Twinlike models for kinks and compactons in flat and warped spacetime

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This work deals with the presence of twinlike models in scalar field theories. We show how to build distinct scalar field theories having the same extended solution, with the same energy density and linear stability. Here, however, we start from a given but generalized scalar field theory, and we construct the corresponding twin model, which also engenders generalized dynamics. We investigate how the twinlike models arise in both flat and curved spacetimes. In the curved spacetime, we consider a braneworld model with the warp factor controlling the spacetime geometry with a single extra dimension of infinite extent. In particular, we study linear stability in both flat and curved spacetimes, and in the case of curved spacetime—in both the gravity and the scalar field sectors—for the two braneworld models.

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I. INTRODUCTION

Topological structures are of great interest in high-energy physics [1,2] and in other areas of nonlinear science [3–5]. In high-energy physics, these structures are kinks, vortices, monopoles, and other field configurations. These are finite-energy field configurations that solve the equations of motion of the respective models in the corresponding spacetime dimensions. Kinks are the simplest structures, and they are usually constructed under the presence of real scalar fields in (1, 1) spacetime dimensions.

In this work we focus on the presence of kinks in models described by real scalar fields, drawing attention to the recent issue concerning the investigation of twinlike models. This issue was first considered in Ref. [6], and then studied in a diversity of contexts in the Refs. [7–11]. In the papers on twinlike models [6–11], the key issue was to construct and investigate twinlike models starting from a standard model, and then to introduce the twin model, which usually describes generalized or nonstandard dynamics. The important point is that it is sometimes possible to find a standard model and another model, with generalized dynamics, both of which have the same defect solution and energy density. The two models are then twinlike models. However, in Ref. [9] it was shown that it is possible to have twinlike models with the same stability behavior. We call this the strong condition; that is, there are twin models—if they have the same solution—with the same energy density. However, there are models that are twins in the strong sense that also have the same behavior concerning linear stability.

To enlarge the scope of this work, we also investigate models of the Randall-Sundrum type [12] in the presence of scalar fields, as suggested in Ref. [13]. The issue was investigated in several works [14,15], and here we consider the scalar field with generalized dynamics, leading us to a mathematical framework which is much more complicated than in the case of standard dynamics. In spite of this, we introduce a complete investigation of linear stability, both in flat and curved spacetimes.

The main issue here is to open another route to deal with twinlike models. Indeed, we consider the construction of twinlike models, but now we start from a generalized model instead of using a standard field theory. The issue is of current interest, mainly because models with generalized dynamics have been used to account for the presence of dark energy and dark matter, and to test possible modifications of general relativity. However, this is not a simple question due to the intricacy of the models to be investigated. To ease the investigation, we follow Refs. [16–18]. In particular, we introduce a new function $W = W(\phi)$, from which we obtain simple first-order equations that help us to study and solve the equations of motion. The presence of $W(\phi)$ allows for supersymmetric extensions, as was previously investigated in Refs. [19,20].

We organize the current paper as follows. In Sec. II we introduce the procedure, starting from a generalized model. In Sec. III we investigate other generalized models to show that the procedure is general, and that it works for models other than that introduced in the previous section. In Sec. IV we extend the procedure to the braneworld scenario, in the case of warped geometry with a single extra dimension of infinite extent. Finally, in Sec. V we end the work with some comments and conclusions.

II. THE NEW PROCEDURE

Let us start by following the lines of Ref. [7]. We consider the model described by a single real scalar field ϕ, with the nonstandard Lagrange density

$$\mathcal{L} = \frac{2^{n-1}}{n} X|X|^{n-1} - U(\phi), \tag{1}$$

where

$$X \equiv \frac{1}{2}\partial_\mu\phi\partial^\mu\phi$$

and $U(\phi)$ is the potential that identifies the theory. In this work we deal with bidimensional spacetime, with metric $ds^2 = dt^2 - dx_1^2$, using $x_0 = t$ and $x^1 = -x_1 = x$. Here we take $\hbar = c = 1$ and we assume that the field and coordinates are all dimensionless.

The equation of motion for this theory is

$$2^{n-1}\partial_\mu(|X|^{n-1}\partial^\mu\phi) + U_\phi = 0, \tag{2}$$

where $U_\phi = dU/d\phi$ and the energy-momentum tensor is given by

$$T_{\mu\nu} = -g_{\mu\nu}\mathcal{L} + \mathcal{L}_X\partial_\mu\phi\partial_\nu\phi, \tag{3}$$

where $\mathcal{L}_X = \partial\mathcal{L}/\partial X$. Here we are interested in static solutions, $\phi = \phi(x)$, so we have that

$$T_{00} = \rho = \frac{1}{2n}\phi'^{2n} + U, \tag{4}$$

$$T_{11} = \frac{(2n-1)}{2n}\phi'^{2n} - U(\phi). \tag{5}$$

Moreover, the equation of motion (2) becomes

$$(\phi'^{2n-1})' = U_\phi|_{\phi=\phi_s(x)}, \tag{6}$$

where $|_{\phi=\phi_s}$ indicates that we have to consider the field as static, that is, $\phi = \phi(x)$. This fact will be denoted from now on by $|_s$. This equation can be integrated once to obtain

$$\frac{(2n-1)}{2n}\phi'^{2n} - U(\phi)|_s = C, \tag{7}$$

where C is a constant that can be identified with the stress tensor T_{11}. The stability of the static solution imposes that $C = 0$, and this makes the energy density T_{00} take the form

