PHYSICAL REVIEW D **86,** 125004 (2012)

Josephson vortices and the Atiyah-Manton construction

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(Received 3 August 2012; published 3 December 2012)

We show that sine-Gordon solitons appear in the low-energy effective theory of a domain wall in a $U(1)$ gauge theory with two charged complex scalar fields with masses if we introduce the Josephson interaction term between the scalar fields. We identify these sine-Gordon solitons as vortices or $\mathbf{C}P^1$ sigma model instantons in the bulk, which are absorbed into the domain wall world volume. These vortices can be called Josephson vortices since they appear in Josephson junctions of two superconductors. This setup gives a physical realization of a lower-dimensional analogue of Atiyah-Manton construction of Skyrmions from instanton holonomy.

DOI: 10.1103/PhysRevD.86.125004 PACS numbers: 11.27.+d, 11.30.Pb, 12.10.−g, 98.80.Cq

I. INTRODUCTION

The Skyrme model was proposed to describe nucleons as solitons (Skyrmions) in the pion effective field theory or the chiral Lagrangian [1]. Although the nucleons are now known as bound states of quarks, the idea of the Skyrme model is still attractive. In fact, the Skyrme model is still valid as the low-energy description of QCD, for instance, in holographic QCD [2,3]. One of the difficulties of the Skyrme model is that no Skyrme solutions are available because of the nonintegrability of the equation of motion, though construction of approximate solutions was proposed [4]. Among several proposals, Atiyah and Manton gave a particularly interesting proposal that Skyrmion solutions can be approximated by the holonomy of Yang-Mills instantons [5]. It has been applied, for instance, to calculate the force between two Skyrmions from the two-instanton holonomy [6]. While the physical meaning of this ansatz was unclear for long time, a physical "proof" of the Atiyah-Manton ansatz was presented some years ago [7]. One can consider a non-Abelian domain wall in a certain $U(2)$ gauge theory in $d = 5 + 1$ dimensions [8], the low-energy effective theory of which is the chiral Lagrangian at the leading order in its $d = 3 + 1$-dimensional world volume. The next leading order contains the Skyrme term [7], which implies that the domain wall world-volume theory is the Skyrme model admitting Skyrmions within it. It was shown that these Skyrmions are nothing but Yang-Mills instantons in the bulk point of view. Since we perform the integration along the codimension of the wall to obtain the effective wall world-volume theory, it gives a physical explanation of the Atiyah-Manton ansatz.

On the other hand, a lower-dimensional analogue of the Atiyah-Manton ansatz was also proposed [9,10]. It was proposed that the sine-Gordon soliton can be approximately constructed as the holonomy of a $\mathbf{C}P^1$ instanton in $d = 2 + 0$ dimensions or a lump in $d = 2 + 1$ dimensions. Since exact solutions of the sine-Gordon solitons are available, the lower-dimensional Atiyah-Manton ansatz can be checked analytically, unlike the original Atiyah-Manton construction. It may help us to understand better or to refine the original proposal by Atiyah and Manton.

In this paper, we give a physical realization of the lower-dimensional Atiyah-Manton construction. We consider the $U(1)$ gauge theory coupled with two charged complex scalar fields ϕ_1 and ϕ_2 with masses in $d = 2 + 1$ dimensions, which reduces to the $\mathbf{C}P^1$ model in the strong gauge coupling limit. This model can be supersymmetric by properly adding bosonic and fermionic fields [11]. This model is known to admit a domain wall solution [12,13]. We add a deformation term $\phi^{1*}\phi^2$ in the original Lagrangian which breaks supersymmetry. This term is known as the Josephson term in the Josephson junction of two superconductors with two condensates ϕ^1 and ϕ^2. We show that this term induces the sine-Gordon potential in the effective theory of the $d = 1 + 1$-dimensional domain wall world volume. We find that the sine-Gordon soliton in the domain wall world volume is nothing but an instanton or a lump in the $\mathbf{C}P^1$ model or a vortex in the gauge theory in the $d = 2 + 1$-dimensional bulk. We call this object the Josephson vortex. This terminology is borrowed from the Josephson junction.

Kinks inside a domain wall were also studied in supersymmetric gauge theories [14–17]. In particular, our work is closely related to a previous work [16], in which an $\mathcal{N} = 1$ supersymmetry preserving deformation term of $\mathcal{N} = 2$ supersymmetry was considered in $d = 3 + 1$. The domain wall is precisely the same as ours [12,13] without the deformation. In that model, too, the effective theory of the domain wall is the sine-Gordon model, and the flux absorbed in the domain wall is a sine-Gordon soliton. However, the crucial difference with ours is that the minimum flux inside the domain wall is half-quantized in that case, while it is unit quanta in our case.

