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Two flavor massless Schwinger model on a torus at a finite chemical potential

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We study the thermodynamics of the two flavor massless Schwinger model on a torus at a finite chemical potential. We show that the physics only depends on the isospin chemical potential, and there are marked deviations from a free fermion theory. We argue that spatial inhomogeneities can develop in the system at very low temperatures.

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I. INTRODUCTION

A study of QCD in the presence of a finite chemical potential is important for our understanding of quark matter at finite density [1]. Lattice QCD provides a nonperturbative approach to this problem, and there has been extensive work performed on this topic [2,3]. The fermion determinant in a fixed gauge field background is complex (it can be made real by summing over a gauge field and its complex conjugate, but the result is not necessarily positive) and therefore suffers from the so-called ''sign problem.'' QCD wth a finite isospin chemical potential does not suffer from the sign problem, and the physics of this model has been explored [4,5].

The Schwinger model (QED in two dimensions) has played the role of a very useful toy model for QCD in four dimensions. Generalized Thirring models have been studied in detail at finite temperature and finite chemical potential [6,7]. One main result that applies to the Schwinger model is the independence on the chemical potential. This can be seen as a consequence of the integral over toron fields [8,9] in the path integral formalism. The issue at hand is imposing Gauss's law in the path integral formalism [10]. Imposing Gauss's law in the Hamiltonian formalism results in the condition that the timelike component of the electromagnetic potential vanishes at spatial infinity [11]. This amounts to setting the toron field in one direction on the torus to zero in the path integral [12]. This would allow for states with net total charge to be present but would break the $U(1)$ global symmetry associated with the Polyakov loop in the timelike direction, placing the theory in a deconfined phase. We will study the two flavor massless Schwinger model on a finite torus in the presence of a chemical potential. We will integrate over the toron fields. As expected, the theory will be independent of the chemical potential that couples to the total charge but will depend on the isospin chemical potential.

We start with a definition of the model on a finite torus and state the result for the fermion determinant in a fixed gauge field background using the zeta-function regularization [13]. We will address the integration of the fermion determinant over the toron fields and show that the result is independent of the chemical potential that couples to the total charge. We will proceed to address the physics of the isospin chemical potential.

II. THE GRAND CANONICAL PARTITION FUNCTION

A. Model basics

Let l be the circumference of the spatial circle and let β be the inverse temperature. We will use l to set all scales in the theory and define $\tau = \frac{l}{\beta}$ as the dimensionless temperature. The physical gauge coupling is set to $\frac{e}{l}$ where e is dimensionless.

The Hodge decomposition of the $U(1)$ gauge field on a $l \times \beta$ torus is

$$A_1(x_1, x_2) = \frac{2\pi h_1}{l} + \partial_1 \chi(x_1, x_2) - \partial_2 \phi(x_1, x_2) - \frac{2\pi k}{l\beta} x_2$$
$$A_2(x_1, x_2) = \frac{2\pi h_2}{\beta} + \partial_2 \chi(x_1, x_2) + \partial_1 \phi(x_1, x_2), \quad (1)$$

where $-\frac{1}{2} \le h_\mu < \frac{1}{2}$ are the toron fields and $\chi(x_1, x_2)$ generates gauge transformations. The electric field density is

$$E(x_1, x_2) = \frac{2\pi k}{l\beta} + \partial^2 \phi(x_1, x_2), \quad (2)$$

where $\phi(x_1, x_2)$ is a periodic function on the torus with no zero momentum mode, and k is the integer-valued topological charge. The gauge action is

$$S_g = \frac{2\pi^2 \tau k^2}{e^2} + \frac{l^2}{2e^2} \int d^2x (\partial^2 \phi)^2. \quad (3)$$

The determinant of a massless Dirac fermion is zero unless $k = 0$, and the determinant for $k = 0$ using zeta-function regularization is [7,13]

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$$Z_f(\phi, h_\mu, \mu_i, q_i) = e^{\frac{q_i^2}{2\pi}\int d^2x\phi\partial^2\phi} \frac{1}{\eta^4(\tau)} \times \sum_{n_1,n_2=-\infty}^{\infty} e^{-\pi\tau(n_1+q_ih_2-i\frac{\mu_i}{\tau})^2} \times e^{-\pi\tau(n_2+q_ih_2-i\frac{\mu_i}{\tau})^2} e^{2\pi i q_i h_1(n_1-n_2)}, \quad (4)$$

where q_i is the integer-valued charge of the fermion and $\frac{2\pi\mu_i}{l}$ is the chemical potential. The Dedekind eta function, $\eta(\tau)$ is given by

$$\eta(\tau) = e^{-\frac{\pi\tau}{12}} \prod_{n=1}^{\infty}(1 - e^{-2\pi\tau n}). \quad (5)$$

There is an infinite normalization factor that has been removed by the zeta-function regularization from the above formula. This factor does depend on τ but only in a trivial manner, to shift the zero point energy.