$$\rho(x) = \phi'^{2n}, \tag{8a}$$

or

$$\rho(x) = \frac{2n}{2n-1}U(\phi(x)). \tag{8b}$$

Let us now follow the procedure introduced in Ref. [18] to write the Eq. (8a) in another form that is much more convenient for studying braneworld models. The key step is to introduce a new function, $W = W(\phi)$, such that

$$\mathcal{L}_X\phi' = W_\phi. \tag{9}$$

This fact leads us to

$$\phi' = W_\phi^{\frac{1}{2n-1}}, \tag{10}$$

and so we can write

$$U(\phi) = \frac{2n-1}{2n}W_\phi^{\frac{2n}{2n-1}}. \tag{11}$$

Therefore, the energy density has the form

$$\rho(x) = W_\phi^{\frac{2n}{2n-1}}. \tag{12}$$

We illustrate this procedure by introducing two choices for the function $W(\phi)$. First, we choose the following n-dependent function:

$$W(\phi) = \phi\,{}_2F_1\left(\frac{1}{2}, -2n+1; \frac{3}{2}; \phi^2\right). \tag{13}$$

This is a polynomial function of degree $4n-1$. The presence of the hypergeometric function ${}_2F_1$ introduces a new and general form in which to write $W(\phi)$; for example, we can write

$$W(\phi) = \phi - \frac{1}{3}\phi^3, \tag{14a}$$

$$W(\phi) = \phi - \phi^3 + \frac{3}{5}\phi^5 - \frac{1}{7}\phi^7, \tag{14b}$$

for $n = 1$ and $n = 2$, respectively.

In Ref. [17], it was shown that the two choices given by Eqs. (14) leave us with a kinklike solution. Here we investigate the behavior of the model characterized by the general parameter n.

Using Eq. (11), we obtain the potential

$$U(\phi) = \frac{2n-1}{2n}(1-\phi^2)^{2n}. \tag{15}$$

For the first potential Eq. (10) can be written as $\phi' = 1 - \phi^2$, whose solution is

$$\phi(x) = \tanh(x). \tag{16}$$

The energy density is given by

$$\rho(x) = \mathrm{sech}^{4n}(x). \tag{17}$$

Although the solution does not depend on the parameter n, the energy density does. The asymptotic behavior for different values of n is

$$\rho(x) = 2^n e^{-nx} + \cdots \tag{18}$$

is clear that the asymptotic behavior of the solutions and energy densities slows down with increasing n. The energy is

$$E = \frac{\sqrt{\pi}\Gamma(2n)}{\Gamma(2n+\frac{1}{2})}. \tag{19}$$

For $n = 1$ and $n = 2$, the energy is $E = 4/3$ and $E = 32/35$, respectively.

Now we introduce the second function, given by

$$W(\phi) = \phi\,{}_2F_1\left(\frac{1}{2}, -n+\frac{1}{2}; \frac{3}{2}; \phi^2\right). \tag{20}$$

For example, we can write this function as

$$W(\phi) = \frac{\phi}{2}\sqrt{|1-\phi^2|} + \frac{1}{2}\arcsin(\phi), \tag{21a}$$

$$W(\phi) = \phi\sqrt{|1-\phi^2|}\left(\frac{5}{8} - \frac{1}{4}\phi^2\right) + \frac{3}{8}\arcsin(\phi), \tag{21b}$$

for $n = 1$ and $n = 2$, respectively.

In Ref. [17], we investigated the case $n = 2$. Here we use Eq. (11) to obtain the general potential

$$U(\phi) = \frac{2n-1}{2n}|1-\phi^2|^n. \tag{22}$$

The choice (20) leads us to $\phi' = \sqrt{|1-\phi^2|}$, which does not depend on n. It supports the compact solution

$$\phi(x) = \begin{cases} 1 & \text{for } x < -\frac{\pi}{2}, \\ \sin(x) & \text{for } -\frac{\pi}{2} \le x \le \frac{\pi}{2}, \\ -1 & \text{for } x > \frac{\pi}{2}, \end{cases} \tag{23}$$

with the respective energy density

$$\rho(x) = \begin{cases} 0 & \text{for } x < -\frac{\pi}{2}, \\ \cos^{2n}(x) & \text{for } -\frac{\pi}{2} \le x \le \frac{\pi}{2}, \\ 0 & \text{for } x > \frac{\pi}{2}. \end{cases} \tag{24}$$

Here we note that the solution and energy density have a compact structure for all n. These kinds of structures have also been studied in Refs. [21,22]. The energy of the solution is

$$E = \frac{\sqrt{\pi}\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}. \tag{25}$$

For $n = 1$ and $n = 2$, the energy is $E = \pi/2$ and $E = 3\pi/4$, respectively.

A. Linear stability

Let us now investigate linear stability. We introduce a small fluctuation $\eta(x,t)$ about the static solution $\phi(x)$; that is, we write

$$\phi(x,t) = \phi(x) + \eta(x,t), \tag{26}$$

where $\phi(x)$ is solution of the static equation (6). With this, we obtain, at first order in η,

$$\partial_\mu\left[\phi'^{2n-2}\left(\partial^\mu\eta - 2(n-1)\partial^\mu\phi\frac{\partial_\nu\phi\partial^\nu\eta}{\phi'^2}\right)\right] + U_{\phi\phi}\eta = 0.$$

Since $\phi = \phi(x)$ is a static solution, we can assume that $\eta(x,t) = \eta_s(x)\cos(\omega t)$. Thus, we have

$$-(2n-1)[\phi'^{2n-2}\eta_s']' + U_{\phi\phi}|_s\eta_s = \omega^2\phi'^{2n-2}\eta_s. \tag{27}$$

