$$(\tilde{\tau}^4)_{mn} = \frac{N-1}{2}\delta_{mn}. \tag{3.31}$$

2. The 't Hooft-Polyakov type solutions

Here we will assume $\gamma \geq \beta \geq \alpha$ for simplicity, since the other cases also give similar results. The Nahm data obtained from the 't Hooft-Polyakov type solution (2.14) are

$$T^1(s) = F_1 \cdot \frac{\sigma^1}{2},$$
$$T^2(s) = F_2 \cdot \frac{\sigma^2}{2}, \tag{3.32}$$
$$T^3(s) = F_3 \cdot \frac{\sigma^3}{2} + \frac{\alpha^2 - \beta^2 - \gamma^2}{2},$$

where

$$F_1(s) = 2(f_1 f_4 - f_2 f_3),$$
$$F_2(s) = 2(f_3 f_1 - f_2 f_4), \tag{3.33}$$
$$F_3(s) = 2(f_1 f_2 - f_3 f_4).$$

This T^I is the Nahm data for the 't Hooft-Polyakov monopole centered at $(t^1, t^2, t^3) = (0, 0, (\alpha^2 - \beta^2 - \gamma^2)/2)$. It is well known that the Nahm data for the 't Hooft-Polyakov monopole is explicitly written as

[12] The D1-brane or the D3-brane are used for the Nahm data $T^I(s)$ interpreted as the D1–D3 bound state, although we do not know a precise relation between this system and the M2–M5 bound state in the ABJM action considered in this paper.

$$s_* - s = \int_{-\infty}^{F_1(s)} \frac{\mathrm{d}F}{\sqrt{(F^2+a)(F^2+b)}}, \tag{3.34}$$

$$F_2(s) = -\sqrt{a + (F_1)^2}, \qquad F_3(s) = -\sqrt{b + (F_1)^2}, \tag{3.35}$$

where the two parameters

$$a = (F_2)^2 - (F_1)^2 = 4\alpha^2(\gamma^2 - \beta^2), \quad b = (F_3)^2 - (F_1)^2 = 4\beta^2(\gamma^2 - \alpha^2), \tag{3.36}$$

the signs of F_I and the integration constant of (3.34) are determined by (3.33) with (2.16), (2.18), and (2.19).[13] Using the first kind of elliptic integral (2.21) and (3.34) is written as

$$\left| s_* - \frac{s_N}{2} - s \right| = \frac{1}{\sqrt{\max(a,b)}} \mathcal{F}\left(\arcsin\sqrt{\frac{(F_1)^2}{(F_1)^2 + \min(a,b)}}, \sqrt{\frac{|a-b|}{\max(a,b)}}\right), \tag{3.37}$$

where

$$s_N = \frac{2}{\sqrt{b}} \mathcal{F}\left(\arcsin(1), \sqrt{\frac{(b-a)}{b}}\right) = \frac{1}{\sqrt{\beta^2(\gamma^2-\alpha^2)}} \mathcal{F}\left(\arcsin(1), \sqrt{\frac{\gamma^2(\beta^2-\alpha^2)}{\beta^2(\gamma^2-\alpha^2)}}\right). \tag{3.38}$$

The schematic profiles of F_1 and F_2 are shown in Fig. 3.

Now we consider physical interpretation of the solution (3.32). The Nahm data contain the translation moduli proportional to the identity matrix with the parameter

$$\frac{\alpha^2 - \beta^2 - \gamma^2}{2}. \tag{3.39}$$

This is interesting because the shift by the identity matrix of the solution of the BPS equation (2.1) does not give another solution in general. Other than this, there are only two parameters

$$a = 4\alpha^2(\gamma^2 - \beta^2), \quad b = 4\beta^2(\gamma^2 - \alpha^2), \tag{3.40}$$

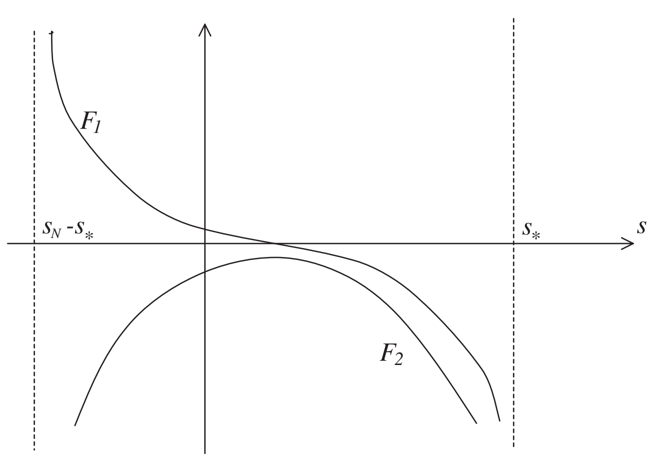

FIG. 3. Profiles of F_1 and F_2.

which represent the distance between the D3-branes s_N (3.38) and the "shape" of the D1-branes as we will see in the next subsection.

