Spinor coupling to the weak Poincare gauge theory of gravity in three dimensions

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The Dirac Lagrangian is minimally coupled to the most general $R + T + T^2$-type Lagrangian in $(1 + 2)$-dimensions. The field equations are obtained from the total Lagrangian by a variational principle. The space-time torsion is calculated algebraically in terms of the Dirac condensate plus coupling coefficients. A family of circularly symmetric rotating exact solutions which is asymptotically three-dimensional anti–de Sitter is obtained. Finally, Banados-Teitelboim-Zanelli-like solutions are discussed.

I. INTRODUCTION

Although it is well known that general relativity is a classically trivial theory in three dimensions, the proposition of topologically massive gravity of Deser *et al.* [1] made it nontrivial and thus considerably increased theoretical interest in 3D gravity. In the meantime, the discovery of Banados-Teitelboim-Zanelli (BTZ) black holes [2] enhanced 3D gravity efforts, see, e.g., Refs. [3–10] and references therein. The motivations for those investigations can be listed briefly as the study of the properties of the quantum fields in curved space-times [11], inflation [12], and the dS/CFT correspondence [13,14].

On the other hand, the non-Riemannian formulation is another approach to be followed to obtain a dynamical 3D theory of gravity. There is plenty of literature on 3D gravity with torsion. The first possibility along this route is the Einstein-Cartan theory. Nevertheless, it is nondynamic in the absence of matter. Thus it is amended by the inclusion of a Chern-Simons term. Then, Mielke and Baekler generalized the topological massive gauge model of gravity by adding a new translational Chern-Simons term to the standard (rotational) one [15]. This generalization with or without matter attracted a lot of attention in the literature, see, for example, Refs. [16–20] and references therein.

On the contrary, the number of published works on the spinor coupled 3D gravity model with/without torsion is much less, to our knowledge [20–22]. Our initial aim is to fill in this gap. Nevertheless, for the first time in the literature, we investigate 3D gravity which is formulated in terms of *the most* general nonpropagating torsion. That is, we write a Lagrangian in the form of $R + T + T^2$ which is also called the weak Poincare gauge theory of gravity. Thus our gravity Lagrangian contains six parameters: a, λ, k_1, k_2, k_3, b. When the Dirac spinor is minimally coupled to it, k_2 disappears and one of k_1 or k_3 can be dropped without loss of generality. Also, b gives contributions to both the bare cosmological constant and the mass of the Dirac spinor.

The paper is organized as follows. Since we will be using the coordinate independent exterior forms, in Sec. II we introduce our notations and conventions. In Sec. III, after we couple minimally the Dirac Lagrangian to the gravitational Lagrangian, we obtain the first and second field equations and the Dirac equation by varying the total Lagrangian with respect to the coframe, the connection, and the adjoint of Dirac spinor, respectively. Before closing this section, we solve torsion from the second equation and insert the findings to the first equation. After that, in Sec. we reduce our theory to a Riemannian one. Section IV starts with a circularly symmetric and rotating metric ansatz. Then we write explicitly the Dirac equation and cast the first equation as five coupled differential equations. In order to see whether there is an exact solution to our model, in Sec. IV A we restrict ourselves to a special case $\alpha = \gamma$ by tracing the technique in Ref. [22]. Here we obtain a family of solutions which goes to AdS_3 as $r \to \infty$. In Sec. IV B we consider BTZ-like solutions and do find one, but only for the case of vanishing Dirac condensate.