We follow Ref. [16] to rewrite the above equation as a Schrödinger-like equation. To do this, we introduce

$$u(z) = (2n-1)^{\frac{1}{4}}\phi'^{(n-1)}\eta_s(\sqrt{2n-1}z), \tag{28}$$

which allows us to write Eq. (27) as

$$-u_{zz}(z) + v(z)u(z) = \omega^2 u(z), \tag{29}$$

where

$$v(z) = \frac{nU_{\phi\phi}}{\phi_z^{2n-2}} - \frac{n(n-1)}{(2n-1)}\frac{U_\phi^2}{\phi_z^{4n-2}}. \tag{30}$$

Then, using Eqs. (10) and (11) we obtain

$$v(z) = nW_\phi^{-\frac{2n-3}{2n-1}}W_{\phi\phi\phi} - \frac{n(n-2)}{2n-1}W_\phi^{-\frac{4(n-1)}{2n-1}}W_{\phi\phi}^2. \tag{31}$$

For kinklike solutions given by Eq. (13) the potential $v(z)$ is

$$v(z) = 4(2n-1)n^2 - 2n(4n^2-1)\mathrm{sech}^2(\sqrt{2n-1}z). \tag{32}$$

This is the modified Pöschl-Teller potential [23]. This potential supports the zero mode and other $2n-1$ bound states, with energies $E_k = (2n-1)k(4n-k)$, for $k = 0, 1, \ldots, 2n-1$. All the others states of the model, with $w \ge 4n^2$, are not bounded.

It is interesting to see that we could associate the parameter n with the number of bound states of the Schrödinger-like potential. As one knows, the number of bound states could in principle affect the rate of energy loss by radiation in dynamical processes such as, for example, in a kink-antikink collision in comparison with the usual ϕ^4 model (which is obtained with $n = 1$). See, e.g., Ref. [24].

For the compacton solutions given by Eq. (20), we get

$$v(z) = n\lambda^2\begin{cases} \infty & \text{for } z < -\frac{\pi}{2\lambda}, \\ -n + (n-1)\sec^2(\lambda z) & \text{for } -\frac{\pi}{2\lambda} \le z \le \frac{\pi}{2\lambda}, \\ \infty & \text{for } z > \frac{\pi}{2\lambda}, \end{cases} \tag{33}$$

where $\lambda = \sqrt{2n-1}$. This is the Pöschl-Teller potential [23]. The interesting feature of this potential is that it only supports bound states, and for $k = 0, 1, 2, \ldots$, the corresponding eigenvalues are given by $E_k = 4(2n-1)k(n+k)$. The radiation of energy for a collision between compactons is then completely different from the case of kinks.

III. TWINLIKE MODELS

Let us now introduce a new family of twinlike models. We first recall that twinlike models are distinct models having the same solution and energy density. The main objective here is then to introduce a family of models which is twin to the family of models investigated in the previous section. We follow the lines of Ref. [9] and consider the theory

$$\mathcal{L} = -U(\phi)F(Y), \tag{34}$$

where Y is defined as

$$Y = -\frac{2^{n-1}}{n}\frac{X|X|^{n-1}}{U(\phi)}. \tag{35}$$

We note that if $n = 1$, we get back to the theory defined in Ref. [9]; also, for $F(Y) = 1 + Y$ we obtain the model introduced in Eq. (1) above.

This new model has the following equation of motion:

$$n\partial_\mu\left(U\frac{Y}{X}F_Y\partial^\mu\phi\right) - U_\phi F + YF_Y U_\phi = 0, \tag{36}$$

and the energy-momentum tensor is given by

$$T_{\mu\nu} = g_{\mu\nu}U(\phi)F(Y) - nU\frac{Y}{X}F_Y\partial_\mu\phi\partial_\nu\phi, \tag{37}$$

where $F_Y = dF/dY$.

As before, here we are interested in static field configurations; so, the equation of motion becomes

$$-2n\left(U\frac{Y}{\phi'}F_Y\right)' + U_\phi F - YF_Y U_\phi = 0. \tag{38}$$

We suppose that $v_i (i = 1, 2, \ldots, n)$ is a set of static and uniform solutions of the equation of motion, meaning that $U'(v_i)$ has to vanish. Also, we use the energy density and take $U(v_i) = 0$ to make the energy itself vanish for the static and uniform solutions. (Recall that the same conditions work for the standard model.)

Since we are considering a new family of models, we guide ourselves with the null energy condition (NEC), that is, we take $T_{\mu\nu}n^\mu n^\nu \geq 0$, where n_μ is a null vector obeying $g_{\mu\nu}n^\mu n^\nu = 0$. This restriction leads to $F_Y \geq 0$ for the general field configuration $\phi(x, t)$, which is supposed to solve the equation of motion (36). Moreover, for static solutions, the energy-momentum tensor gives

$$T_{00} = UF, \tag{39a}$$
$$T_{11} = -UF + 2nYF_Y U. \tag{39b}$$

Equation (38) can be integrated once to give

$$2nYF_Y - F = \frac{C}{U}. \tag{40}$$

Again, C is a constant identified with the stress tensor T_{11}. Furthermore, we have

$$Y = \frac{1}{2n}\frac{\phi'^{2n}}{U(\phi)}. \tag{41}$$

Equation (40) can be written in the form

$$\phi'^{2n} = 2nG\left(\frac{C}{U}\right)U(\phi), \tag{42}$$

where G is a function with inverse $G^{-1}(Y) = 2nYF_Y - F$.