It is noted that $s_N \geq s_*$. This is seen by comparing (2.23) with $(\alpha_1, \alpha_2, \alpha_3) = (\gamma, \alpha, \beta)$ and (3.38). They differ only in the first argument of $\mathcal{F}$, with which $\mathcal{F}$ monotonically increases. Thus, since the first argument of $\mathcal{F}$ in s_*

$$\arcsin\sqrt{1 - \frac{\alpha^2}{\gamma^2}} \tag{3.41}$$

is smaller than that in s_N, one obtains the inequality. Indeed, for the solution, from (3.33), F_1 is given by

$$F_1(s) = 2\left(f_1\sqrt{f_1^2 + \gamma^2} - \sqrt{(f_1^2 + \alpha^2)(f_1^2 + \beta^2)}\right), \tag{3.42}$$

then $F_1(s_*) = -\infty$, while

$$F_1(0) = \gamma^2 - \alpha^2 - \beta^2, \tag{3.43}$$

which is finite because of the cancellation of the $(f_1)^2$ terms. This means that the locations of the two M5-branes are different from the ones of the (hypothetical) D3-branes at least naively. An extremal case is $\alpha \leq \beta = \gamma$, where, despite M2-branes suspending between two finitely separated M5-branes, the corresponding D1-branes are attached to a D3-brane and extended to infinity (Fig. 4). This point is interesting and we speculate that, related to the D3-NS5 system, this point would be interpreted naturally. However, by now we have not found any concrete interpretation.

On the other hand, T^4 is given by

$$T^4(s) = 2(f_1 f_2 + f_3 f_4) \cdot \frac{\sigma^3}{2} + 2f_1^2 + \frac{\alpha^2 + \beta^2 + \gamma^2}{2}, \tag{3.44}$$

which is divergent at $s = 0$ and $s = s_*$. Thus the extended Nahm data should be considered to be defined between the $s = 0$ and $s = s_*$.

[13] Thus, there must be a mathematical identities relating a bilinear of the elliptic functions and a single elliptic function. Unfortunately, however, we cannot show these identities explicitly here.

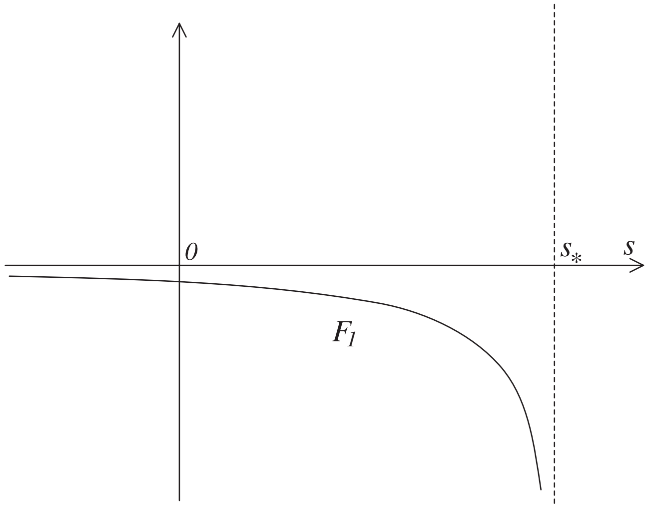

FIG. 4. Profile of F_1 for $\alpha \leq \beta = \gamma$.

