Slowly rotating relativistic stars in scalar-tensor gravity

Hajime Sotani[*]

Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606-8502, Japan
(Received 3 August 2012; published 19 December 2012)

We consider the slowly rotating relativistic stars with a uniform angular velocity in scalar-tensor gravity, and examine the rotational effect around such compact objects. For this purpose, we derive a second order differential equation describing the frame dragging in scalar-tensor gravity and solve it numerically. As a result, we find that the total angular momentum is proportional to the angular velocity even in scalar-tensor gravity. We also show that one can observe the spontaneous scalarization in rotational effects as well as the other stellar properties, if the cosmological value of the scalar field is zero. On the other hand, if the cosmological value of the scalar field is nonzero, the deviation from general relativity can be seen in a wide range of coupling constants. Additionally, we find that, independently of the cosmological value of the scalar field, the deviation from general relativity becomes larger with more massive stellar models. Thus, via precise observations of astronomical phenomena associated with rotating relativistic stars, one may probe not only the gravitational theory in the strong-field regime, but also the existence of a scalar field.

I. INTRODUCTION

The theory of general relativity proposed by Einstein is the theory describing gravity. In order to verify general relativity, many experiments have been performed up to now and nothing indicates the failure of this theory. However, most of these experiments are performed in a weak gravitational field, such as the Solar System. This means that the gravitational theory in a strong gravitational field is still unconstrained by astronomical observations. Namely, the gravitational theory might be different from general relativity in a strong gravitational field. If so, one can probe the gravitational theory by observing the deviation from predictions in general relativity. Actually, the development of technology will enable us to observe the phenomena related to compact objects with high accuracy. Such observations may be used as a test of the gravitational theory in a strong gravitational field. So far, many attempts and possibilities to test the gravitational theory in a strong gravitational field have been suggested [1–6]. Hopefully, the reliability of gravitational theory even in a strong gravitational field will be observationally revealed in the near future.

Scalar-tensor gravity is a natural extension of general relativity, and one of the simplest alternative gravitational theories, where gravity is mediated by long-range scalar fields in addition to the usual tensor field in general relativity [1,7]. The scalar field plays an essential role in explaining the accelerated expansion phases of the Universe, such as the inflation scenario. Additionally, the scalar-tensor theory in gravity can be obtained from the low energy limit of string and/or other gauge theories. In scalar-tensor gravity, one adopts the basic assumption that the scalar and gravitational fields, φ and $g_{*\mu\nu}$, are

coupled to matter via an effective metric defined as $\tilde{g}_{\mu\nu} = A^2(\varphi)g_{*\mu\nu}$, where the function form of the conformal factor is still unclear. The Brans-Dicke theory is the simplest version of scalar-tensor gravity, which assumes that $A(\varphi) = \exp(\alpha\varphi)$ [8]. The parameter in the conformal factor, α, can often be connected to the Brans-Dicke parameter, ω_{BD}, as $\alpha^2 = (2\omega_{BD} + 3)^{-1}$, and the expected deviation from general relativity becomes of the order α^2 [9]. The experiments in the solar system set severe constraints on the parameter, i.e., $\omega_{BD} \gtrsim 40000$ or $\alpha < 10^{-5}$ [10]. On the other hand, Damour and Esposito-Farèse adopted the different function forms of the conformal factor, i.e., $A(\varphi) = \exp(\alpha\varphi + \beta\varphi^2/2)$. Then, they showed that massive neutron stars in scalar-tensor gravity can significantly deviate from the expectations of general relativity [9,11]. In particular, they pointed out that the sudden deviation in neutron star properties from the predictions in general relativity is possible for specific values of the coupling constants, even if the value of α is considerably small. This phenomenon is referred to as "spontaneous scalarization." Subsequently, Harada studied more details with linear analysis and with the technique of catastrophe theory, and found that spontaneous secularization is possible for $\beta \lesssim -4.35$ [12]. Recently, Freire *et al.* have ruled out the parameter range of $\beta < -5$ with the observations of the pulsar white dwarf binary [13]. This constraint is quite strict, but we still have a small chance to see spontaneous scalarization. So, we focus on the range of $\beta \gtrsim -5$ in this article.

Previously, several possible ways to distinguish scalar-tensor gravity from general relativity by using astronomical observations have been suggested, e.g., with the redshift lines of the X and γ rays radiated from the surface of neutron stars [14] and with the spectrum of gravitational waves emitted from neutron stars [15,16]. In this article,

---

*hajime.sotani@nao.ac.jp

we examine a different approach to probe the gravitational theory. That is, we focus on the rotational effect around neutron stars, where we especially consider slowly rotating neutron stars with a uniform angular velocity [17]. A similar analysis in general relativity was originally done by Hartle [19], and subsequently a lot of studies associated with the stellar rotation have been examined. For instance, taking into account the rotational effect, the additional oscillation family referred to as *r* modes can be excited, and it might be possible to obtain the stellar information from their spectrum [20,21]. In this sense, this article could become a first step toward considering the rotational effect around relativistic stars in a more complicated system in scalar-tensor gravity. We should remark that a similar analysis of the tensor-vector-scalar (TeVeS) theory has been done [6]. However, in this analysis, they did not consider the dependence on the scalar coupling, because in TeVeS the dependence on the vector coupling is stronger than that on the scalar coupling. In addition, for simplicity, they did not consider the dependence on the cosmological value of the scalar field. So, in this article, we will examine the dependence on the scalar coupling as well as the dependence on the cosmological value of the scalar field, although scalar-tensor gravity is one of the specific cases in TeVeS.