In the limit that the domain wall is infinitely heavy, our model is close to a Josephson junction of two superconductors of two condensations ϕ_1 and ϕ_2 sandwiching an insulator. Vortices in the bulk are absorbed into the insulator, becoming Josephson vortices or fluxons; see Ref. [18] as a review. As in our case, dynamics of Josephson vortices can be described by the sine-Gordon equation. Josephson vortices also appear in high-T_c superconductors with multilayered structures [19] and in two coupled Bose-Einstein condensates [20].

In addition, a kink inside a domain wall appears in several systems in condensed matter physics: a Bloch line in a Bloch wall in magnetism [21], chiral p-wave superconductors, and a Mermin-Ho vortex within a domain wall in ^{3}He superfluid (see Fig. 16.9 of Ref. [22]). Therefore, our method of a field theoretical approach may be applied to these condensed matter systems.

This paper is organized as follows. After our model is given in Sec. II, we present the main results in Sec. III; we construct the domain wall effective theory by the moduli approximation of Manton [23] and find it to be the sine-Gordon model when we add the Josephson term in the original theory. We then construct sine-Gordon kinks and show that they carry instanton (lump) charge in the bulk. Section IV is devoted to a summary and discussion. An application to the Atiyah-Manton construction is briefly discussed.

II. THE MODEL

We consider the $U(1)$ gauge theory coupled with two charged complex scalar fields $\phi^1(x)$ and $\phi^2(x)$ with masses and real scalar field $\Sigma(x)$ in $d = 2 + 1$ dimensions. The Lagrangian which we consider is given by

$$\mathcal{L} = -\frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} + \frac{1}{e^2}(\partial_\mu\Sigma)^2 + |D_\mu\Phi|^2 - V \tag{1}$$

$$V = \frac{e^2}{2}(\Phi^\dagger\Phi - v^2)^2 + \Phi^\dagger(\Sigma\mathbf{1}_2 - M)^2\Phi - \beta^2\Phi^\dagger\sigma_x\Phi, \tag{2}$$

where e is the gauge coupling, complex scalar fields are written as $\Phi^T = (\phi^1, \phi^2)$, and the masses are given by $M = \mathrm{diag}(m_1, m_2)$ with $m_1 > m_2$.

We refer to the last term in the potential

$$\mathcal{L}_J = \beta^2\Phi^\dagger\sigma_x\Phi = \beta^2\phi^{1*}\phi^2 + \mathrm{c.c.} \tag{3}$$

as the "Josephson" interaction term, because it appears in the Josephson junction of two superconductors with two condensations ϕ^1 and ϕ^2. In the limit $\beta = 0$, the model enjoys $\mathcal{N} = 4$ supersymmetry (with eight supercharges) in $d = 2 + 1$ with appropriately adding scalar fields $\tilde{\Phi} = (\tilde{\phi}^1, \tilde{\phi}^2)$ and fermion superpartners. In this case, the Josephson term breaks supersymmetry explicitly. In this paper, supersymmetry is not essential apart from technical reasons [24].

For explicit calculation, we work in the strong gauge coupling limit $e^2 \to \infty$ in which the model reduces to the $\mathbf{C}P^1$ model with potential terms, but the results in this paper do not rely on this limit. By rewriting $\Phi^T = (1, u)/\sqrt{1 + |u|^2}$ with complex projective coordinate u, the Lagrangian becomes

$$\mathcal{L} = \frac{\partial_\mu u^*\partial^\mu u - m^2|u|^2}{(1 + |u|^2)^2} + \beta^2 D_x, \qquad D_x \equiv \frac{u + u^*}{1 + |u|^2}, \tag{4}$$

with the mass $m \equiv m_1 - m_2$. Here, D_x is a moment map of the isometry generated by σ_x. With $\beta = 0$, this model is known as the massive $\mathbf{C}P^1$ model with the potential term of the Killing vector squared corresponding to the isometry generated by σ_z. It is a truncated version of a supersymmetric sigma model with eight supercharges [11]. The potential of this model

$$V = \frac{m^2|u|^2}{(1 + |u|^2)^2} - \frac{\beta^2(u + u^*)}{1 + |u|^2} \tag{5}$$

admits two discrete vacua $u = 0$ and $u = \infty$ (for $\beta < m$).