II. MATHEMATICAL PRELIMINARIES

We specify the space-time geometry by a triplet (M, g, ∇) where M is a three-dimensional differentiable manifold equipped with a metric tensor

$$g = \eta_{ab} e^a \otimes e^b \tag{1}$$

of signature $(-, +, +)$. e^a is an orthonormal coframe dual to the frame vectors X_a, that is $e^a(X_b) \equiv \iota_b e^a = \delta_b^a$ where $\iota_b := \iota_{X_b}$ denotes the interior product. A metric compatible connection ∇ can be specified in terms of connection 1-forms $\omega^a{}_b$ satisfying $\omega_{ba} = -\omega_{ab}$. Then the Cartan structure equations

$$de^a + \omega^a{}_b \wedge e^b = T^a, \tag{2}$$

$$d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b = R^a{}_b \tag{3}$$

define the space-time torsion 2-forms T^a and curvature 2-forms $R^a{}_b$, respectively. Here d denotes the exterior derivative and $\wedge$ the wedge product. We fix the orientation of space-time by choosing the volume 3-form $*1 = e^0 \wedge e^1 \wedge e^3$

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where * is the Hodge star map. In three-dimensional space-times with Lorentz signature for any p-form $^{**} = -1$. We will use the abbreviation $e^{ab\cdots} := e^a \wedge e^b \wedge \cdots$. It is possible to decompose the connection 1-forms in a unique way as

$$\omega^a{}_b = \tilde{\omega}^a{}_b + K^a{}_b, \tag{4}$$

where $\tilde{\omega}^a{}_b$ are the zero-torsion Levi-Civita connection 1-forms satisfying

$$de^a + \tilde{\omega}^a{}_b \wedge e^b = 0, \tag{5}$$

and $K^a{}_b$ are the contortion 1-forms satisfying

$$K^a{}_b \wedge e^b = T^a. \tag{6}$$

Correspondingly, the full curvature 2-form is decomposed as the Riemannian part plus torsional contributions:

$$R^a{}_b = \tilde{R}^a{}_b + \tilde{D}K^a{}_b + K^a{}_c \wedge K^c{}_b, \tag{7}$$

where $\tilde{R}^a{}_b$ is the Riemannian curvature 2-form and

$$\tilde{D}K^a{}_b = dK^a{}_b + \tilde{\omega}^a{}_c \wedge K^c{}_b - \tilde{\omega}^c{}_b \wedge K^a{}_c.$$

As seen above, we label the Riemannian quantities by a tilde.

We are using the formalism of Clifford algebra $C\ell_{1,2}$-valued exterior forms. $C\ell_{1,2}$ algebra is generated by the relation among the orthonormal basis $\{\gamma_0, \gamma_1, \gamma_2\}$

$$\gamma^a \gamma^b + \gamma^b \gamma^a = 2\eta^{ab}. \tag{8}$$

One particular representation of the γ^α's is given by the following Dirac matrices:

$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$\gamma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{9}$$

In this case a Dirac spinor Ψ can be represented by a two-component column matrix. Thus we write explicitly the covariant exterior derivative of Ψ, its Dirac conjugate, and the curvature of the spinor bundle, respectively,

$$D\Psi = d\Psi + \frac{1}{2}\sigma_{ab}\Psi\omega^{ab}, \qquad D\bar{\Psi} = d\bar{\Psi} - \frac{1}{2}\bar{\Psi}\sigma_{ab}\omega^{ab},$$
$$D^2\Psi = \frac{1}{2}R^{ab}\sigma_{ab}\Psi, \tag{10}$$

where $\sigma_{ab} := \frac{1}{4}[\gamma_a, \gamma_b] = \frac{1}{2}\epsilon_{abc}\gamma^c$ are the generators of the Lorentz group. Overline figures the Dirac adjoint, $\bar{\Psi} := \Psi^\dagger \gamma_0$. We frequently make use of the identity

$$\gamma_c \sigma_{ab} + \sigma_{ab}\gamma_c = \epsilon_{abc}. \tag{11}$$

III. THE WEAK POINCARE GAUGE THEORY OF GRAVITY

The field equations of our model are obtained by varying the action

$$I[e^a, \omega^{ab}, \bar{\Psi}] = \int_M (L_G + L_D), \tag{12}$$

where L_G signifies the gravitational Lagrangian density 3-form

$$L_G = \frac{a}{2}R_{ab} \wedge {}^*e^{ab} + \lambda^*1 + \frac{k_1}{2}T^a \wedge {}^*T_a + \frac{k_2}{2}\mathcal{V} \wedge {}^*\mathcal{V}$$
$$+ \frac{k_3}{2}\mathcal{A} \wedge {}^*\mathcal{A} + \frac{b}{2}T^a \wedge e_a, \tag{13}$$