This article is organized as follows. In the next section, we mention the fundamental parts of scalar-tensor gravity and the equilibrium of nonrotating relativistic stars in scalar-tensor gravity. In Sec. III, we derive the differential equation describing the frame dragging in scalar-tensor gravity. Additionally, we numerically show the rotational effect in this section. Then, we draw a conclusion in Sec. IV. In this article, we adopt the units of $c = G = 1$, where c and G denote the speed of light and the gravitational constant, respectively, and the metric signature is $(-, +, +, +)$.

II. STELLAR MODELS IN SCALAR-TENSOR GRAVITY

A. Scalar-tensor gravity

The scalar-tensor gravity is a natural extension of general relativity, where gravity is mediated not only by a metric tensor but also by a massless scalar field φ. The total action of such a gravitational theory is given by [7]

$$S = \frac{1}{16\pi G_*} \int \sqrt{-g_*}(R_* - 2g_*^{\mu\nu}\varphi_{,\mu}\varphi_{,\nu})d^4x$$
$$+ S_m[\Psi_m, A^2(\varphi)g_{*\mu\nu}], \qquad (2.1)$$

where G_* is the bare gravitational coupling constant, $g_{*\mu\nu}$ is the Einstein metric, R_* is the Ricci scalar constructed with $g_{*\mu\nu}$, Ψ_m represents matter fields collectively, and S_m denotes the matter action. It is known that the formulation of field equations is better in the "Einstein frame" described by the Einstein metric. But a test particle moves

on the geodesic in the "Jordan frame" described by the metric $\tilde{g}_{\mu\nu}$, which is defined as $\tilde{g}_{\mu\nu} = A^2(\varphi)g_{*\mu\nu}$ [7]. So, the Jordan frame is often referred to as the physical frame. Hereafter, quantities with an asterisk are related to the Einstein frame, while quantities with a tilde denote those in the physical frame.

By varying the total action S with respect to $g_{*\mu\nu}$, one can obtain the field equation for the tensor field:

$$G_{*\mu\nu} = 8\pi G_* T_{*\mu\nu} + 2\left(\varphi_{,\mu}\varphi_{,\nu} - \frac{1}{2}g_{*\mu\nu}g_*^{\alpha\beta}\varphi_{,\alpha}\varphi_{,\beta}\right),$$
$$(2.2)$$

where $T_{*\mu\nu}$ is the energy-momentum tensor in the Einstein frame, which is associated with that in the physical frame as

$$T_*^{\mu\nu} \equiv \frac{2}{\sqrt{-g_*}}\frac{\delta S_m}{\delta g_{*\mu\nu}} = A^6(\varphi)\tilde{T}^{\mu\nu}. \qquad (2.3)$$

Similarly, by varying S with respect to φ, one obtains the field equation for the scalar field:

$$\Box_*\varphi = -4\pi G_*\alpha(\varphi)T_*, \qquad (2.4)$$

where $\Box_*$ denotes the d'Alembertian of $g_{*\mu\nu}$, and $\alpha(\varphi)$ and T_* are defined as

$$\alpha(\varphi) \equiv \frac{d\ln A(\varphi)}{d\varphi}, \qquad (2.5)$$

$$T_* \equiv T_{*\mu}{}^\mu = T_*^{\mu\nu}g_{*\mu\nu}. \qquad (2.6)$$

We should remark that scalar-tensor gravity for $\alpha(\varphi) = 0$ reduces to general relativity. In addition to Eqs. (2.2) and (2.4), one gets another equation from the energy-momentum conservation law, i.e., $\tilde{\nabla}_\mu\tilde{T}^\mu{}_\nu = 0$, such as

$$\nabla_{*\mu}T_*^\mu{}_\nu = \alpha(\varphi)T_*\nabla_{*\nu}\varphi. \qquad (2.7)$$

In particular, in this article, we adopt the same form of conformal factor $A(\varphi)$ as in Ref. [9], i.e.,

$$A(\varphi) = e^{\beta\varphi^2/2}, \qquad (2.8)$$

where β is a real constant. Thus, $\alpha(\varphi)$ can be written as $\alpha(\varphi) = \beta\varphi$, and scalar-tensor gravity with $\beta = 0$ corresponds to general relativity.

B. Nonrotating relativistic stellar models in scalar-tensor gravity

The equilibrium configurations of nonrotating relativistic stars in scalar-tensor gravity have been calculated in Refs. [7,12,15]. The metric describing the static, spherically symmetric relativistic stars can be written as

$$ds_*^2 = g_{*\mu\nu}dx^\mu dx^\nu$$
$$= -e^{2\Phi(r)}dt^2 + e^{2\Lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$
$$(2.9)$$

where $e^{-2\Lambda} = 1 - 2\mu(r)/r$, and $\mu(r)$ corresponds to the mass function. Therefore, the physical metric is

$$d\tilde{s}^2 = \tilde{g}_{\mu\nu}dx^\mu dx^\nu$$
$$= -A^2 e^{2\Phi}dt^2 + A^2 e^{2\Lambda}dr^2 + A^2 r^2(d\theta^2 + \sin^2\theta d\phi^2).$$
$$(2.10)$$

We assume that the stellar matter is a perfect fluid, i.e., $\tilde{T}_{\mu\nu} = (\tilde{\rho} + \tilde{P})\tilde{U}_\mu\tilde{U}_\nu + \tilde{P}\tilde{g}_{\mu\nu}$, where $\tilde{\rho}$, $\tilde{P}$, and $\tilde{U}_\mu$ are the total energy density, the pressure, and the four-velocity of the fluid in the physical frame, respectively.