We define the Fourier components of $\phi(x_1, x_2)$ according to

$$\phi(x_1, x_2) = \frac{e}{4\pi^2\tau^{\frac{3}{2}}} \sum_{k_1,k_2=-\infty}^{\prime\infty} e^{\frac{2\pi i}{\beta}(\frac{k_1}{\tau}x_1+k_2x_2)} \tilde{\phi}(k_1, k_2), \quad (6)$$

with $\tilde{\phi}(-k_1, -k_2) = \tilde{\phi}^*(k_1, k_2)$, and the prime over sum implies that $k_1 = k_2 = 0$ is excluded. Then

$$\frac{l^2}{2e^2}\int d^2x(\partial^2\phi)^2 = \frac{1}{2}\sum_{k_1,k_2=-\infty}^{\prime\infty} |\phi(k_1, k_2)|^2\left(k_2^2 + \frac{1}{\tau^2}k_1^2\right)^2 \quad (7)$$

and

$$\frac{q_i^2}{2\pi}\int d^2x\phi(\partial^2\phi) = \frac{e^2q_i^2}{8\pi^3\tau^2}\sum_{k_1,k_2=-\infty}^{\prime\infty} |\phi(k_1, k_2)|^2\left(k_2^2 + \frac{1}{\tau^2}k_1^2\right). \quad (8)$$

B. Bosonic and toronic partition functions

We will consider the two flavor Schwinger model with $q_1 = q_2 = 1$. We write the partition function as

$$Z(\mu_1, \mu_2, \tau, e) = Z_b(\tau, e)Z_t(\mu_1, mu_2, \tau), \quad (9)$$

where the first factor is the bosonic (ϕ) partition function and the second factor is the toronic (h_μ) partition function.

Since the total action (gauge and fermionic contribution) is a quadratic function in ϕ, the bosonic partition function is

$$Z_b(\tau, e) = \prod_{k_1,k_2=-\infty}^{\prime\infty} \sqrt{\frac{1}{(k_2^2 + \frac{1}{\tau^2}k_1^2)\left(k_2^2 + \frac{1}{\tau^2}\left[k_1^2 + \frac{e^2}{2\pi^3}\right]\right)}}. \quad (10)$$

Starting from (4) and after a little bit of algebra, the toronic partition function can be reduced to

$$Z_t(\mu_1, \mu_2, \tau) = \eta^{-4}(\tau) \sum_{m_1,m_2,m_3=-\infty}^{\infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} dh_2 e^{-\pi\tau[(m_2-m_1+2m_3+2h_2-i\frac{\mu_1+\mu_2}{\tau})^2+(m_1-i\frac{\mu_1-\mu_2}{\tau})^2+m_2^2]}. \quad (11)$$

Consider the integral

$$Z_3\left(k, \frac{\mu_1 + \mu_2}{\tau}\right) = \sum_{m_3=-\infty}^{\infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} dh_2 e^{-\pi\tau(k+2m_3+2h_2-i\frac{\mu_1+\mu_2}{\tau})^2}. \quad (12)$$

Viewing $z = h_2 - i\frac{\mu_1+\mu_2}{2\tau}$ as a complex variable, we see that the integrand is analytic in z and periodic under $z \to z + 1$. Therefore, the integral is independent of $\frac{\mu_1+\mu_2}{\tau}$. We explicitly see that the partition function does not depend on the chemical potential that couples to the total charge. Note that the integrand is positive definite if we set $(\mu_1 + \mu_2) = 0$, but this is not the case for a general $(\mu_1 + \mu_2)$. One will encounter a sign problem if one tries to compute the integral numerically with $(\mu_1 + \mu_2)$ not equal to zero. The integral is the same for all even k and the same for all odd k. We can write the reduced integral as

$$Z_3^k(\tau) = \sum_{m_3=-\infty}^{\infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} dh_2 e^{-\pi\tau(k+2m_3+2h_2)^2}; \quad k = 0, 1. \quad (13)$$