For the other extended Nahm data $\tilde{T}^M$, we have

$$\begin{aligned}
\tilde{T}^1(s) &= 2(f_1 f_4 + f_2 f_3) \cdot \frac{\sigma^1}{2}, \\
\tilde{T}^2(s) &= -2(f_3 f_1 + f_2 f_4) \cdot \frac{\sigma^2}{2}, \\
\tilde{T}^3(s) &= -2(f_1 f_2 + f_3 f_4) \cdot \frac{\sigma^3}{2} + \frac{\alpha^2 - \beta^2 - \gamma^2}{2}, \\
\tilde{T}^4(s) &= 2(f_1 f_2 - f_3 f_4) \cdot \frac{\sigma^3}{2} - 2f_1^2 - \frac{\alpha^2 + \beta^2 + \gamma^2}{2},
\end{aligned} \tag{3.45}$$

where the parameters a and b which determine the Nahm data $\tilde{T}^I = \tilde{F}^I \sigma^I/2$ are the same as the ones for T^I. However, $\tilde{T}^I$ is divergent at $s = 0$ and is finite at $s = s_*$ in contrast to the profile of T^I.

As we said in this section, with the parameters α, β, γ as (2.29), with $\gamma_0 = 4\pi v/k$, and taking the $k \to \infty$ limit, one can interpret the Nahm data as actually representing the D2–D4 bound state. In this case a, b are $a = \alpha_0^2/4\pi^2$ and $b = \beta_0^2/4\pi^2$. The parameter of the translation (3.39) is $-2\pi v^2/k$, which means that by (3.23), the center of the two D2-branes is at the origin. Since $\alpha/\gamma = 0$, by (3.41) the inequality $s_N \geq s_*$ is saturated, and the positions of the D4-branes coincide with those of the M5-branes.

B. Conserved quantities by Lax formula

It is known that the Nahm equation (3.5) can be written as the Lax form and then there are infinitely many conserved quantities. This implies that the BPS equation (2.1) for the M2–M5 bound state also has infinitely many conserved quantities because of the map between the solutions of the two sets of the equations shown in this section.

The Nahm equation for the T^I is equivalent to the equation in the Lax form:

$$\dot{A} = [A, B], \tag{3.46}$$

where

$$A(s;\lambda) = \frac{k}{2\pi}\left(T^3 + \frac{\lambda}{2}(T^1 + iT^2) - \frac{1}{2\lambda}(T^1 - iT^2)\right), \tag{3.47}$$

$$B(s;\lambda) = -T^3 - \lambda(T^1 + iT^2), \tag{3.48}$$

for $\forall \lambda \in \mathbb{C}$. This enables us to write down the infinitely many conserved quantities [29] (which do not need to be independent of each other)

$$E_n(\lambda) = \mathrm{Tr}(A^n), \tag{3.49}$$

$$\dot{E}_n(\lambda) = 0. \tag{3.50}$$

In terms of the original variables Y^a, we find a simple factorized form:[14]

$$\begin{aligned}
A(s;\lambda) &= Y^1 Y^{1\dagger} - \frac{1}{\lambda} Y^1 Y^{2\dagger} + \lambda Y^2 Y^{1\dagger} - Y^2 Y^{2\dagger} \\
&= (Y^1 + \lambda Y^2)\left(Y^{1\dagger} - \frac{1}{\lambda} Y^{2\dagger}\right).
\end{aligned} \tag{3.53}$$

By the symmetry transformation, $E_n(\lambda)$ transforms in the simple way. Indeed, it is invariant under the $U(N) \times U(N)$. Under the $SU(2)$ global transformation, $Y^a \to \Lambda^{ab}$, where

$$\Lambda = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \tag{3.54}$$

and $|a|^2 + |b|^2 = 1$, we have

$$E_n(\lambda) \to v^n E_n(\lambda'), \tag{3.55}$$

where

$$v = |a|^2 - |b|^2 - \lambda \bar{a}\bar{b} + \frac{1}{\lambda} ab, \tag{3.56}$$

$$\lambda' = \frac{b + \lambda \bar{a}}{a - \lambda \bar{b}}. \tag{3.57}$$

Some of the conserved quantities represent the spacetime configuration of the D1–D3 system. The center of the D1-branes may be defined as

$$\langle t^I \rangle = \frac{1}{N} \mathrm{Tr}(T^I), \tag{3.58}$$

[14]The conserved quantities can be constructed also from $\tilde{T}^I$ as

$$\begin{aligned}
\tilde{A}(s;\lambda) &= \frac{k}{2\pi}\left(\tilde{T}^3 + \frac{\lambda}{2}(\tilde{T}^1 + i\tilde{T}^2) - \frac{1}{2\lambda}(\tilde{T}^1 - i\tilde{T}^2)\right) \\
&= (Y^{1\dagger} + \lambda Y^{2\dagger})\left(Y^1 - \frac{1}{\lambda} Y^2\right).
\end{aligned} \tag{3.51}$$