Using Eqs. (2.2), (2.4), and (2.7), one can obtain the Tolman-Oppenheimer-Volkoff (TOV) equations in scalar-tensor gravity [7,12], such as

$$\mu' = 4\pi G_* r^2 A^4 \tilde{\rho} + \frac{1}{2}r(r - 2\mu)\Psi^2, \qquad (2.11)$$

$$\Phi' = 4\pi G_* \frac{r^2 A^4 \tilde{P}}{r - 2\mu} + \frac{1}{2}r\Psi^2 + \frac{\mu}{r(r - 2\mu)}, \qquad (2.12)$$
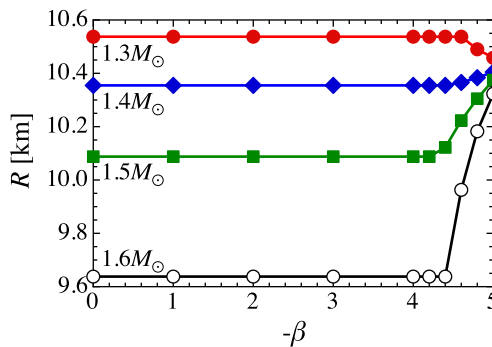
$$\varphi' = \Psi, \qquad (2.13)$$

$$\Psi' = 4\pi G_* \frac{rA^4}{r - 2\mu}[\alpha(\tilde{\rho} - 3\tilde{P}) + r(\tilde{\rho} - \tilde{P})\Psi]$$
$$- \frac{2(r - \mu)}{r(r - 2\mu)}\Psi, \qquad (2.14)$$

$$\tilde{P}' = -(\tilde{\rho} + \tilde{P})[\Phi' + \alpha\Psi], \qquad (2.15)$$

where the prime denotes a derivative with respect to r. In order to close the system of equations, one needs to prepare the relationship between $\tilde{\rho}$ and $\tilde{P}$, i.e., the equation of state (EOS). In this article, we adopt the polytropic EOS as in Refs. [7,12,15]:

$$\tilde{P} = Kn_0 m_b\left(\frac{\tilde{n}}{n_0}\right)^\Gamma, \qquad (2.16)$$

$$\tilde{\rho} = \tilde{n}m_b + \frac{\tilde{P}}{\Gamma - 1}, \qquad (2.17)$$

where $\tilde{n}$ is the baryon number density in the physical frame, while m_b and n_0 are some constants given by $m_b = 1.66 \times 10^{-24}$ g and $n_0 = 0.1$ fm^{-3}. In order to fit this EOS to the tabulated data of realistic EOS known as EOS A [22] and EOS II [23], we especially adopt $\Gamma = 2.46$ and $K = 0.00936$ for EOS A and $\Gamma = 2.34$ and $K = 0.0195$ for EOS II.

The stellar equilibrium can be determined as follows. With the boundary conditions at the stellar center as $\mu(0) = 0$, $\Phi(0) = \Phi_c$, $\tilde{\rho}(0) = \tilde{\rho}_c$, $\varphi(0) = \varphi_c$, and $\Psi(0) = 0$, the TOV equations with the EOS are integrated outward and the stellar surface is determined at $\tilde{P} = 0$ as $R = A(\varphi_s)r_s$, where φ_s and r_s denote the values of φ and r at $\tilde{P} = 0$. For $r \geq r_s$, the TOV equations with $\tilde{\rho} = \tilde{P} = 0$ are integrated outward. Then, the central values of Φ_c and φ_c can be determined so that the calculated solutions should agree with the asymptotic behaviors as

$$g_{*\mu\nu} = \eta_{\mu\nu} + \frac{2M_{\text{ADM}}}{r}\delta_{\mu\nu} + \mathcal{O}\left(\frac{1}{r^2}\right), \qquad (2.18)$$

$$\varphi(r) = \varphi_0 + \frac{Q}{r} + \mathcal{O}\left(\frac{1}{r^2}\right), \qquad (2.19)$$

where $\eta_{\mu\nu}$, M_{ADM}, φ_0, and Q correspond to the Minkowskian metric, the Arnowitt-Deser-Misner (ADM) mass, the cosmological value of scalar field, and the total scalar charge, respectively [24]. In practice, with the matching conditions at the stellar surface between the interior and exterior solutions, one can determine the values of M_{ADM}, φ_0, and Q as

$$M_{\text{ADM}} = \frac{r_s^2\Phi_s'}{G_*}\left(1 - \frac{2\mu_s}{r_s}\right)^{1/2}$$
$$\times \exp\left[-\frac{\Phi_s'}{\sqrt{(\Phi_s')^2 + \Psi_s^2}}\text{arctanh}\left(\frac{\sqrt{(\Phi_s')^2 + \Psi_s^2}}{\Phi_s' + 1/r_s}\right)\right],$$
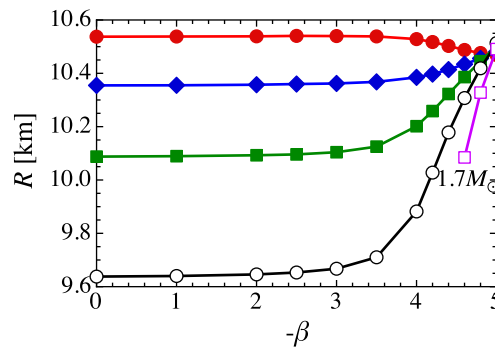$$(2.20)$$