Setting the dimensionless isospin chemical potential equal to

$$\mu_I = 2\pi(\mu_1 - \mu_2), \quad (14)$$

we have

$$Z_t(\mu_I, \tau) = \eta^{-4}(\tau)e^{\frac{\mu_I^2}{4\pi\tau}} \sum_{m_1,m_2=-\infty}^{\infty} \cos(m_1\mu_I)e^{-\pi\tau(m_1^2+m_2^2)} \times Z_3^{\mathrm{mod}(m_2-m_1,2)}(\tau). \quad (15)$$

Let

$$Z_2^0(\tau) = \sum_{m_2=-\infty}^{\infty} [e^{-\pi\tau(2m_2)^2}Z_3^0(\tau) + e^{-\pi\tau(2m_2+1)^2}Z_3^1(\tau)];$$

$$Z_2^1(\tau) = \sum_{m_2=-\infty}^{\infty} [e^{-\pi\tau(2m_2+1)^2}Z_3^0(\tau) + e^{-\pi\tau(2m_2)^2}Z_3^1(\tau)]. \quad (16)$$

The final expression for the toronic partition function is

$$Z_t(\mu_I,\tau) = \eta^{-4}(\tau) e^{\frac{\mu_I^2}{4\pi\tau}} \sum_{k=0}^{1} \sum_{m_1=-\infty}^{\infty} \cos((2m_1+k)\mu_I) \times e^{-\pi\tau(2m_1+k)^2} Z_2^k(\tau). \tag{17}$$

C. Thermodynamic observables

The only contribution to the isospin number comes from the toronic partition function and is

$$N_I = \tau \frac{\partial \ln Z(\mu_I,\tau,e)}{\partial \mu_I} = \frac{\mu_I}{2\pi} - f(\mu_I,\tau), \tag{18}$$

where

$$f(\mu_I,\tau) = \tau \frac{\sum_{k=0}^{1}\sum_{m_1=-\infty}^{\infty}(2m_1+k)\sin((2m_1+k)\mu_I)e^{-\pi\tau(2m_1+k)^2}Z_2^k(\tau)}{\sum_{k=0}^{1}\sum_{m_1=-\infty}^{\infty}\cos((2m_1+k)\mu_I)e^{-\pi\tau(2m_1+k)^2}Z_2^k(\tau)}. \tag{19}$$

The dimensionless energy is

$$E(\tau;N_I,e) = -\frac{\partial \ln Z(\mu_I,\tau,e)}{\partial \frac{1}{\tau}}\Bigg|_{\frac{\mu_I}{\tau}} = E_b(\tau;e) + E_t(\tau;N_I) \tag{20}$$

and the contributions from the bosonic partition function and the toronic partition function are written separately. The result from the toronic partition function is

$$\frac{E_t(\tau;N_I)-E_t(0;0)}{\pi} = \frac{1}{3}(\tau^2+1) - \sum_{n=1}^{\infty}\frac{8n\tau^2}{e^{2\pi n\tau}-1} + N_I^2 - f^2(\mu_I,\tau) + g(\mu_I,\tau), \tag{21}$$

where

$$g(\mu,\tau) = \tau^2 \frac{\sum_{k=0}^{1}\sum_{m_1=-\infty}^{\infty}\cos((2m_1+k)\mu_I)\frac{d}{d(\pi\tau)}[e^{-\pi\tau(2m_1+k)^2}Z_2^k(\tau)]}{\sum_{k=0}^{1}\sum_{m_1=-\infty}^{\infty}\cos((2m_1+k)\mu_I)e^{-\pi\tau(2m_1+k)^2}Z_2^k(\tau)}. \tag{22}$$

The first term is the Fermi energy that grows quadratically with the isospin number, and second term is the leading- order low temperature correction that is positive and quadratic in the temperature.

The result from the bosonic partition function is

$$E_b(\tau;e) - E_b(0;e) = \tau - \frac{e}{\sqrt{2\pi}}\left[\tanh\frac{e}{\sqrt{2\pi\tau}} - 1\right] - 4\pi\sum_{k_1=1}^{\infty}k_1\left[\tanh\frac{\pi k_1}{\tau} - 1\right]. \tag{23}$$

D. Free fermions

In order to understand the results for the two flavor massless Schwinger model, it is useful to recall that the partition function for free fermions in one dimension is given by

$$\ln Z_f = \frac{2l}{\pi}\int_0^\infty dp[\beta p + \ln(1+e^{-\beta(p-\mu_f)}) + \ln(1+e^{-\beta(p+\mu_f)})], \tag{24}$$

where μ_f is the chemical potential for free fermions which we set to $\frac{\mu_I}{2l}$ in order to be consistent with the two flavor notation in (14). The free fermion isospin number is given by

$$N_f = \tau \frac{\partial \ln Z_f}{\partial \mu_I}\Bigg|_\beta = \frac{\mu_I}{2\pi}. \tag{25}$$