For stressless solutions, that is, for $C = 0$, we have that $2nYF_Y = F$, and if we assume that $G(0) = c$, with c constant and real, we find that $Y = c$. With this result, we can rewrite Eq. (42) in the form

$$\phi'^{2n} = 2ncU(\phi). \tag{43}$$

Here we note that the solution $\phi(x)$ of this equation is the same solution $\phi_s(x)$ of Eq. (7), which appears in the previous model, with the position changed as $x \to \sqrt{m}x$, with $m = [c(2n-1)]^{1/n}$. This means that we can write

$$\phi(x) = \phi_s(\sqrt{m}x), \tag{44}$$

and now the thickness of the solution is given by

$$\delta = \delta_s/\sqrt{m}. \tag{45}$$

Thus, the solution is thicker or thinner, depending on whether the value of c is less or greater than unity. We also note that c cannot be negative; furthermore, only stressless solutions have the specific form given by Eq. (44).

The energy density of the stressless solution (43) takes the form

$$\rho(x) = F(c)U(\phi(x)). \tag{46}$$

The energy is then $E = F(c)\int_{-\infty}^{\infty} dxU(\phi(x))$, or

$$E = \frac{F(c)}{[c(2n-1)]^{1/(2n)}}\int_{-\infty}^{\infty} dyU(\phi_s(y)) \tag{47}$$

$$= \frac{(2n-1)F(c)}{2n[c(2n-1)]^{1/(2n)}}E_s, \tag{48}$$

where E_s is the energy for $F = 1 + Y$.

The solutions with a nonvanishing T_{11} are different from the corresponding solutions of the previous model because they do not have the form given by Eq. (44). We recall that for $T_{11} = C$, only the stressless solutions are stable [16]. Usually, the energy of the other possible solutions are divergent, and the solutions have oscillatory or divergent profiles. We find the same behavior in the standard model.

Now we use again the formalism introduced in Refs. [16,17] to rewrite the energy density for the generalized model. Assuming that Eq. (9) is valid, we have that

$$\phi' = F_Y^{-\frac{1}{2n-1}}W_\phi^{\frac{1}{2n-1}}. \tag{49}$$

The potential is given by

$$U(\phi) = \frac{F_Y^{-\frac{2n}{2n-1}}}{2nY}W_\phi^{\frac{2n}{2n-1}}, \tag{50}$$

and we obtain the energy density in the form

$$\rho(x) = \frac{FF_Y^{-\frac{2n}{2n-1}}}{2nY}W_\phi^{\frac{2n}{2n-1}}. \tag{51}$$

Now, using Eq. (40) with $C = 0$ we have

$$\rho(x) = F_Y^{-\frac{1}{2n-1}} W_\phi^{\frac{2n}{2n-1}}. \tag{52}$$

For $m = 1$, i.e.,

$$c = \frac{1}{2n-1}, \tag{53}$$

we have to impose

$$F_Y((2n-1)^{-1}) = 1 \tag{54a}$$

in order to make Eqs. (49) and (52) identical to Eqs. (10) and (12), respectively. This also imposes that

$$F((2n-1)^{-1}) = \frac{2n}{2n-1}. \tag{54b}$$

The Eqs. (54a) and (54b) are the general restrictions on $F(Y)$ required to make the model the twin of the previous model.

A. Linear stability

Let us again investigate linear stability by introducing small fluctuations $\eta(x, t)$ in the static solution $\phi(x)$, as was done in Sec. II A. Using (26) in (36) we obtain

$$\begin{aligned} &-\partial_\mu\bigg[\phi'^{2n-2}\bigg(-2(n-1)F_Y\partial^\mu\phi\frac{\partial_\alpha\phi\partial^\alpha\eta}{\phi'^2} \\ &\quad + F_Y\partial^\mu\eta + F_{YY}\Delta Y\partial^\mu\phi\bigg)\bigg] \\ &= U_{\phi\phi}(F - YF_Y)\eta + U_\phi(F - YF_Y)_Y\Delta Y, \end{aligned} \tag{55}$$

where

$$\Delta Y \equiv Y\bigg(-\frac{2n}{\phi'^2}\partial_\beta\phi\partial^\beta\eta - \frac{U_\phi}{U}\eta\bigg). \tag{56}$$

Taking $\eta(t, x) = \eta_s(x)\cos(\omega t)$ we get

$$\begin{aligned} &-[q(x)[2nF_{YY}Y + (2n-1)F_Y]\eta_s']' \\ &= \bigg[U_{\phi\phi}(YF_Y - F) - \bigg(\phi'^{2n-1}F_{YY}Y\frac{U_\phi}{U}\bigg)' \\ &\quad - F_{YY}Y^2\frac{U_\phi^2}{U}\bigg]\eta_s + \omega^2 F_Y q(x)\eta_s, \end{aligned} \tag{57}$$

where $q(x) \equiv \phi'^{2n-2}$.

In the case of a stressless solution we can use Eq. (40) with $C = 0$ to transform (57) into the form

$$-[q(x)\eta_s']' + U_{\phi\phi}Y\eta_s = \frac{\omega^2}{A^2}q(x)\eta_s, \tag{58}$$

where

$$A^2 = \frac{2nF_{YY}Y + (2n-1)F_Y}{F_Y}. \tag{59}$$

We note that A is constant for a stressless solution. Also, it is required that A is positive if we want to ensure the hyperbolicity of the differential equation.