FIG. 1 (color online).   For EOS A, the stellar radius R is plotted as a function of $-\beta$ with the fixed ADM mass for $\varphi_0 = 0$ (left panel) and for $\varphi_0 = 0.01$ (right panel).
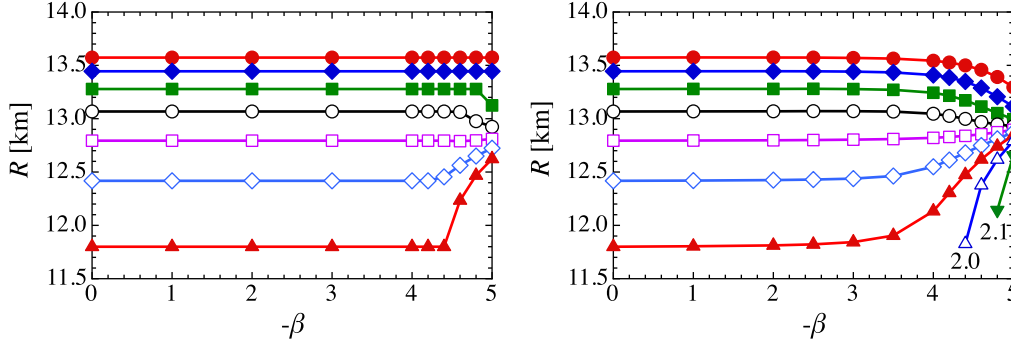
FIG. 2 (color online). Similar to Fig. 1, but for EOS II. The labels denote the adopted stellar mass in units of $M_\odot$, where the seven solid lines from top to bottom correspond to the cases of $M/M_\odot = 1.3, 1.4, 1.5, 1.6, 1.7, 1.8$, and 1.9.

$$\varphi_0 = \varphi_s + \frac{\Psi_s}{\sqrt{(\Phi_s')^2 + \Psi_s^2}} \, \text{arctanh} \left[\frac{\sqrt{(\Phi_s')^2 + \Psi_s^2}}{\Phi_s' + 1/r_s} \right],$$

$$\tag{2.21}$$

$$Q = -\frac{\Psi_s}{\Phi_s'} M_{\text{ADM}}, \tag{2.22}$$

where the quantities with the subscript s denote their values at $r = r_s$ [9]. At last, choosing the values of two constants β and φ_0, the stellar models become one parameter family with $\tilde{\rho}_c$ (or M_{ADM}). In particular, in this article, we adopt $\varphi_0 = 0.01$ as a example for $\varphi_0 \neq 0$.

We show the dependence of the stellar radius R on the coupling constant $-\beta$ with the fixed ADM mass in Fig. 1 for EOS A and in Fig. 2 for EOS II. In both figures, the left and right panels correspond to the cases of $\varphi_0 = 0$ and $\varphi_0 = 0.01$, respectively. From these figures, one can observe the spontaneous scalarization in R at around $\beta \simeq -4.4$ for $\varphi_0 = 0$, while the stellar radii are changing smoothly for $\varphi_0 \neq 0$. Namely, even for $-4.4 < \beta < 0$, the stellar properties in scalar-tensor gravity with $\varphi_0 \neq 0$ are different from those in general relativity (for the case of $\beta = 0$).

III. SLOWLY ROTATING RELATIVISTIC STARS IN SCALAR-TENSOR GRAVITY

A. Rotating dragging

In order to see the rotational effect around relativistic stars in scalar-tensor gravity, we consider a slowly rotating stellar model with a uniform angular velocity, $\tilde{\Omega}$, where the rotational axis is put on $\theta = 0$. For this purpose, we make an assumption to keep only the linear effects in the angular velocity, which is the same treatment as in general relativity [19]. With this assumption, the stellar model is still spherically symmetric, because the deformation of stellar shape due to rotation is of the order $\tilde{\Omega}^2$. Similarly, the rotational effects in the components of the metric except for $\tilde{g}_{t\phi}$ become of the order $\tilde{\Omega}^2$. So, the metric describing a slowly rotating system in the physical frame can be written as

$$d\tilde{s}^2 = -A^2 e^{2\Phi} dt^2 + A^2 e^{2\Lambda} dr^2 + A^2 r^2 (d\theta^2 + \sin^2\theta d\phi^2)$$
$$- 2\omega r^2 A^2 \sin^2\theta dt d\phi, \tag{3.1}$$

where the last term appears due to the rotational effect. Hereafter, in order to specify the rotational effects, the deviation from the nonrotating stellar model will be expressed by using variables with δ, such as $\delta\tilde{g}_{t\phi} = -\omega r^2 A^2 \sin^2\theta$. The deviations of the pressure, density, and scalar field due to the rotation are also of the order $\mathcal{O}(\tilde{\Omega}^2)$, i.e., $\delta\tilde{P} = \mathcal{O}(\tilde{\Omega}^2)$, $\delta\tilde{\rho} = \mathcal{O}(\tilde{\Omega}^2)$, and $\delta\varphi = \mathcal{O}(\tilde{\Omega}^2)$, because those properties should behave in the same way under a reversal in the direction of rotation. With such an ordering, one can show that $\delta\tilde{g}_{\mu\nu} = A^2 \delta g_{*\mu\nu}$ and $\delta T_{*\mu\nu} = A^2 \delta\tilde{T}_{\mu\nu}$ within the linear order of $\tilde{\Omega}$. Additionally, the deviation of fluid velocity in the physical frame can be described as

$$\delta\tilde{U}^\mu = (0, 0, 0, \tilde{\Omega}\tilde{U}^t), \tag{3.2}$$

where $\tilde{U}^t$ denotes the t component of the four-velocity for a nonrotating stellar model, i.e., $\tilde{U}^t = e^{-\Phi}/A$. Consequently, the nonzero component of the energy-momentum tensor of the order $\tilde{\Omega}$ is only $\delta\tilde{T}_{t\phi}$, which is given as

$$\delta\tilde{T}_{t\phi} = r^2 A^2 (\tilde{\rho} + \tilde{P})(\omega - \tilde{\Omega})\sin^2\theta - \tilde{P}\omega r^2 A^2 \sin^2\theta. \tag{3.3}$$