After subtracting the zero point energy, the dimensionless energy of free fermions at low temperatures is given by

$$\frac{E_f(\tau;N_f)-E_f(0,0)}{\pi} = N_f^2 + \frac{1}{3}\tau^2 + \cdots. \tag{26}$$

III. RESULTS AND DISCUSSION

We proceed to compare the results for the two flavor Schwinger model with that for free fermions. The result for the isospin number in (18) is plotted in Fig. 1 for several

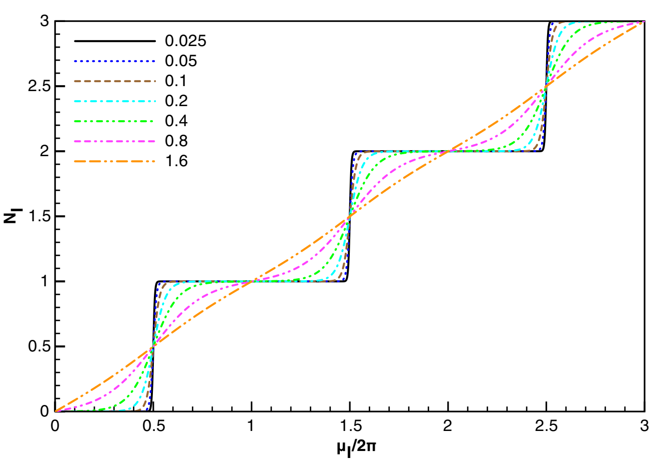

FIG. 1 (color online). Plot of $N_I(\mu_I,\tau)$ versus μ_I [cf. (18)] for several different values of τ.

different values of temperature. The linear behavior in (25) expected of two flavors of free fermions in (25) is the first term in (18), and this is achieved only in the high temperature limit, where the contribution from the second term goes to zero. The first term is the naïve contribution from two flavors of free fermions. The second term is the result of integrating the effect of boundary conditions over all possible choices.

We can use Fig. 1 to see how the isospin chemical potential depends on the temperature at a fixed isospin number. The quasiperiodicity seen in the figure is a consequence of $f(\mu_I + 2\pi, \tau) = f(\mu_I, \tau)$ in (19). Furthermore, $f(\mu_I, \infty) = 0$, and

$$\frac{\mu_I(\infty)}{2\pi} = N_I, \tag{27}$$

like for free fermions. On the other hand,

$$\lim_{\tau\to 0} f(\mu_I, \tau) = \begin{cases} \frac{\mu_I}{2\pi} & \text{if } 0 < \mu_I < \pi \\ \frac{\mu_I}{2\pi} - 1 & \text{if } \pi < \mu_I < \pi \end{cases}. \tag{28}$$

Therefore,

$$\frac{\mu_I(0)}{\pi} = \lceil N_I \rceil \tag{29}$$

for all noninteger values of N_I and $\lceil N_I \rceil$ is the ceiling function. The behavior for nonzero and finite temperatures is to interpolate between (27) and (29) as shown in Fig. 2. The behavior is shown for values of N_I in the range $0 \le N_I \le 3$ in steps of 0.1. Plots are color coded to show periodicity of $f(\mu_I, \tau)$. Since $f(2n\pi, \tau) = 0$ for all values of $\tau > 0$ and any integer n, we see that integer values of N_I are special and behave like free fermions for all temperatures.

Since the partition function is independent of $(\mu_1 + \mu_2)$, the net charge is zero. However, we can maintain the system at a nonzero isospin number, N_I. Since the system can exchange particles with the reservoir, the expectation value of the isospin number need not be an integer. Let us assume that we start at a high temperature with a fixed N_I assumed to take an integer value. Since the chemical potential does not change with temperature for this case and remains the unique value for this particular value of N_I, the system will remain homogeneous at all temperatures. Now consider noninteger values of N_I. The system will be homogeneous at high temperatures since the chemical potential is different for different values of N_I. As the system is cooled and brought down to zero temperature, different values of N_I can coexist as long as the different values all have the same ceiling value, $\lceil N_I \rceil$, since they all have the same chemical potential at zero temperature [cf. (29)]. The system is bound to form inhomogeneities at zero temperature.