Just for convenience, we can rewrite the Lagrangian in Eq. (4) to another form. Introducing a three-vector of scalar fields by $\mathbf{n}(x) \equiv \Phi^\dagger\vec{\sigma}\Phi$ with the Pauli matrices $\vec{\sigma}$, the Lagrangian can be rewritten in the form of the $O(3)$ model:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\mathbf{n}\cdot\partial^\mu\mathbf{n} - m^2(1 - n_z^2) + \beta^2 n_x, \quad \mathbf{n}^2 = 1. \tag{6}$$

This model is known as the Heisenberg ferromagnet with anisotropy with two easy axes.

III. SINE-GORDON SOLITONS FROM $\mathbf{C}P^1$ INSTANTONS INSIDE A DOMAIN WALL

A. Domain wall solution

For a while, we consider the case $\beta = 0$, and we turn on it later. There are two discrete vacua $u = 0$ and $u = \infty$. Let us construct a domain wall perpendicular to the x^1 axis, interpolating these two vacua. The Bogomol'nyi completion for the domain wall can be obtained as

$$E = \int dx^1 \frac{|\partial_1 u \mp mu|^2 \pm m(u^*\partial_1 u + u\partial_1 u^*)}{(1 + |u|^2)^2} \geq |T_{\mathrm{wall}}|, \tag{7}$$

where ∂_i denotes the differentiation with respect to x^i. Here, T_{wall} is the topological charge which characterizes the wall:

$$T_{\mathrm{wall}} = m\int dx^1 \frac{u^*\partial_1 u + u\partial_1 u^*}{(1 + |u|^2)^2} = \frac{m}{2}\left[\frac{1 - |u|^2}{1 + |u|^2}\right]_{x^1=-\infty}^{x^1=+\infty}. \tag{8}$$

Among all configurations with a fixed boundary condition, that is, with a fixed topological charge T_{wall}, the most stable configurations with the least energy saturate the inequality (7) and satisfy the Bogomol'nyi-Prasad-Sommereld (BPS) equation

$$\partial_1 u \mp mu = 0, \tag{9}$$

which is obtained by $|\ldots|^2 = 0$ in Eq. (7). This BPS equation can be immediately solved as [12,13]

$$u_{\rm dw} = e^{\pm m(x^1 - X) + i\varphi}, \tag{10}$$

with the width $\Delta x^1 = 1/m$ and the tension

$$|T_{\rm wall}| = m, \tag{11}$$

where $\pm$ denotes a domain wall and an antidomain wall. Here, X and φ are real constants called moduli parameters which are Nambu-Goldstone modes associated with broken translational and internal $U(1)$ symmetries, respectively.

B. Low-energy effective theory on domain wall world volume

Next, let us construct the effective field theory of the domain wall [+ signature in Eq. (10)]. According to Manton [23], the effective theory on the domain wall can be obtained by promoting the moduli parameters to fields $X(x^i)$ and $\varphi(x^i)$ on the domain wall world volume x^i ($i = 0, 2$) and by performing the integration over the codimension $x \equiv x^1$:

$$\mathcal{L}_{\rm dw\ eff} = \int_{-\infty}^{+\infty} dx \frac{e^{2mx}}{(1 + e^{2mx})^2}[(\partial_i X)^2 + (\partial_i \varphi)^2] = \frac{1}{2m}[(\partial_i X)^2 + (\partial_i \varphi)^2] - m, \tag{12}$$

where the constant term recovers the domain wall tension. This is just a free field theory, or a nonlinear sigma model with the target space $\mathbf{R} \times S^1$.

Let us turn on the Josephson term ($\beta \neq 0$). We work in the parameter region $\beta \ll m$. We assume that the domain wall solution (10) is not deformed. The domain wall effective action is deformed by

$$\Delta\mathcal{L} = \beta^2 \int_{-\infty}^{+\infty} dx \frac{e^{mx+i\varphi} + e^{mx-i\varphi}}{1 + e^{2mx}} = \frac{2}{m}\int_{-\infty}^{+\infty} dx \frac{e^{mx}}{1 + e^{2mx}} 2\cos\varphi = \frac{\pi\beta^2}{m}\cos\varphi. \tag{13}$$

Finally, we thus obtain the domain wall effective theory as

$$\mathcal{L}_{\rm dw\ eff} = \frac{1}{2m}[(\partial_i X)^2 + (\partial_i \varphi)^2 + 2\pi\beta^2\cos\varphi] = \frac{1}{2m}[(\partial_i X)^2 + (\partial_i \varphi)^2 + \tilde\beta^2\cos\varphi], \tag{14}$$

with $\tilde\beta^2 \equiv 2\pi\beta^2$ apart from the constant term. This is the sine-Gordon model with the additional field X.