At last, the rotational effect ω can be determined by solving the following equation:

$$\delta G_{*\mu\nu} = 8\pi G_* \delta T_{*\mu\nu} - e^{-2\Lambda}\Psi^2 \delta g_{*\mu\nu}, \tag{3.4}$$

which comes from the Einstein equation (2.2). We should remark that another field equation (2.4) cannot tell us any information about the rotational effect in the order of $\tilde{\Omega}$, because $\delta\varphi \sim \mathcal{O}(\tilde{\Omega}^2)$, $\delta\alpha \sim \mathcal{O}(\tilde{\Omega}^2)$, and $\delta T_* \sim \mathcal{O}(\tilde{\Omega}^2)$. One can show that only the (t, ϕ) component in Eq. (3.4) becomes a nontrivial equation, which can be written as

$$\omega'' + \left(\frac{4}{r} - \Phi' - \Lambda'\right)\omega'$$
$$- 2\left[\Phi'' + (\Phi' - \Lambda')\left(\Phi' + \frac{1}{r}\right) + \Psi^2\right]\omega$$
$$+ \frac{1}{r^2}e^{2\Lambda}\left(\omega_{,\theta\theta} + 3\omega_{,\theta}\frac{\cos\theta}{\sin\theta}\right)$$
$$+ 16\pi G_* A^4 e^{2\Lambda}[\tilde{P}\omega + (\tilde{\rho} + \tilde{P})(\tilde{\Omega} - \omega)] = 0. \quad (3.5)$$

In general, $\omega \sim \mathcal{O}(\tilde{\Omega})$ can be expressed as $\omega(r, \theta) = -\omega(r)/\sin\theta \partial_\theta P_\ell(\cos\theta)$, where P_ℓ is the Legendre polynomial of order ℓ. With this expression, the bracket of the fourth term in Eq. (3.5) can be reduced to

$$\omega_{,\theta\theta} + 3\omega_{,\theta}\frac{\cos\theta}{\sin\theta} = (\ell + 2)(\ell - 1)\omega\frac{\partial_\theta P_\ell}{\sin\theta}. \quad (3.6)$$

Furthermore, in order to reproduce the general relativity limit of scalar-tensor gravity, i.e., $\beta \to 0$, we especially adopt the case for $\ell = 1$ in this article. In practice, one cannot adopt an arbitrary value of ℓ except for $\ell = 1$ to construct the distribution of ω which satisfies the regularity condition at the stellar center [19]. Finally, the equation to solve becomes

$$\omega'' + \left(\frac{4}{r} - \Phi' - \Lambda'\right)\omega'$$
$$- 2\left[\Phi'' + (\Phi' - \Lambda')\left(\Phi' + \frac{1}{r}\right) + \Psi^2\right]\omega$$
$$+ 16\pi G_* A^4 e^{2\Lambda}[\tilde{P}\omega + (\tilde{\rho} + \tilde{P})(\tilde{\Omega} - \omega)] = 0. \quad (3.7)$$

Note that Eq. (3.7) for $\beta = 0$ agrees with the well-known equation describing the frame dragging in general relativity [19].

B. Numerical results

In order to determine the rotational effect $\omega(r)$ with Eq. (3.7), one needs to prepare the boundary conditions at the stellar center and spatial infinity, i.e., the regularity condition at $r = 0$ and the asymptotic flatness far from the central object. Since one can expand ω as $\omega(r) = \omega_{c0} + \omega_{c2}r^2 + \cdots$ in the vicinity of the stellar center, Eq. (3.7) is integrated outward from $r = 0$ with the boundary conditions of $\omega(0) = \omega_{c0}$ and $\omega'(0) = 0$. Then, one can find the exact value of ω_{c0} in such a way that the solution $\omega(r)$ should satisfy the asymptotic behavior, i.e.,

$$\omega(r) = \frac{2J}{r^3} + \mathcal{O}\left(\frac{1}{r^4}\right), \quad (3.8)$$

where J is some constant. In practice, we adopt the boundary condition of $1 + 3\omega/(\omega' r) = 0$, which can be derived from Eq. (3.8), at the numerical boundary far from the central object. In general relativity, the distribution of $\omega(r)$ outside the star can be analytically written as $\omega(r) = 2J/r^3$, where the constant J corresponds to the total angular momentum of the central object, and J is related to the angular velocity for slow rotation as
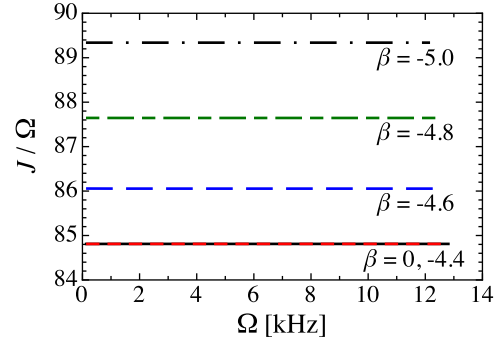

FIG. 3 (color online). Dependence of $J/\tilde{\Omega}$ on the angular velocity $\tilde{\Omega}$ for the stellar models with EOS A and with $M_{\text{ADM}} = 1.4M_\odot$ and $\varphi_0 = 0$, where the solid line corresponds to the case in the general relativity, while the broken lines correspond to the results in the scalar-tensor gravity with different values of β.