and L_D denotes the (Hermitian) Dirac Lagrangian density 3-form

$$L_D = \frac{i}{2}(\bar{\Psi}^*\gamma \wedge D\Psi - D\bar{\Psi} \wedge {}^*\gamma\Psi) + im\bar{\Psi}\Psi^*1, \tag{14}$$

with the definitions $\mathcal{V} = \iota_a T^a$ and $\mathcal{A} = T^a \wedge e_a$. Here the gravitational constants a, k_1, k_2, k_3, mass m, and the Dirac field Ψ have the dimension of length^{-1}, the gravitational constant b has the dimension of length^{-2}, and the cosmological constant λ has the dimension of length^{-3}. When all k_1, k_2, k_3, b coefficients are zero, it corresponds to the well-known Einstein-Cartan-Dirac theory with cosmological constant. The Hermiticity of the Lagrangian (14) leads to a charge current which admits the usual probabilistic interpretation. L_G is the most general gravity Lagrangian with nonpropagating torsion in three dimensions. It is also called the weak Poincare gauge theory of gravity in three dimensions. We remind that a term containing an odd number of the Hodge star has even parity and its coefficient is *scalar*, and with an even number of the Hodge star, it has odd parity and its factor is *pseudoscalar*. Correspondingly, we notice that a, k_1, k_2, k_3, m are scalar, but b is pseudoscalar. $\frac{b}{2}T^a \wedge e_a$ is known as the *translational Chern-Simons term* which corresponds to the usual (rotational) Chern-Simons 3-form $(1/2)(\omega^a{}_b \wedge d\omega^b{}_a + (2/3)\omega^a{}_b \wedge \omega^b{}_c \wedge \omega^c{}_a)$ for the curvature [15].

We obtain the field equations via independent variations with respect to e^a, ω^{ab}, $\bar{\Psi}$. Thus e^a variation yields the first equation

$$-\frac{a}{2}\epsilon_{abc}R^{bc} - \lambda^*e_a - bT_a - \frac{k_1}{2}[2D^*T_a + \iota_a(T^b \wedge {}^*T_b)$$
$$- 2(\iota_a T^b) \wedge {}^*T_b] + \frac{k_2}{2}[2D(\iota_a{}^*\mathcal{V}) - \iota_a(\mathcal{V} \wedge {}^*\mathcal{V})$$
$$- 2(\iota_a T^b) \wedge (\iota_b{}^*\mathcal{V})] - \frac{k_3}{2}[2D(e_a \wedge {}^*\mathcal{A})$$
$$+ \iota_a(\mathcal{A} \wedge {}^*\mathcal{A}) - 2(\iota_a T^b) \wedge (e_b \wedge {}^*\mathcal{A})] = \tau_a, \tag{15}$$

ω^{ab} variation yields the second equation

$$-\frac{a}{2}\epsilon_{abc}T^c + \frac{b}{2}e_{ab} + \frac{k_1}{2}(e_a \wedge {}^*T_b - e_b \wedge {}^*T_a)$$
$$-\frac{k_2}{2}(e_a \wedge \iota_b{}^*\mathcal{V} - e_b \wedge \iota_a{}^*\mathcal{V}) + k_3 e_{ab} \wedge {}^*\mathcal{A} = \Sigma_{ab}, \tag{16}$$

and $\bar{\Psi}$ variation yields the Dirac equation

$$^*\gamma \wedge \left(D - \frac{1}{2}\mathcal{V}\right)\Psi + m\Psi^*1 = 0, \tag{17}$$

where $\Sigma_{ab} = -\mathcal{S}e_{ab}$ is the Dirac angular momentum 2-form with $\mathcal{S} := \frac{i}{4}\bar{\Psi}\Psi$, and τ_a is the Dirac energy-momentum 2-form

$$\tau_a = \frac{i}{2}{}^*e_{ba} \wedge [\bar{\Psi}\gamma^b(D\Psi) - (D\bar{\Psi})\gamma^b\Psi] + im\bar{\Psi}\Psi^*e_a. \tag{18}$$