C. The sine-Gordon soliton inside the domain wall

Next, we construct a sine-Gordon kink in the domain wall effective theory and identify what it is in the bulk. The Bogomol'nyi completion for the energy density corresponding to the Lagrangian in Eq. (14) is obtained (for $X = 0$) as

$$\begin{aligned} 2mE &= (\partial_2\varphi)^2 + \tilde\beta^2\left(\sin^2\frac{\varphi}{2} - 1\right) \\ &= \left(\partial_2\varphi \pm \tilde\beta\sin\frac{\varphi}{2}\right)^2 \mp 2\tilde\beta\partial_2\varphi\sin\frac{\varphi}{2} - \tilde\beta^2 \\ &\geq 2m|t_{\rm SG}| - \tilde\beta^2, \end{aligned} \tag{15}$$

with the topological charge density

$$t_{\rm SG} \equiv \frac{\tilde\beta}{m}\partial_2\varphi\sin\frac{\varphi}{2} = -\frac{2\tilde\beta}{m}\partial_2\left(\cos\frac{\varphi}{2}\right). \tag{16}$$

The inequality is saturated by the BPS equation

$$\partial_2\varphi \pm \tilde\beta\sin\frac{\varphi}{2} = 0. \tag{17}$$

For instance, the one-kink solution can be given as

$$\varphi(x^2) = 4\arctan\exp\frac{\tilde\beta}{4}(x^2 - Y) + \frac{\pi}{2}, \tag{18}$$

with the position Y in the x^2 coordinate. The topological charge for this solution is

$$T_{\rm SG} = \int dx^2 t_{\rm SG} = \frac{4\tilde\beta}{m}. \tag{19}$$

The width of the sine-Gordon kink is $\Delta x^2 = 1/\tilde\beta$ so that we have a relation $\Delta x^1/\Delta x^2 \sim m/\tilde\beta$. The total configuration is schematically drawn in Fig. 1(a). In Fig. 1(b), we plot the spin texture of the $\mathbf{C}P^1$ target space for this configuration.

What is this solution in the $d = 2 + 1$-dimensional bulk theory? We now show that this is a $\mathbf{C}P^1$ instanton (lump) in $d = 2 + 1$. Let us calculate the topological lump (instanton) charge by ($a = 1, 2$):

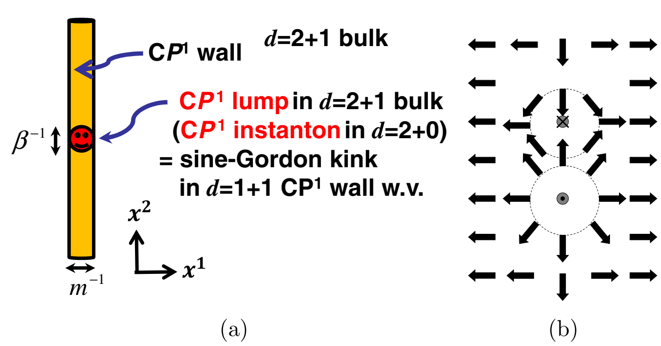

FIG. 1 (color online). A sine-Gordon soliton in the domain wall describing the $\mathbf{C}P^1$ lump inside the domain wall. (a) Schematic configuration in the entire space. (b) Points in the $\mathbf{C}P^1$ target space are denoted by three-dimensional arrows. The north and south poles are denoted by the left and right arrows, respectively.

$$
\begin{aligned}
T_{\text{lump}} &= \int d^2x \frac{i(\partial_1 u^* \partial_2 u - \partial_2 u^* \partial_1 u)}{(1+|u|^2)^2} \\
&= \oint dx^a \frac{-i(u^* \partial_a u - (\partial_a u^*) u)}{2(1+|u|^2)} \\
&= \oint dx^a \frac{|u|^2}{1+|u|^2} \partial_a \varphi \\
&= \int dx^2 \partial_2 \varphi|_{x^1=+\infty} = [\varphi]^{(x^1,x^2)=(+\infty,+\infty)}_{(x^1,x^2)=(+\infty,-\infty)} \\
&= 2\pi k.
\end{aligned} \tag{20}
$$

Here, we have used $\partial_1 \varphi = 0$ at $x^2 = \pm\infty$ in the third-to-last equality and the k winding of the phase φ for k sine-Gordon kinks in the last equality. This precisely shows the coincidence between the topological charges for k lumps and k sine-Gordon kinks.