$$J = I\tilde{\Omega}. \quad (3.9)$$

Here, the constant of proportionality I corresponds to the relativistic generalization of momentum of inertia for a slowly rotating system [19]. On the other hand, in scalar-tensor gravity, we plot the value of $J/\tilde{\Omega}$ as a function of the angular velocity $\tilde{\Omega}$ for the stellar models with EOS A and with $M_{\text{ADM}} = 1.4M_\odot$ in Fig. 3 for $\varphi_0 = 0$ and in Fig. 4 for $\varphi_0 \neq 0$. In both figures, the right endpoints of lines correspond to the allowed maximum angular velocity for each stellar model. From these figures, one can see that the value of $J/\tilde{\Omega}$ is independent of $\tilde{\Omega}$ even in scalar-tensor gravity. That is, Eq. (3.9) can hold even in scalar-tensor gravity. This is the same result as in TeVeS [6]. We remark that in Fig. 3 the lines for $\beta \geq -4.4$ are completely equivalent to the line for $\beta = 0$, because the stellar models with $\varphi_0 = 0$ for $\beta \geq -4.4$ are the same as those in general relativity (for $\beta = 0$).

The distribution of $\omega(r)$ for the stellar model with EOS A and with $M_{\text{ADM}} = 1.4M_\odot$ and $\tilde{\Omega} = 1$ kHz is plotted in Fig. 5 for $\varphi_0 = 0$ and in Fig. 6 for $\varphi_0 = 0.01$. In both cases, we can obviously observe the difference of ω inside the star in scalar-tensor gravity from the prediction in
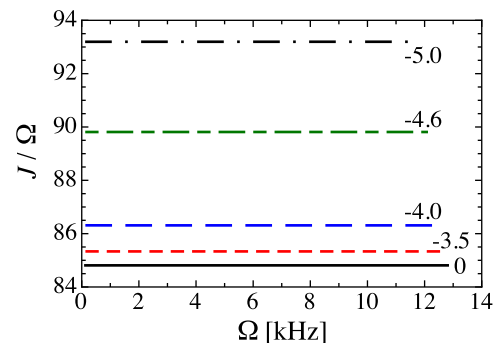

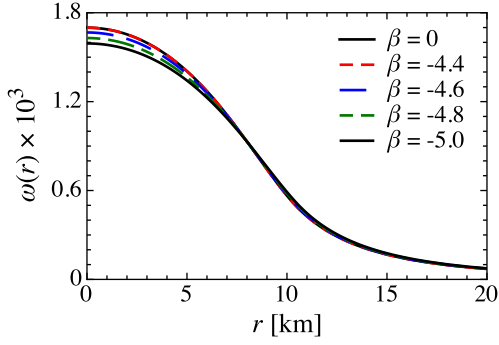FIG. 4 (color online). Similar to Fig. 3, but for $\varphi_0 = 0.01$. The labels attached on each line denote values of β.

FIG. 5 (color online). Distribution of $\omega(r)$ for $\varphi_0 = 0$ with different values of β, where the adopted stellar model is $M_{\rm ADM} = 1.4 M_\odot$ and $\tilde{\Omega} = 1$ kHz for EOS A.
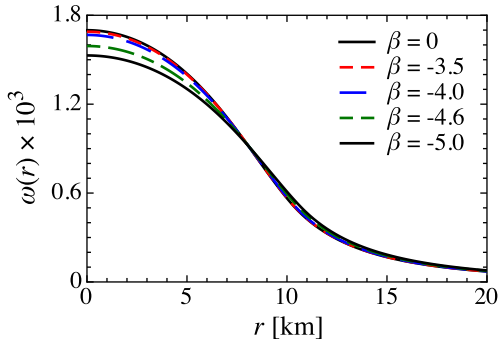

FIG. 6 (color online). Similar to Fig. 5, but for $\varphi_0 = 0.01$.

general relativity. In fact, compared with the case in general relativity, one can see the decrease of the central value of ω in scalar-tensor gravity up to 6.3% in Fig. 5 and 10.0% in Fig. 6. Thus, via the observations such as the stellar oscillations and/or the radiated gravitational waves, one could probe the gravitational theory in the strong-field regime. Meanwhile, although the deviation outside the star may be almost negligible, it still exists. Now, we should remark on the difference between the results in scalar-tensor gravity and in TeVeS. Unlike the results in scalar-tensor gravity, the value of $\omega(r)$ in TeVeS becomes

larger than that in general relativity in the vicinity of the stellar center, and becomes smaller in the outer region (see Fig. 3 in Ref. [6]). This means that one might be able to probe the gravitational theory via the observations of a binary system composed of relativistic stars with quite high accuracy.

Furthermore, we plot the momentum of inertia I as a function of the coupling constant $-\beta$ with the fixed ADM mass in Fig. 7 for EOS A and in Fig. 8 for EOS II, where the left and right panels are results for $\varphi_0 = 0$ and $\varphi_0 = 0.01$, respectively. From both figures, it is found that the dependence of I on $-\beta$ becomes stronger as the stellar mass increases. One can also observe that, compared with the case of $\varphi_0 = 0$, the case of $\varphi_0 \neq 0$ depends strongly on the scalar coupling β. For example, for EOS A, the value of I in the scalar-tensor gravity becomes up to 1.4%, 5.3%, 11.0%, and 19.7% larger than that in general relativity for $M = 1.3 M_\odot$, $1.4 M_\odot$, $1.5 M_\odot$, and $1.6 M_\odot$ in the left panel of Fig. 7, while up to 4.7%, 9.9%, 16.8%, and 27.4% larger for $M = 1.3 M_\odot$, $1.4 M_\odot$, $1.5 M_\odot$, and $1.6 M_\odot$ in the right panel of Fig. 7. We should emphasize that such a dependence of I on the gravitational theory is completely different from that in the case of TeVeS, i.e., the value of I in TeVeS becomes smaller than that in general relativity [6]. Additionally, for the case of $\varphi_0 = 0$ (left panels in Figs. 7 and 8), only if β is less than around -4.4, I in scalar-tensor gravity can deviate from I in general relativity. Namely, the spontaneous scalarization can be observed in I as well as the other stellar properties. On the other hand, for the case of $\varphi_0 \neq 0$ (right panels in Figs. 7 and 8), one can observe the deviation in I even for $-4.4 \lesssim \beta < 0$. Now, in order to clarify the deviation in I for $\varphi_0 = 0.01$, we define the relative deviation Δ as