We now proceed to use (18) and (21) to compute the toronic contribution to the energy as a function of the temperature at fixed isospin number. Consider the zero temperature limit in order to extract the Fermi energy as a function of the isospin number. Since $f(2n\pi, 0)$ and $g(2n\pi, 0)$ are zero, it follows that the Fermi energy for integer values of isospin are given by the free fermion value. As τ goes to zero, $g(\mu, \tau)$ approaches a nonzero limit as long as N_I is not an integer. As a consequence, the Fermi energy is given by

$$E_F(N_I) = \lfloor N_I \rfloor^2 + (2\lfloor N_I \rfloor + 1)(N_I - \lfloor N_I \rfloor), \tag{30}$$

and it linearly interpolates between the free fermion values at integer values of N_I. We were unable to analytically obtain an explicit expression for the linear coefficient in (21). We numerically evaluated it and found that the second term in (21) contributes $\frac{2\tau}{\pi}$, and the last two terms in (21) contribute $-\frac{3\tau}{2\pi}$. The leading behavior of the toronic contribution to the energy at low temperature is

$$\frac{E_t(\tau; N_I) - E_t(0;0)}{\pi} = E_F(N_I) + \frac{\tau}{2\pi} + \cdots. \tag{31}$$

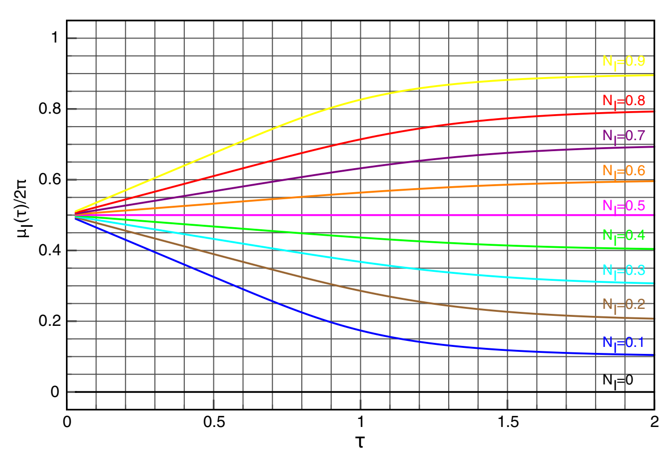

FIG. 2 (color online). Plot of $\mu_I(\tau)$ versus τ [cf. (18)] for several different values of N_I in steps of 0.1 starting from 0 and ending in 3.

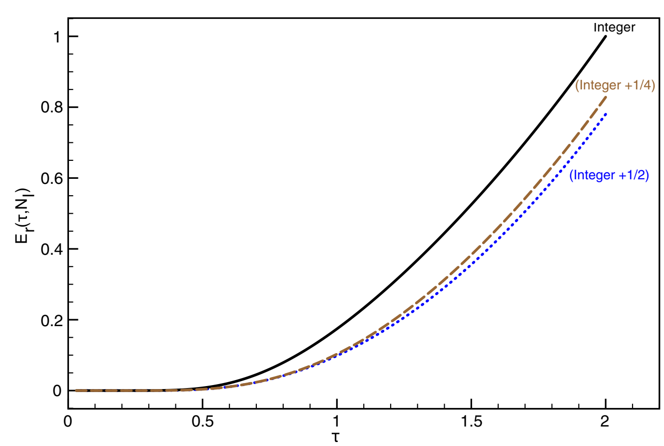

FIG. 3 (color online). Plot of $E_r(\tau; N_I)$ versus τ [cf. (32)] for $N_I = 0.1, 0.2, 0.3, 0.4, 0.5$. The color coding is the same as Fig. 2.

This is qualitatively different from the free fermion result where the leading term is quadratic in τ. The linear coefficient of $\frac{1}{2\pi}$ in (31) gets modified to $\frac{3}{2\pi}$ for the total energy when we include the leading contribution from the bosonic partition function in (23).

The higher-order corrections in τ to the energy from the toronic partition function,

$$E_r(\tau, N_I) = \frac{E_t(\tau; N_I) - E_t(0;0)}{\pi} - E_F(N_I) - \frac{\tau}{2\pi}, \qquad (32)$$

is plotted in Fig. 3 as a function of τ for various values of $N_I(N_I = 0.1, 0.2, 0.3, 0.4, 0.5)$. Due to quasiperiodicity, $E_r(\tau, 1 - N_I) = E_r(\tau, N_I)$ for $(0 < N_I \leq 0.5)$. In addition, $E_r(\tau, N_I + 1) = E_R(\tau, N_I)$.

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