Equivalently, this charge can be rewritten as the vortex charge

$$
T_{\text{vortex}} = \int d^2x F_{12} = \oint dx^a A_a = T_{\text{lump}}, \tag{21}
$$

with the (auxiliary) $U(1)$ gauge field

$$
A_\mu = \frac{i}{2}(\Phi^\dagger \partial_\mu \Phi - (\partial_\mu \Phi^\dagger)\Phi) = \frac{-i(u^* \partial_\mu u - (\partial_\mu u^*) u)}{2(1+|u|^2)}. \tag{22}
$$

If we work in finite gauge coupling e instead of taking the infinite coupling limit, lumps are replaced with vortices with the charge in Eq. (21) counting the magnetic fluxes, where A_μ is a dynamical gauge field which cannot be written as Eq. (22).

Although the charges and the numbers of the sine-Gordon kinks in the wall and the lumps in the bulk coincide, the more detailed information, such as their shape, can be deformed. In fact, the spin texture of the sine-Gordon kink in the domain wall shows that the lump is split into a pair of a vortex and an antivortex. Each of them has a half lump charge so that they are fractional lumps, that is, merons.

We have used the Bogomol'nyi completion to obtain the domain wall and sine-Gordon kinks. However, the composite state is not BPS anymore because the Josephson term breaks supersymmetry. This implies the existence of the static interaction between the domain wall and the vortex in the bulk. Although both the sine-Gordon topological charge in Eq. (19) and the lump charge in Eq. (20) are proportional to the soliton number, their coefficients do not coincide. The former can be interpreted as the kink energy on the domain wall and the latter as the vortex energy in the bulk. We thus find that the energies of the vortex are smaller inside the wall than in the bulk in the small β regime ($\beta \ll m$) which we are working in. Therefore, we conclude that there exists the attraction between the vortex in the bulk and the domain wall and that the vortex is absorbed into the domain wall world volume, becoming the stable Josephson vortex.

D. Extension and related model

We can extend our model to $U(1)$ gauge theory coupled with N (more than two) charged complex scalar fields $\phi^i(x)$ $(i = 1, \ldots, N)$ with $M = \text{diag}(m_1, \ldots, m_N)$ with $m_i > m_{i+1}$. A natural choice of Josephson terms may be introduced between two neighboring pairs [25]:

$$
\mathcal{L}_J = \sum_{i=1}^{N-1} \beta_i^2 \phi^{i*} \phi^{i+1} + \text{c.c.} \tag{23}
$$

In the absence of the Josephson terms, the model reduces to the massive $\mathbf{C}P^{N-1}$ model, admitting $N-1$ parallel domain walls [27,28]. With the Josephson terms, this describes arrays of N Josephson junctions. Vortices ($\mathbf{C}P^{N-1}$ instantons or lumps) in various components will be absorbed in each domain wall, which should be studied elsewhere.

Finally let us make comments on the previous work [16], where a nonlinear sigma model on the target space $(T^*)\mathbf{C}P^1$ is considered. With the mass matrix $M = \text{diag}(m, -m)$, the model admits a domain wall whose effective theory is a free theory, a sigma model on $\mathbf{C}^* = \mathbf{R} \times S^1$ in $d = 1+1$ as ours. On the other hand, with the mass matrix

$$
M = \begin{pmatrix} m & -\beta/\sqrt{2} \\ \beta/\sqrt{2} & -m \end{pmatrix},
$$

the model admits a domain wall whose effective theory is the sine-Gordon model, namely, $\mathbf{C}^*$ with a potential $V = -(\beta^2 \xi/m)\cos^2\sigma$. In this case, one sine-Gordon kink carries the half quantized flux of $U(1)$ gauge theory or the half lump (instanton) charge of the $\mathbf{C}P^1$ model. Therefore, after one vortex in the bulk is absorbed into the domain wall, it splits into two sine-Gordon kinks in this case. On the other hand, in our model, the numbers of the sine-Gordon kinks in the domain wall and the instantons (lumps) in the bulk correspond to each other one-to-one.