$$\Delta = (I_{\rm ST} - I_{\rm GR})/I_{\rm GR}, \qquad (3.10)$$

where $I_{\rm ST}$ and $I_{\rm GR}$ denote the momentum of inertia defined as Eq. (3.9) in scalar-tensor gravity and in general relativity, respectively. The calculated Δ is plotted as a function of $-\beta$ in Fig. 9, where the left and right panels correspond to the results for EOS A and for EOS II. This figure obviously shows the statement mentioned above, i.e., the deviation
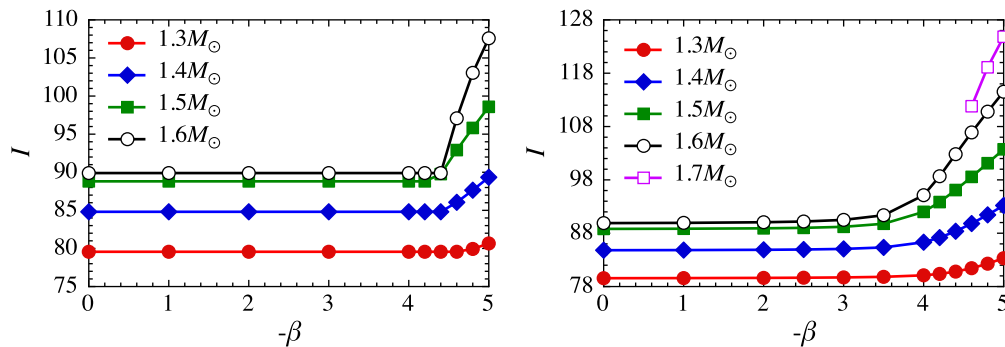

FIG. 7 (color online). For EOS A, the momentum of inertia I is plotted as a function of $-\beta$ with the fixed ADM mass for $\varphi_0 = 0$ (left panel) and for $\varphi_0 = 0.01$ (right panel).
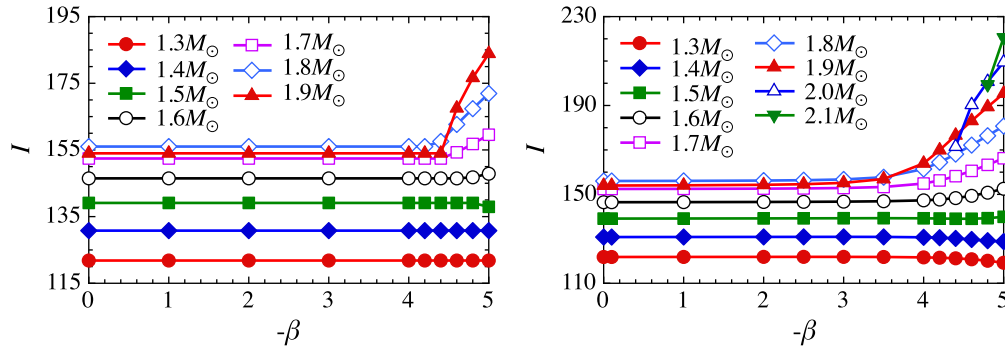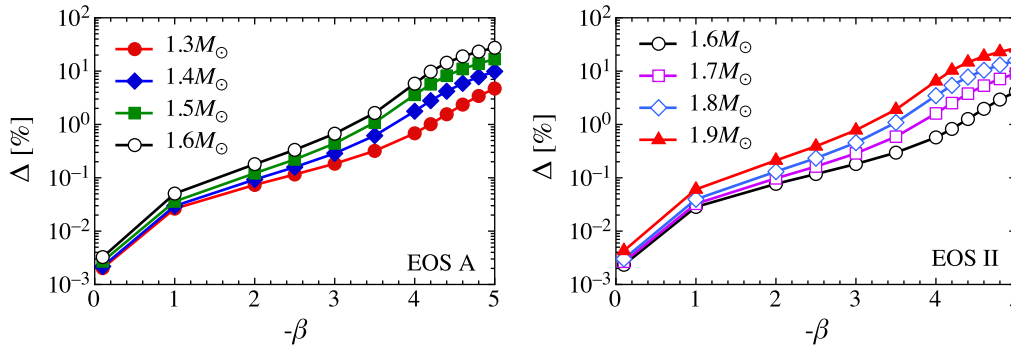
FIG. 8 (color online). Similar to Fig. 7, but for EOS II.

FIG. 9 (color online). Relative deviation of I in scalar-tensor gravity compared with I in general relativity. The left and right panels correspond to the cases of EOS A and EOS II, respectively.

from general relativity becomes larger with a more massive stellar model. Meanwhile, comparing stellar models with the same mass but different EOS, one can see that the stellar model with softer EOS becomes a larger deviation from the prediction in general relativity. At last, one can clearly see the deviation from general relativity even for $-4.4 \lesssim \beta < 0$, which is around 0.01%–10%. This deviation might be small, but we still have a chance to probe the gravitational theory in the strong-field regime via accurate observations around relativistic objects.