Again, we introduce the suggested exchange of variables

$$u(z) = F_Y^{\frac{1}{2}} A^{\frac{1}{2}} \phi'^{n-1}\eta_s(Az). \tag{60}$$

Here we get

$$-u_{zz}(z) + v_2(z)u(z) = \omega^2 u(z), \tag{61}$$

where

$$v_2(z) = nA^2 Y\bigg(\frac{U_{\phi\phi}}{\phi_z^{2n-2}} - (n-1)Y\frac{U_\phi^2}{\phi_z^{4n-2}}\bigg)_{\phi=\phi_s(Az)}. \tag{62}$$

Now, using Eqs. (49) and (50) we obtain

$$v_2(z) = \frac{A^2 F_Y^{-\frac{2}{2n-1}}}{2n-1}\left(n\frac{W_{\phi\phi\phi}}{W_\phi^{\frac{2n-3}{2n-1}}} - \frac{n(n-2)}{2n-1}\frac{W_{\phi\phi}^2}{W_\phi^{\frac{4(n-1)}{2n-1}}}\right). \tag{63}$$

We note that if we impose the twin conditions (54), we obtain $A^2 = 2n - 1 + 2n(2n-1)^{-1}F_{YY}$. With this, we obtain the relation $v_2(x) = v(x)$ if we choose

$$F_{YY}((2n-1)^{-1}) = 0. \tag{64}$$

Thus, we see that it is possible to find twinlike models starting from nonstandard theories. This is a new result, since previously one usually started from a standard field theory in order to construct the related twinlike model.

B. Examples

Let us now specify the function $F(Y)$ in order to illustrate how the formalism introduced above works. The first model we consider is

$$F(Y) = a + bY|Y|^{k-1}. \tag{65}$$

Here we suppose that $k \geq 1$ and a, b are real numbers.

To consider the stressless solution, we write

$$c = \bigg(\frac{a}{2nkb - b}\bigg)^{1/k}. \tag{66}$$

We note that

$$F_Y(c) = bk\bigg(\frac{a}{2nkb - b}\bigg)^{\frac{k-1}{k}} \quad \text{and} \quad F(c) = \frac{2nka}{2nk-1}. \tag{67}$$

The conditions (54) give

$$a = \frac{2nk-1}{k(2n-1)} \quad \text{and} \quad b = \frac{(2n-1)^{k-1}}{k}. \tag{68}$$

The function given by Eq. (65) can be written as

$$F(Y) = \frac{2nk-1}{k(2n-1)} + \frac{(2n-1)^{k-1}}{k}Y|Y|^{k-1}, \tag{69}$$

and so the Lagrange density is

$$\mathcal{L} = (2n-1)^{k-1}\frac{2^{(n-1)k}}{kn^k}\frac{X|X|^{nk-1}}{U^{k-1}} - \frac{2nk-1}{k(2n-1)}U(\phi). \tag{70}$$

To investigate linear stability we have to consider

$$A^2 = 2nk - 1, \tag{71}$$

and so we note that the condition (64) requires that $k = 1$. However, in this case the twin theory is identical to the original model.

We now introduce the second family of models, which obeys the strong condition. Let us consider the following function:

$$F(Y) = 1 + Y + G\left(Y - \frac{1}{2n-1}\right). \tag{72}$$

For the three conditions (same solution, same energy density, and same stability behavior) to be valid, one imposes that

$$G(0) = G'(0) = G''(0) = 0. \tag{73}$$

We can write a general function which obeys these three conditions; it has the form

$$F(Y) = 1 + Y + \sum_{i>2}\beta_i\left(Y - \frac{1}{2n-1}\right)^i, \tag{74}$$

where all the β_i are real parameters.

IV. BRANEWORLD MODELS

Let us now investigate how the twinlike models studied above can be used to represent generalized braneworld models. Here we follow along the lines of Ref. [18]. In this context, we consider an action in five dimensions that describes gravity coupled to a scalar field in the form

$$S = \int d^5x\sqrt{g}\left(-\frac{1}{4}R + \mathcal{L}(\phi, X)\right), \tag{75}$$

where we are using $4\pi G = 1$ and

$$X = \frac{1}{2}\nabla_M\phi\nabla^M\phi, \tag{76}$$

with $M, N = 0, 1, 2, 3, 4$ running on the five-dimensional spacetime. The equation of motion which we obtain is given by

$$G^{NM}\nabla_N\nabla_M\phi + 2X\mathcal{L}_{X\phi} - \mathcal{L}_\phi = 0, \tag{77}$$

where

$$G^{NM} = \mathcal{L}_{XX}\nabla^M\phi\nabla^N\phi + g^{MN}\mathcal{L}_X. \tag{78}$$

The energy-momentum tensor T_{MN} has the form

$$T_{MN} = -g_{MN}\mathcal{L} + \mathcal{L}_X\nabla_M\phi\nabla_N\phi. \tag{79}$$

The line element of the five-dimensional spacetime can be written as $ds^2 = e^{2\mathcal{A}}\eta_{\mu\nu}dx^\mu dx^\nu - dy^2$, where $\mathcal{A}$ is used to describe the warp factor. We suppose that both $\mathcal{A}$ and ϕ are static, such that they only depend on the extra dimension y, that is, $\mathcal{A} = \mathcal{A}(y)$ and $\phi = \phi(y)$. In this case, the equation of motion for the scalar field reduces to

$$(2X\mathcal{L}_{XX} + \mathcal{L}_X)\phi'' - (2X\mathcal{L}_{X\phi} - \mathcal{L}_\phi) = -4\mathcal{L}_X\phi'\mathcal{A}'. \tag{80}$$

Moreover, from the Einstein equations we get

$$\mathcal{A}'' = \frac{4}{3}X\mathcal{L}_X, \tag{81a}$$

$$\mathcal{A}'^2 = \frac{1}{3}(\mathcal{L} - 2X\mathcal{L}_X), \tag{81b}$$

where $X = -\phi'^2/2$ for the static configuration, as before.