For future convenience, by using the Dirac equation (17) and its conjugate $(D - \frac{1}{2}\mathcal{V})\bar{\Psi} \wedge {}^*\gamma - m\bar{\Psi}^*1 = 0$, we rewrite the Dirac energy-momentum 2-form as

$$\tau_a = -\frac{i}{2}[\bar{\Psi}\gamma_b(D_a\Psi) - (D_a\bar{\Psi})\gamma_b\Psi]^*e^b$$
$$= -\frac{i}{2}[\bar{\Psi}\gamma_b(\partial_a\Psi) - (\partial_a\bar{\Psi})\gamma_b\Psi]^*e^b + \mathcal{S}\omega_{bc,a}e^{bc}, \tag{19}$$

where $D_a := \iota_a D$, $\partial_a := \iota_a d$ and $\omega_{bc,a} := \iota_a\omega_{bc}$.

Now we solve the second Eq. (16) for torsion

$$T^a = \mathcal{P}^*e^a \quad \text{where } \mathcal{P} = \frac{2\mathcal{S} + b}{-a + 2(k_1 + 3k_3)}. \tag{20}$$

Then we calculate $\mathcal{V} = 0$ and $\mathcal{A} = 3\mathcal{P}^*1$. By substituting these results into the first Eq. (15) we obtain

$$\frac{a}{2}\epsilon_{abc}R^{bc} + (k_1 + 3k_3)e_a \wedge d\mathcal{P}$$
$$+ \left[\lambda + b\mathcal{P} - \frac{1}{2}(k_1 + 3k_3)\mathcal{P}^2\right]^*e_a + \tau_a = 0. \tag{21}$$

The manner in which $(k_1 + 3k_3)$ appears in Eqs. (20) and (21) makes it clear that one can set $k_1 = 0$ or $k_3 = 0$ without loss of generality. Instead, we redefine $k_1 + 3k_3 = c$. For later use, we also note that substitution of (20) into (6) yields the following expression for the contortion:

$$K_{ab} = -\frac{\mathcal{P}}{2}{}^*e_{ab}. \tag{22}$$

A. Reduction to a Riemannian theory

To gain physical insight on the coupling parameters and torsion, we reformulate the theory in terms of Riemannian quantities. Firstly, we decompose the concerned quantities by using (20) and (22) repeatedly,

$$R_{ab} \wedge {}^*e^{ab} = \tilde{R}_{ab} \wedge {}^*e^{ab} + \frac{3}{2}\mathcal{P}^{2*}1 + \text{mod}(d), \tag{23}$$

$$T^a \wedge {}^*T_a = -3\mathcal{P}^{2*}1, \tag{24}$$

$$\mathcal{A} \wedge {}^*\mathcal{A} = -9\mathcal{P}^{2*}1, \tag{25}$$

$$T^a \wedge e_a = 3\mathcal{P}^*1, \tag{26}$$

$$D\Psi = \tilde{D}\Psi + \frac{\mathcal{P}}{4}\gamma\Psi, \tag{27}$$

$$D\bar{\Psi} = \tilde{D}\bar{\Psi} - \frac{\mathcal{P}}{4}\bar{\Psi}\gamma. \tag{28}$$