IV. CONCLUSION

In order to examine the rotational effect around relativistic stars in scalar-tensor gravity, we consider slowly rotating relativistic objects with a uniform angular velocity. From the Einstein equations, we derive a second order differential equation governing the frame dragging, and solve it numerically with appropriate boundary conditions. As a result, we find that, similar to general relativity, the total angular momentum is proportional to the angular velocity for the slow rotation. Additionally, we find that spontaneous scalarization can arise in rotational effects as well as the other stellar properties for $\varphi_0 = 0$, where φ_0 denotes the cosmological value of the scalar field. On the other hand, for $\varphi_0 \neq 0$, we can observe the deviation from general relativity in a wide range of the coupling constant β. Comparing the results in this article with those in the

case of TeVeS, we find that the dependence on gravitational theory is completely different. In particular, we find an obvious difference in the momentum of inertia. That is, compared with the prediction in general relativity, the momentum of inertia in TeVeS becomes smaller, while that in scalar-tensor gravity becomes larger. Thus, via the astronomical observations around relativistic stars with high accuracy, one could probe not only the gravitational theory in the strong-field regime, but also the existence of a scalar field. At last, we adopt simple stellar models in this article, but to compare with the actual observational data, we need to consider the more realistic stellar models, where we might take into account the magnetic effects [25] and the effects of crust and/or the more exotic phase [26,27]. Considering such additional properties, one could obtain further constraints in the theory.

ACKNOWLEDGMENTS

[1] C. M. Will, *Theory and Experiment in Gravitational Physics* (Cambridge University Press, Cambridge England, 1993).

[2] C. M. Will, Living Rev. Relativity **4**, 4 (2001).

[3] D. Psaltis, Living Rev. Relativity **11**, 9 (2008).

[4] P. D. Lasky, H. Sotani, and D. Giannios, Phys. Rev. D **78**, 104019 (2008).

[5] H. Sotani, Phys. Rev. D **79**, 064033 (2009); **80**, 064035 (2009); **83**, 124030 (2011).

[6] H. Sotani, Phys. Rev. D **81**, 084006 (2010); **82**, 124061 (2010).

[7] T. Damour and G. Esposito-Farèse, Classical Quantum Gravity **9**, 2093 (1992).

[8] C. Brans and R. H. Dicke, Phys. Rev. **124**, 925 (1961).

[9] T. Damour and G. Esposito-Farèse, Phys. Rev. Lett. **70**, 2220 (1993).

[10] G. Esposito-Farèse, arXiv:gr-qc/0402007.

[11] T. Damour and G. Esposito-Farèse, Phys. Rev. D **54**, 1474 (1996).

[12] T. Harada, Prog. Theor. Phys. **98**, 359 (1997); Phys. Rev. D **57**, 4802 (1998).

[13] P. C. C. Freire, N. Wex, G. Esposito-Farèse, J. P. W. Verbiest, M. Bailes, B. A. Jacoby, M. Kramer, I. H. Stairs, J. Antoniadis, and G. H. Janssen, Mon. Not. R. Astron. Soc. **423**, 3328 (2012).

[14] S. DeDeo and D. Psaltis, Phys. Rev. Lett. **90**, 141101 (2003).

[15] H. Sotani and K. D. Kokkotas, Phys. Rev. D **70**, 084026 (2004).

[16] H. Sotani and K. D. Kokkotas, Phys. Rev. D **71**, 124038 (2005).

[17] Previously, Reizlin also considered the rotational effect in the Brans-Dicke theory [18], but, as mentioned in the text, we cannot expect a significant deviation in this case from the prediction in general relativity.

[18] V. I. Reizlin, Sov. Phys. J. **24**, 401 (1981).

[19] J. B. Hartle, Astrophys. J. **150**, 1005 (1967).

[20] E. Gaertig and K. D. Kokkotas, Phys. Rev. D **80**, 064026 (2009).

[21] E. Gaertig, K. Glampedakis, K. D. Kokkotas, and B. Zink, Phys. Rev. Lett. **107**, 101102 (2011).

[22] W. D. Arnett and R. L. Bowers, Astrophys. J. Suppl. Ser. **33**, 415 (1977).

[23] J. Diaz-Alonso and J. M. Ibañez-Cabanell, Astrophys. J. **291**, 308 (1985).

[24] The original notification in Ref. [9] for the total scalar charge is ω_A. But, in order to distinguish it from the rotational effect ω, we use different notation Q in this article.

[25] H. Sotani, K. D. Kokkotas, and N. Stergioulas, Mon. Not. R. Astron. Soc. **375**, 261 (2007); **385**, L5 (2008); H. Sotani, A. Colaiuda, and K. D. Kokkotas, *ibid.* **385**, 2161 (2008); H. Sotani and K. D. Kokkotas, *ibid.* **395**, 1163 (2009).

[26] H. Sotani, Mon. Not. R. Astron. Soc. **417**, L70 (2011); H. Sotani, K. Nakazato, K. Iida, and K. Oyamatsu, Phys. Rev. Lett. **108**, 201101 (2012); arXiv:1210.0955 [Mon. Not. R. Astron. Soc. Lett. (to be published)].

[27] H. Sotani, N. Yasutake, T. Maruyama, and T. Tatsumi, Phys. Rev. D **83**, 024014 (2011); H. Sotani, T. Maruyama, and T. Tatsumi, arXiv:1207.4055.