However, they are not independent of $E_n(\lambda)$ as seen from

$$\tilde{E}_n(\lambda) = \mathrm{Tr}(\tilde{A}^n) = E_n(-1/\lambda). \tag{3.52}$$

which is written by $E_1(\lambda)$ as

$$\langle t^1\rangle = \frac{4\pi}{kN}\,\mathrm{Re}[E_1]_1, \qquad \langle t^2\rangle = \frac{4\pi}{kN}\,\mathrm{Im}[E_1]_1, \qquad \langle t^3\rangle = \frac{2\pi}{kN}[E_1]_0, \tag{3.59}$$

where $[E_n]_l$ is given by

$$E_n(\lambda) = \sum_{l\in\mathbf{Z}}[E_n]_l\lambda^l. \tag{3.60}$$

We can also consider the parameters which may represent how the shape of the D1-branes is squashed in the t^I plane:

$$\delta_I^2 = \frac{1}{N}\,\mathrm{Tr}((T^I - \langle t^I\rangle)^2. \tag{3.61}$$

These are written by $E_2(\lambda)$ as

$$\begin{aligned}\delta_1^2 - \delta_2^2 &= \frac{16\pi^2}{k^2N}(\mathrm{Re}[E_2]_2) - \langle t^1\rangle^2 + \langle t^2\rangle^2,\\ \delta_2^2 - \delta_3^2 &= \frac{4\pi^2}{k^2N}(-2\,\mathrm{Re}[E_2]_2 - [E_2]_0) - \langle t^2\rangle^2 + \langle t^3\rangle^2,\\ \delta_3^2 - \delta_1^2 &= \frac{4\pi^2}{k^2N}(-2\,\mathrm{Re}[E_2]_2 + [E_2]_0) - \langle t^3\rangle^2 + \langle t^1\rangle^2.\end{aligned} \tag{3.62}$$

For the funnel type solutions, we obtain

$$[A(\lambda)]_{mn} = -\frac{kC^2}{4\pi}(N-m)\delta_{mn} \tag{3.63}$$

by evaluating it at $s\to\infty$. Thus, the center of the D1-branes is

$$\langle t^1\rangle = \langle t^2\rangle = 0, \qquad \langle t^3\rangle = -\frac{C^2}{4}(N-1), \tag{3.64}$$

and the shape parameters are given by

$$\delta_1^2 - \delta_2^2 = 0, \qquad \delta_2^2 - \delta_3^2 = -\frac{C^4(N^2-1)}{48}, \qquad \delta_3^2 - \delta_1^2 = \frac{C^4(N^2-1)}{48}, \tag{3.65}$$

which show that the D1-branes are squashed to the t^3 direction. For the 't Hooft-Polyakov type solutions, we find

$$Y^1 + \lambda Y^2 = \sqrt{\frac{k}{4\pi}}\begin{bmatrix}\lambda(\gamma+\beta) & \alpha\\ -\alpha & \lambda(\gamma-\beta)\end{bmatrix} \tag{3.66}$$

at $s = s_*/2$. Thus we can compute

$$A(\lambda) = -\frac{k}{4\pi}\begin{bmatrix}(\gamma+\beta)^2-\alpha^2 & \left(\lambda-\frac{1}{\lambda}\right)\alpha\beta + \left(\lambda+\frac{1}{\lambda}\right)\alpha\gamma\\ \left(\lambda-\frac{1}{\lambda}\right)\alpha\beta - \left(\lambda+\frac{1}{\lambda}\right)\alpha\gamma & (\gamma-\beta)^2-\alpha^2\end{bmatrix} \tag{3.67}$$

and

$$\langle t^1\rangle = \langle t^2\rangle = 0, \qquad \langle t^3\rangle = -\frac{\beta^2+\gamma^2-\alpha^2}{2}, \tag{3.68}$$

$$\delta_1^2 - \delta_2^2 = \alpha^2(\beta^2-\gamma^2), \qquad \delta_2^2-\delta_3^2 = \gamma^2(\alpha^2-\beta^2), \qquad \delta_3^2-\delta_1^2 = \beta^2(\gamma^2-\alpha^2). \tag{3.69}$$

In both cases, one can have the solution centered at an arbitrary point by adjusting the parameters of the solution and $SU(2)$ rotation. Thus we explicitly see that the moduli corresponding to the translation, which is realized trivially in the D2–D4 case, also exists in the M2-M5 case.

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