Here since $\text{mod}(d) := (\tilde{D}K_{ab}) \wedge {}^*e^{ab} = d(K_{ab} \wedge {}^*e^{ab})$ is an exact form, it can be discarded. Also, $\tilde{D}\Psi$ is defined as $\tilde{D}\Psi = d\Psi + \frac{1}{2}\sigma_{ab}\Psi\tilde{\omega}^{ab}$, and similarly so is $\tilde{D}\bar{\Psi}$. When we insert all those findings into the total Lagrangian $L = L_G + L_D$, we obtain a new Riemannian Lagrangian which is equivalent to the weak Poincare gauge theory of gravity,

$$\tilde{L} = \frac{a}{2}\tilde{R}_{ab} \wedge {}^*e^{ab} + \rho_a \wedge T^a + \left[\lambda + \frac{3b^2}{4(2c - a)}\right]^*1$$
$$+ \frac{i}{2}(\bar{\Psi}^*\gamma \wedge \tilde{D}\Psi - \tilde{D}\bar{\Psi} \wedge {}^*\gamma\Psi)$$
$$+ i\left[m + \frac{3(\mathcal{S} + b)}{4(2c - a)}\right]\bar{\Psi}\Psi^*1, \tag{29}$$

where ρ_a is a Lagrange multiplier 1-form constraining torsion to zero. As seen above, pseudoscalar coupling coefficient b shifts the bare cosmological constant and the mass of the Dirac particle. In fact, the Dirac field gains mass through torsional interactions.

IV. CIRCULARLY SYMMETRIC ROTATING SOLUTIONS

We consider the metric

$$g = -f^2(r)dt^2 + h^2(r)dr^2 + r^2(w(r)dt + d\phi)^2, \tag{30}$$

in plane polar coordinates (t, r, ϕ). Here the metric function $w(r)$ is concerned with rotation. We use the notation and the techniques introduced in Ref. [22]. The following choice of the orthonormal basis 1-forms

$$e^0 = f(r)dt, \qquad e^1 = h(r)dr, \qquad e^2 = r(w(r)dt + d\phi) \tag{31}$$

leads to the Levi-Civita connection 1-forms

$$\tilde{\omega}^0{}_1 = \alpha e^0 - \frac{\beta}{2} e^2, \quad \tilde{\omega}^0{}_2 = -\frac{\beta}{2} e^1,$$

$$\tilde{\omega}^1{}_2 = -\frac{\beta}{2} e^0 - \gamma e^2, \tag{32}$$

where we defined

$$\alpha = \frac{f'}{fh}, \qquad \beta = \frac{rw'}{fh}, \qquad \gamma = \frac{1}{rh}. \tag{33}$$

Here prime denotes the derivative with respect to r. Then we write explicitly the full connection 1-forms with the substitution of (22) and (32) into Eq. (4)

$$\omega_{01} = -\alpha e^0 + \frac{\beta - \mathcal{P}}{2} e^2, \qquad \omega_{02} = \frac{\beta + \mathcal{P}}{2} e^1,$$

$$\omega_{12} = -\frac{\beta + \mathcal{P}}{2} e^0 - \gamma e^2. \tag{34}$$

Under the assumption of $\Psi = \Psi(r)$ we calculate the curvature 2-forms

$$R^0{}_1 = \left(-\frac{\alpha'}{h} - \alpha^2 + \frac{3\beta^2}{4} + \frac{\mathcal{P}^2}{4}\right) e^{01}$$
$$+ \left(\frac{\mathcal{P}' - \beta'}{2h} - \beta\gamma\right) e^{12},$$

$$R^0{}_2 = \left(-\alpha\gamma + \frac{\mathcal{P}^2 - \beta^2}{4}\right) e^{02},$$

$$R^1{}_2 = \left(\frac{\mathcal{P}' + \beta'}{2h} + \beta\gamma\right) e^{01}$$
$$+ \left(-\frac{\gamma'}{h} - \gamma^2 + \frac{\mathcal{P}^2 - \beta^2}{4}\right) e^{12}. \tag{35}$$