To get to the first-order framework, we suppose that

$$\mathcal{A}' = -\frac{1}{3}W(\phi). \tag{82}$$

In this case, Eqs. (81a) and (81b) lead us to, respectively,

$$\phi'\mathcal{L}_X = \frac{1}{2}W_\phi, \tag{83}$$

$$\mathcal{L} - 2X\mathcal{L}_X = \frac{1}{3}W^2(\phi). \tag{84}$$

In the case of the theory (1), the equation of motion (80) becomes

$$(2n-1)\phi'^{2n-2}\phi'' + 4\phi'^{2n-1}\mathcal{A}' = U_\phi. \tag{85}$$

We use Eq. (83) to write

$$\phi' = 2^{\frac{1}{1-2n}}W_\phi^{\frac{1}{2n-1}}, \tag{86}$$

with the potential

$$U(\phi) = \frac{2n-1}{n2^{\frac{4n-1}{2n-1}}}W_\phi^{\frac{2n}{2n-1}} - \frac{1}{3}W^2(\phi) \tag{87}$$

and the energy density

$$T_{00} = e^{2\mathcal{A}}\left(2^{\frac{2n}{1-2n}}W_\phi^{\frac{2n}{2n-1}} - \frac{1}{3}W^2(\phi)\right). \tag{88}$$

Now we have to find the twin model. For this, let us consider a scalar field theory governed by the following Lagrange density:

$$\mathcal{L} = -U(\phi)F(Y) + f(\phi), \tag{89}$$

where Y was defined in Eq. (35), and $f(\phi)$ is to be determined. We use Eqs. (83) and (84) to write, respectively,

$$\phi' = \frac{1}{(2F_Y)^{\frac{1}{2n-1}}} W_\phi^{\frac{1}{2n-1}} \tag{90}$$

and

$$\phi'^{2n} = \frac{F}{F_Y} U(\phi), \tag{91}$$

where we have used $f(\phi) = W^2/3$.

We can rewrite Eq. (91) in the form

$$F = 2nYF_Y. \tag{92}$$

The Lagrange density of the twin brane model then has the following form:

$$\mathcal{L} = -U(\phi)F(Y) + \frac{1}{3}W^2(\phi). \tag{93}$$

Moreover, the energy density is

$$T_{00} = e^{2\mathcal{A}}\left[\frac{2^{\frac{2n}{1-2n}}}{F_Y^{\frac{1}{2n-1}}} W_\phi^{\frac{2n}{2n-1}} - \frac{1}{3}W^2(\phi)\right], \tag{94}$$

which exactly reproduces the previous expression (88) if

$$F_Y(c) = 1 \tag{95a}$$

and, consequently,

$$F(c) = 2n(2n-1)^{-1}. \tag{95b}$$

Thus, the two models have the same solution and the same energy density: these are the two conditions required for the models to be twinlike models.

A. Brane stability

The investigation of the linear stability of the brane-world model can be done following Ref. [18]. The metric is perturbed in the form

$$ds^2 = e^{2A(y)}(\eta_{\mu\nu} + h_{\mu\nu}(y,x))dx^\mu dx^\nu - dy^2, \tag{96}$$

and the scalar field in the form

$$\phi = \phi(y) + \tilde{\phi}(y,x). \tag{97}$$

For the starting model, given by Eq. (1), the first-order contributions to the energy-momentum tensor are

$$\begin{aligned}
\bar{T}^{(1)}_{\mu\nu} &= \frac{\eta_{\mu\nu}e^{2\mathcal{A}}}{3} W_\phi\Big[(n-1)\tilde{\phi}' - 2^{\frac{1}{1-2n}} W_\phi^{\frac{2-2n}{2n-1}} W_{\phi\phi}\tilde{\phi} \\
&\quad + \frac{4}{3}W\tilde{\phi}\Big] - 2e^{2\mathcal{A}}h_{\mu\nu}\Big[2^{\frac{1-4n}{2n-1}} W_\phi^{\frac{2n}{2n-1}} - \frac{1}{3}W^2\Big], \\
\bar{T}^{(1)}_{\mu 4} &= \frac{1}{2}W_\phi \nabla_\mu \tilde{\phi}, \\
\bar{T}^{(1)}_{44} &= \frac{2^{\frac{1}{1-2n}}}{3} W_\phi^{\frac{1}{2n-1}} W_{\phi\phi}\tilde{\phi} - \frac{4}{9}W_\phi W\tilde{\phi} + \frac{2n-1}{3}W_\phi\tilde{\phi}'.
\end{aligned} \tag{98}$$

The first-order contributions to the Einstein equations are

$$\begin{aligned}
&e^{2\mathcal{A}}\Big(\frac{1}{2}\partial_y^2 - \frac{2}{3}W\partial_y\Big)h_{\mu\nu} - \frac{1}{6}\eta_{\mu\nu}e^{2\mathcal{A}}W\partial_y(\eta^{\alpha\beta}h_{\alpha\beta}) \\
&\quad - \frac{1}{2}\eta^{\alpha\beta}(\partial_\mu\partial_\nu h_{\alpha\beta} - \partial_\mu\partial_\alpha h_{\nu\beta} - \partial_\nu\partial_\alpha h_{\mu\beta}) \\
&\quad = \frac{4e^{2\mathcal{A}}\eta_{\mu\nu}}{3}W_\phi\Big[\frac{(n-1)}{2}\tilde{\phi}' - \frac{W_\phi^{\frac{2-2n}{2n-1}}W_{\phi\phi}}{2^{\frac{1}{2n-1}}}\tilde{\phi} + \frac{4W}{3}\tilde{\phi}\Big]
\end{aligned} \tag{99}$$