IV. SUMMARY AND DISCUSSION

We have constructed a sine-Gordon kink in the domain wall world volume in the $U(1)$ gauge theories coupled with two complex scalar fields ϕ^1 and ϕ^2 with the Josephson interaction term $\phi^{1*}\phi^2$ in $d = 2+1$ dimensions. We have shown the sine-Gordon soliton in the $d = 1+1$-dimensional domain wall world volume is nothing but an instanton or a lump in the $\mathbf{C}P^1$ model or a vortex in the gauge theory in the $d = 2+1$-dimensional bulk. This provides a physical realization of the lower-dimensional Atiyah-Manton construction.

It was proposed in Ref. [9] that a sine-Gordon kink φ (in $d = 1+1$) is well approximated by a holonomy of $\mathbf{C}P^1$ instanton (in $d = 2+0$):

$$(-1)^k \exp[i\varphi(x)] = \exp\left(\int_{-\infty}^{+\infty} A_1(x^1, x^2)dx^1\right), \qquad (24)$$

with the instanton (lump) number k of $\mathbf{C}P^1$ instantons with the auxiliary gauge field A_1 in Eq. (22),

$$\varphi = k\pi + \int_{-\infty}^{+\infty} dx^1 \frac{-i(u^*\partial_1 u - (\partial_1 u^*)u)}{2(1+|u|^2)}. \qquad (25)$$

From Fig. 1(b), we expect that a better approximation will be given by a pair of a meron and an antimeron rather than a cylindrically symmetric lump solution. This deformation may be achieved by considering caloron [29] (see also Ref. [30]), i.e., a periodic lump solution on $\mathbf{R} \times S^1$ with taking the periodicity as the wall width $\Delta x^1 = 1/m$. Another improvement is replacing the $\mathbf{C}P^1$ model with lumps by a $U(1)$ gauge theory with two charged Higgs fields with semilocal vortices. This may give a better approximation because of an exponential rather than power law asymptotic behavior, as discussed in Ref. [9].

If we extend the model to N complex scalar fields, reducing to the massive $\mathbf{C}P^{N-1}$ model in the strong gauge coupling limit, it admits $N-1$ parallel domain walls [27,28]. It remains as an interesting future work how instantons are absorbed into each domain wall. Another interesting extension will be non-Abelian $U(N_C)$ gauge theories with $N_F(>N_C)$ flavors in the fundamental representation ($N_C \times N_F$ matrix of scalar fields). The model reduces to the massive Grassmannian $SU(N_F)/[SU(N_C) \times SU(N_F - N_C) \times U(1)]$ sigma model in the strong gauge coupling limit [31]. Appropriate extension of the Josephson terms is not known. Without the Josephson terms, domain walls in this theory were studied in Ref. [28]. The construction of the effective theory on general domain wall solutions can be found in Ref. [32]. The fate of Grassmannian lumps (or non-Abelian semilocal vortices [33]) should be studied elsewhere.

Yet another interesting extension will be to study what happens in the presence of domain wall junctions or networks [34,35], which are possible if we introduce complex masses M for scalar fields, and domain walls stretched by vortices [35,36]. The effective theories of the domain wall network and the vortices stretched between domain walls were constructed in Refs. [37,38], respectively. How Josephson vortices are absorbed into these composite solitons remains for a future study.

ACKNOWLEDGMENTS

This work is supported in part by Grant-in-Aid for Scientific Research (No. 23740198) and by the "Topological Quantum Phenomena" Grant-in-Aid for Scientific Research on Innovative Areas (No. 23103515) from the Ministry of Education, Culture, Sports, Science and Technology (MEXT) of Japan.

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[24] We have added this term in order to stabilize the flux tube (vortex) absorbed into the domain wall in this theory. For $\beta = 0$, the flux is diluted and disappears inside the domain wall. However, this term is not the unique term to stabilize it; one could consider other renormalizable interactions. In this paper, we consider this term because it is the simplest among them. In fact, this term frequently appears in condensed matter physics such as the Josephson junction of superconductors, high T_c superconductors, and multicomponent Bose-Einstein condensates. The other motivation to consider this term is an application to the Atiyah-Manton construction for which one-to-one correspondence between the flux tube outside the domain wall and the soliton inside the domain wall is required. As discussed in Sec. III D, another term which preserves four supercharges was considered in Ref. [16]. As we will see in the following section, one flux tube outside the domain wall corresponds to one soliton inside the domain wall in our case, while one flux tube corresponds to two solitons inside the domain wall in the case of Ref. [16].

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