The next operation is to write down the Dirac equation (17) and its adjoint

$$\frac{\Psi'}{h} = -\left[\frac{\alpha + \gamma}{2} + \left(\frac{\beta}{4} + \frac{3\mathcal{P}}{4} + m\right)\gamma_1\right]\Psi, \tag{36}$$

$$\frac{\bar{\Psi}'}{h} = -\bar{\Psi}\left[\frac{\alpha + \gamma}{2} - \left(\frac{\beta}{4} + \frac{3\mathcal{P}}{4} + m\right)\gamma_1\right]. \tag{37}$$

Now we can calculate explicitly the Dirac energy-momentum 2-forms by using Eq. (19)

$$\tau_0 = -2\alpha\mathcal{S}e^{01} - \mathcal{S}(\beta + \mathcal{P})e^{12},$$
$$\tau_1 = -(2\mathcal{PS} + 4m\mathcal{S})e^{02}, \tag{38}$$
$$\tau_2 = \mathcal{S}(\beta - \mathcal{P})e^{01} - 2\gamma\mathcal{S}e^{12}.$$

Then the first Eq. (21) turns out to be the following set of the coupled ordinary differential equations:

$$\frac{\beta'}{2h} + \frac{(a - 2c)\mathcal{P}'}{2ah} + \beta\gamma - \frac{2\alpha\mathcal{S}}{a} = 0, \tag{39}$$

$$\frac{\beta'}{2h} - \frac{(a - 2c)\mathcal{P}'}{2ah} + \beta\gamma + \frac{2\gamma\mathcal{S}}{a} = 0, \tag{40}$$

$$-\frac{\alpha'}{h} - \alpha^2 + \frac{3\beta^2}{4} + \frac{(a - 2c)\mathcal{P}^2}{4a} + \frac{\mathcal{S}(\beta - \mathcal{P}) + b\mathcal{P} + \lambda}{a}$$
$$= 0, \tag{41}$$

$$-\frac{\gamma'}{h} - \gamma^2 - \frac{\beta^2}{4} + \frac{(a - 2c)\mathcal{P}^2}{4a} - \frac{\mathcal{S}(\beta + \mathcal{P}) - b\mathcal{P} - \lambda}{a}$$
$$= 0, \tag{42}$$

$$-\alpha\gamma - \frac{\beta^2}{4} + \frac{(a - 2c)\mathcal{P}^2}{4a} + \frac{2\mathcal{PS} + b\mathcal{P} + 4m\mathcal{S} + \lambda}{a}$$
$$= 0. \tag{43}$$

A. $\alpha(r) = \gamma(r)$ case

We firstly restrict our attention to those cases for which

$$\gamma = \alpha = \frac{1}{rh}. \tag{44}$$

From the definitions (33) it immediately follows that

$$f(r) = f_0 r, \tag{45}$$

where f_0 is a constant. Then, using (39)$\pm$(40) we arrive at

$$\beta(r) = \frac{\beta_0}{r^2}, \qquad \mathcal{S}(r) = \frac{\mathcal{S}_0}{r^2}, \tag{46}$$

where β_0 and $\mathcal{S}_0$ are integration constants. But, (41) and (42) causes a constraint among them

$$\beta_0 = -\frac{2}{a} \mathcal{S}_0. \tag{47}$$

Inserting the above results and $\alpha = 1/rh$ into (41)+(42) yields a solution for $h(r)$

$$h(r) = 1/\sqrt{h_0 - A\mathcal{S}_0^2/2r^2 + \Lambda r^2}, \tag{48}$$

where h_0 is an integration constant, A is the shifted coupling constant, and Λ is the effective cosmological constant

$$A = \frac{4(2a - c)}{a^2(a - 2c)}, \qquad \Lambda = \frac{4\lambda(a - 2c) - 3b^2}{4a(a - 2c)}. \tag{49}$$