and

$$\frac{1}{2}\eta^{\alpha\beta}\partial_y(\partial_\alpha h_{\mu\beta} - \partial_\mu h_{\alpha\beta}) = \frac{1}{2}W_\phi\partial_\mu\tilde{\phi} \tag{100}$$

$$\begin{aligned}
&-\frac{1}{2}\Big(\partial_y^2 + \frac{2}{9}W^2\partial_y\Big)\eta^{\alpha\beta}h_{\alpha\beta} \\
&\quad = \frac{1}{3}\frac{1}{2^{\frac{1}{2n-1}}}W_\phi^{\frac{1}{2n-1}}W_{\phi\phi}\tilde{\phi} - \frac{4}{9}W_\phi W\tilde{\phi} + \frac{(2n+1)}{3}W_\phi\tilde{\phi}'.
\end{aligned} \tag{101}$$

The equation of motion for the scalar field leads to

$$\begin{aligned}
&W_\phi^{\frac{2n-2}{2n-1}}e^{2\mathcal{A}}\Box\tilde{\phi} - (2n-1)[W_\phi^{\frac{2n-2}{2n-1}}\tilde{\phi}']' \\
&\quad + \frac{4(2n-1)}{3}WW_\phi^{\frac{2n-2}{2n-1}}\tilde{\phi}' + \frac{2^{\frac{2}{1-2n}}}{2n-1}W_\phi^{\frac{2-2n}{2n-1}}W_{\phi\phi}^2\tilde{\phi} \\
&\quad + \frac{1}{2^{\frac{2}{2n-1}}}W_\phi^{\frac{1}{2n-1}}W_{\phi\phi\phi}\tilde{\phi} - \frac{2^{\frac{4n-3}{2n-1}}}{3}W_{\phi\phi}W\tilde{\phi} - \frac{2^{\frac{4n-3}{2n-1}}}{3}W_\phi^2\tilde{\phi} \\
&\quad = \frac{1}{2^{\frac{1}{2n-1}}}W_\phi\eta^{\alpha\beta}h_{\alpha\beta}.
\end{aligned} \tag{102}$$

For the general model (93), after substituting the two twin conditions (95), one is led to following set of equations:

(i) the energy-momentum components,

$$\begin{aligned}
\bar{T}^{(1)}_{\mu\nu} &= \frac{\eta_{\mu\nu}e^{2\mathcal{A}}}{3}W_\phi\Big[\Big(\frac{nF_{YY}}{2n-1} + n - 1\Big)\tilde{\phi}' \\
&\quad - \Big(1 + \frac{nF_{YY}}{(2n-1)^2}\Big)2^{\frac{1}{1-2n}}W_\phi^{\frac{2-2n}{2n-1}}W_{\phi\phi}\tilde{\phi} + \frac{4}{3}W\tilde{\phi}\Big] \\
&\quad - 2e^{2\mathcal{A}}h_{\mu\nu}\Big[2^{\frac{1-4n}{2n-1}}W_\phi^{\frac{2n}{2n-1}} - \frac{1}{3}W^2\Big],
\end{aligned} \tag{103a}$$

$$\bar{T}^{(1)}_{\mu 4} = \frac{1}{2}W_\phi\nabla_\mu\tilde{\phi}, \tag{103b}$$

$$\begin{aligned}
\bar{T}^{(1)}_{44} &= \frac{2^{\frac{1}{1-2n}}}{3}\Big(1 - 2n\frac{F_{YY}}{(2n-1)^2}\Big)W_\phi^{\frac{1}{2n-1}}W_{\phi\phi}\tilde{\phi} \\
&\quad - \frac{4}{9}W_\phi W\tilde{\phi} + \Big(\frac{2n}{3}\frac{F_{YY}}{2n-1} + \frac{2n-1}{3}\Big)W_\phi\tilde{\phi}';
\end{aligned} \tag{103c}$$

(ii) the Einstein equations,

$$e^{2\mathcal{A}}\left(\frac{1}{2}\partial_y^2-\frac{2}{3}W\partial_y\right)h_{\mu\nu}-\frac{1}{6}\eta_{\mu\nu}e^{2\mathcal{A}}W\partial_y(\eta^{\alpha\beta}h_{\alpha\beta})-\frac{1}{2}\eta^{\alpha\beta}(\partial_\mu\partial_\nu h_{\alpha\beta}-\partial_\mu\partial_\alpha h_{\nu\beta}-\partial_\nu\partial_\alpha h_{\mu\beta})$$
$$=\frac{4e^{2\mathcal{A}}\eta_{\mu\nu}}{3}W_\phi\left[\frac{1}{2}\left(n\frac{F_{YY}}{2n-1}+n-1\right)\tilde{\phi}'-\frac{1}{2^{\frac{2n}{2n-1}}}\left(2+\frac{nF_{YY}}{(2n-1)^2}\right)W_\phi^{\frac{2-2n}{2n-1}}W_{\phi\phi}\tilde{\phi}+\frac{4}{3}W\tilde{\phi}\right] \tag{104}$$

and

$$\frac{1}{2}\eta^{\alpha\beta}\partial_y(\partial_\alpha h_{\mu\beta}-\partial_\mu h_{\alpha\beta})=\frac{1}{2}W_\phi\partial_\mu\tilde{\phi}, \tag{105a}$$

$$-\frac{1}{2}\left(\partial_y^2+\frac{2}{9}W^2\partial_y\right)\eta^{\alpha\beta}h_{\alpha\beta}=-\frac{4}{9}WW_\phi\tilde{\phi}+\frac{1}{3}\left(\frac{1-2n\frac{F_{YY}}{(2n-1)^2}}{2^{\frac{1}{2n-1}}}\right)W_\phi^{\frac{1}{2n-1}}W_{\phi\phi}\tilde{\phi}+\frac{1}{3}\left(\frac{2nF_{YY}}{2n-1}+2n+1\right)W_\phi\tilde{\phi}'; \tag{105b}$$

(iii) and the scalar field equation,

$$W_\phi^{\frac{2n-2}{2n-1}}e^{2\mathcal{A}}\Box\tilde{\phi}-\left(2n\frac{F_{YY}}{2n-1}+2n-1\right)[W_\phi^{\frac{2n-2}{2n-1}}\tilde{\phi}']'+\frac{4}{3}\left(2n\frac{F_{YY}}{2n-1}+2n-1\right)W_\phi^{\frac{2n-2}{2n-1}}W\tilde{\phi}'$$
$$+\left(\frac{1+2n\frac{F_{YY}}{(2n-1)^2}}{2^{\frac{2}{2n-1}}(2n-1)}\right)W_\phi^{\frac{2-2n}{2n-1}}W_{\phi\phi}^2\tilde{\phi}+\left(\frac{2n\frac{F_{YY}}{(2n-1)^2}+1}{2^{\frac{2}{2n-1}}}\right)W_\phi^{\frac{1}{2n-1}}W_{\phi\phi\phi}\tilde{\phi}$$
$$-\frac{2^{\frac{4n-3}{2n-1}}}{3}\left(\frac{2nF_{YY}}{(2n-1)^2}+1\right)W_{\phi\phi}W\tilde{\phi}-\frac{2^{\frac{4n-3}{2n-1}}}{3}W_\phi^2\tilde{\phi}$$
$$=\frac{1}{2^{\frac{1}{2n-1}}}W_\phi\eta^{\alpha\beta}h_{\alpha\beta}. \tag{106}$$

We see that only when

$$F_{YY}=0 \tag{107}$$

are the set of equations equivalent to those corresponding to the starting model. As we know, the study of stability is not a trivial task [25]; however, we can assure here that, using the three conditions—same solution, same energy density, and the strong condition (107)—the linear stabilities of the two models are the same.

In the gravity sector, we can simplify the investigation of stability by considering the transverse traceless components of metric fluctuations,

$$\bar{h}_{\mu\nu}=\left(\frac{1}{2}(\pi_{\mu\alpha}\pi_{\nu\beta}+\pi_{\mu\beta}\pi_{\mu\alpha})-\frac{1}{3}\pi_{\mu\nu}\pi_{\alpha\beta}\right)h^{\alpha\beta}, \tag{108}$$

where $\pi_{\mu\nu}=\eta_{\mu\nu}-\partial_\mu\partial_\nu/\Box$. Indeed, we can check that Eq. (99) reduces to the known equation

$$(\partial_y^2+4\mathcal{A}'\partial_y-e^{-2\mathcal{A}}\Box)\bar{h}_{\mu\nu}=0. \tag{109}$$

The next steps are known: we introduce the z coordinate in order to make the metric conformally flat, with $dz=e^{-A(y)}dy$, and we write

$$H_{\mu\nu}(z)=e^{-ipx}e^{3/2\mathcal{A}(z)}\bar{h}_{\mu\nu}. \tag{110}$$

In this case, the four-dimensional components of $\bar{h}_{\mu\nu}$ obey the Klein-Gordon equation and the metric fluctuations of the brane solution lead to the Schrödinger-like equation

$$[-\partial_z^2+U(z)]H_{\mu\nu}=p^2H_{\mu\nu}, \tag{111}$$

where

$$U(z)=\frac{9}{4}\mathcal{A}'^2(z)+\frac{3}{2}\mathcal{A}''(z). \tag{112}$$

We note that the stability behavior in the gravity sector only depends on the warp factor $\mathcal{A}$, so the first two conditions for the models to be twins—that they have the same solution and energy density—are necessary for the two models to have the same stability behavior in the gravity sector.

Therefore, we can write the following two important conclusions concerning the stability of the two general models, described by Eqs. (1) and (89), in the braneworld context: (i) the stability in the gravity sector is controlled by the warp factor, so the first two conditions for the models to be twins—explicitly, that they have the same solution and energy density—results in the two twin models having the same stability behavior, and (ii) the stabilities in the scalar field sector are in general different, but the strong condition—as given by Eq. (107)—results in the two models having the same stability behavior in the scalar field sector as well.

The above results show that the modifications proposed in the current work are robust and may be of direct interest to high-energy physics.

V. CONCLUSIONS

In this paper we introduced another route to construct twinlike models, starting from one generalized model and generating another. We investigated several examples, showing that the procedure is generic and works for a diversity of models.

To make the investigation stronger, in this paper we also discussed the case of branes with warped geometry, in the scenario with a single extra dimension of infinite extent. Here we also investigated how the first two conditions for the models to be twins—namely, that they have the same solution and the same energy density—and the extra condition, which we called the strong condition, enter the analysis when one investigates stability. The result is that stability in the gravity sector is controlled by the warp factor, so it requires that the models are twins, that is, that they present the same solution with the same energy density. In the scalar field sector, however, the stabilities also have the same behavior if we include the strong condition (107).

The procedure seems to be robust, as it works for several distinct models, is valid in both flat and curved spacetimes, and in the last case is valid for a braneworld model with a single extra dimension of infinite extent.

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