Equation (42) gives a constraint between the integration constants

$$h_0 = \frac{4M}{a} \mathcal{S}_0, \tag{50}$$

where M is a shifted Dirac mass $M = \frac{4m(a-2c)-3b}{4(a-2c)}$. Now we calculate $w(r)$ from ((33)ii)

$$w(r) = -\sqrt{\frac{8f_0^2}{Aa^2}}\arctan\left[\frac{\sqrt{2\Lambda}r^2 + \sqrt{-A\mathcal{S}_0^2 + 8M\mathcal{S}_0 r^2/a + 2\Lambda r^4}}{\mathcal{S}_0\sqrt{A}}\right] + w_0, \tag{51}$$

where w_0 is a constant. Here we notice the consistency condition $A, \Lambda > 0$. Moreover, if we choose

$$w_0 = \sqrt{\frac{2\pi^2 f_0^2}{Aa^2}}, \tag{52}$$

then as $\mathcal{S}_0 \to 0$, $w(r)$ goes to zero. These results have been cross-checked by the computer algebra system REDUCE [23] and its package EXCALC [24].

Our final job is to work out the Dirac equation. Let us consider a Dirac spinor field and its Dirac conjugate

$$\Psi = \begin{pmatrix} \psi_1(r) \\ \psi_2(r) \end{pmatrix}, \quad \bar{\Psi} := \Psi^\dagger \gamma_0 = \begin{pmatrix} -\psi_2^\star(r) & \psi_1^\star(r) \end{pmatrix}, \tag{53}$$

where * denotes complex conjugation and ψ_1, ψ_2 are complex functions. Then Eq. (36) reads in components as follows:

$$\psi_1' = -h\alpha\psi_1 - \frac{\beta + 3\mathcal{P} + 4m}{4}h\psi_2, \tag{54}$$

$$\psi_2' = -h\alpha\psi_2 - \frac{\beta + 3\mathcal{P} + 4m}{4}h\psi_1. \tag{55}$$

We take the combinations $\psi_\pm = \psi_1 \pm \psi_2$ and write a decoupled system of equations

$$\psi_\pm' = -\left(\alpha \pm \frac{\beta + 3\mathcal{P} + 4m}{4}\right)h\psi_\pm. \tag{56}$$

The explicit solutions to these equations are given by

$$\psi_\pm(r) = \frac{C_\pm}{r}e^{\mp[-A\varphi(r) + M\theta(r)]}, \tag{57}$$

where $C_\pm$ are the complex integration constants and

$$\varphi(r) = \frac{a}{4}\int h(r)\mathcal{S}(r)dr = \sqrt{\frac{a^2}{8A}}\arctan\left[\frac{\sqrt{2\Lambda}r^2 + \sqrt{-A\mathcal{S}_0^2 + 8M\mathcal{S}_0 r^2/a + 2\Lambda r^4}}{\mathcal{S}_0\sqrt{A}}\right], \tag{58}$$

$$\theta(r) = \int h(r)dr = \frac{1}{\sqrt{4\Lambda}}\ln\left[\frac{4M\mathcal{S}_0/a + 2\Lambda r^2 + \sqrt{2\Lambda(-A\mathcal{S}_0^2 + 8M\mathcal{S}_0 r^2/a + 2\Lambda r^4)}}{\sqrt{2A\Lambda\mathcal{S}_0^2 + 16M^2\mathcal{S}_0^2/a^2}}\right]. \tag{59}$$

Thus we can write the components of the Dirac spinor as $\psi_1 = (\psi_+ + \psi_-)/2$ and $\psi_2 = (\psi_+ - \psi_-)/2$. Consequently, we write down explicitly the Dirac condensate $\mathcal{S} := \frac{i}{4}\bar{\Psi}\Psi = \frac{i}{8r^2}(C_-^\star C_+ - C_+^\star C_-)$. By comparing this with ((46)ii), we observe

$$\mathcal{S}_0 = \frac{i}{8}(C_-^\star C_+ - C_+^\star C_-). \tag{60}$$

Here we want to remark that if C_I's, $I = -, +$, are the ordinary complex numbers [i.e., $C_I C_J = +C_J C_I$, $(C_I C_J)^\star = C_I^\star C_J^\star$, $C_I^{\star\star} = C_I$] then $\mathcal{S}_0$ is a real number. Similarly, if C_I's are the Grassmann complex numbers [i.e., $C_I C_J = -C_J C_I$, $(C_I C_J)^\star = C_J^\star C_I^\star$, $C_I^{\star\star} = C_I$] then $\mathcal{S}_0$ is again a real number.

B. $h(r) = 1/f(r)$ case

We try to find a family of solutions in the form of $h(r) = 1/f(r)$. By substituting this into the Dirac equation (36) with notation (53) and $\psi_\pm = \psi_1 \pm \psi_2$, we obtain

$$\psi_\pm(r) = \frac{C_\pm}{r}e^{\mp\theta(r)}, \tag{61}$$

where

$$\theta(r) = -\int\left(\frac{M}{f(r)} + \frac{j}{2r^2 f(r)}\right)dr. \tag{62}$$

Here j is a constant. Then we calculate the Dirac condensate as

$$\mathcal{S}(r) = \frac{\mathcal{S}_0}{rf(r)}, \tag{63}$$

where $\mathcal{S}_0 = \frac{i}{8}(C_-^\star C_+ - C_+^\star C_-)$. Now if we choose $\mathcal{S}_0 = 0$, then the set of Eqs. (39)–(43) accepts a family of solutions as follows:

$$f(r) = \sqrt{\Lambda r^2 - \mathcal{M} + j^2/r^2}, \quad h(r) = 1/f(r),$$
$$w(r) = j/r^2, \tag{64}$$

where $\mathcal{M}$ is an integration constant. This looks exactly the same as the very well-known BTZ metric of general relativity.

V. CONCLUSION

We formulated the Dirac coupled gravity theory with *the most* general nonpropagating torsion (the weak Poincare gauge theory of gravity) in $(1 + 2)$-dimensions by using the algebra of exterior differential forms. We obtained the field equations by a variational principle. The space-time torsion was calculated algebraically from the second field equation in terms of the coupling constants and the quadratic spinor invariant, the so-called the Dirac condensate. Further, we reformulated the non-Riemannian theory in terms of Riemannian quantities. Thus we could gain new interpretations on the coupling coefficients and the mass of Dirac field.

We then looked for rotating circularly symmetric solutions and found a particular class of solutions which is asymptotically AdS_3. These solutions exhibit one singularity at the origin and two more at the outer region. In order to obtain the physical meaning of the above singularities, we calculated the following pair of invariants. The first is the curvature scalar

$$\mathcal{R} = \frac{[3b^2(2a - 3c) - 6\lambda(a - 2c)^2]r^4 + [12b(a - c) - 8m(a - 2c)^2]\mathcal{S}_0 r^2 + 12c\mathcal{S}_0^2}{a(a - 2c)^2 r^4}$$

and the second is the quadratic torsion

$$^*(T^a \wedge {}^*T_a) = \frac{3(br^2 + 2\mathcal{S}_0)^2}{(a - 2c)^2 r^4}.$$

As seen above, although the singularities at the outer region are coordinate singularities, the singularity at the origin is essential. Correspondingly, that solution seems to define a black hole with two horizons. We also remark that if one sets $\mathcal{S}_0 = 0$, then both invariants turn out to be constant.

Finally we obtained a BTZ-type solution in the case of vanishing condensate. Although we searched if Eqs. (39)–(43) accepted the BTZ solution when $\mathcal{S}_0 \neq 0$, we were not able to arrive at a definite answer. This fact, however, does not diminish the novelty of our solution because our space-time is still non-Riemannian because of the nonzero torsion, see Eq. (20). Accordingly, the autoparallel curves of our geometry do not coincide with the geodesics of metric (64). We also noticed that the coupling parameter b still shifts the mass term of the Dirac field, see the last parenthesis of (29). That is, even if the Dirac field was massless, it would gain mass through the b-contained interactions.

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