Triangular solution to the general relativistic three-body problem for general masses

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Continuing work initiated in an earlier publication [T. Ichita, K. Yamada, and H. Asada, Phys. Rev. D **83**, 084026 (2011)], we reexamine the post-Newtonian effects on Lagrange's equilateral triangular solution for the three-body problem. For three finite masses, it is found that a triangular configuration satisfies the post-Newtonian equation of motion in general relativity if and only if it has the relativistic corrections to each side length. This post-Newtonian configuration for three finite masses is not always equilateral, and it recovers previous results for the restricted three-body problem when one mass goes to zero. For the same masses and angular velocity, the post-Newtonian triangular configuration is always smaller than the Newtonian one.

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I. INTRODUCTION

One of the classical problems in astronomy and physics is the three-body problem in Newtonian gravity (e.g., Refs. [1–3]).

The gravitational three-body problem is not integrable by the analytical method. As particular solutions, however, Euler and Lagrange found a collinear solution and an equilateral triangular one, respectively.

The solutions for the restricted three-body problem, where one of three bodies is a test mass, are known as Lagrange points L_1, L_2, L_3, L_4, and L_5 [1]. *Lagrange's equilateral triangular solution* has also a practical importance, since it is stable for some cases. Lagrange points L_4 and L_5 for the Sun-Jupiter system are stable and indeed the Trojan asteroids are located there. For the Sun-Earth system, asteroids were also found around L_4 by recent observations [4].

Recently, Lagrange points have attracted renewed interest for relativistic astrophysics [5–10], where they have discussed the relativistic corrections for Lagrange points [5,6] and the gravitational radiation reaction on L_4 and L_5 analytically [7] and by numerical methods [8–10]. It is currently important to reexamine Lagrange points in the framework of general relativity. As a pioneering work, it was pointed out by Nordtvedt that the location of the triangular points is very sensitive to the ratio of the gravitational mass to the inertial one [11]. Along this course, it might be important as a gravity experiment to discuss the three-body coupling terms in the post-Newtonian (PN) force, because some of the terms are proportional to a product of three masses as $M_1 \times M_2 \times M_3$. Such a triple product can appear only for relativistic three- (or more) body systems but cannot for a relativistic compact binary nor a Newtonian three-body system.

It was shown by Ichita, Yamada, and Asada, including the present authors, that a relativistic equilateral triangular solution does not satisfy the equation of motion at the first post-Newtonian (1PN) order except for two cases [12]: (i) three finite masses are equal and (ii) one mass is finite and the other two are zero. Hence, it is interesting to investigate what happens at the 1PN level for three unequal finite masses in Lagrange's equilibrium configuration. For the restricted three-body problem, on the other hand, Krefetz found a relativistic triangular solution by adding the corrections to the position of the third body [5]. For three general finite masses, we shall look for a relativistic equilibrium solution that corresponds to Lagrange's equilateral triangular one.

Throughout this paper, we take the units of $G = c = 1$.

II. NEWTONIAN EQUILATERAL TRIANGULAR SOLUTION

First, we consider the Newtonian gravity among three masses denoted as M_I ($I = 1, 2, 3$) in a circular motion. The location of each mass is written as $\boldsymbol{r}_I$, where we choose the origin of the coordinates as the common center of mass, so that

$$M_1\boldsymbol{r}_1 + M_2\boldsymbol{r}_2 + M_3\boldsymbol{r}_3 = 0. \tag{1}$$

We start by seeing whether the Newtonian equation of motion for each body can be satisfied if the configuration is an equilateral triangle. Let us put $r_{12} = r_{23} = r_{31} \equiv a$, where we define the relative position between masses as

$$\boldsymbol{r}_{IJ} \equiv \boldsymbol{r}_I - \boldsymbol{r}_J, \tag{2}$$

and $r_{IJ} \equiv |\boldsymbol{r}_{IJ}|$ for I, $J = 1$, 2 3. Then, the equation of motion for each mass becomes

$$\frac{d\boldsymbol{r}_I}{dt^2} = -\frac{M}{a^3}\boldsymbol{r}_I, \tag{3}$$

where M denotes the total mass $\sum_I M_I$. Therefore, it is possible that each body moves around the common center of mass with the same orbital period.

Figure 1 shows a triangular configuration for general masses.

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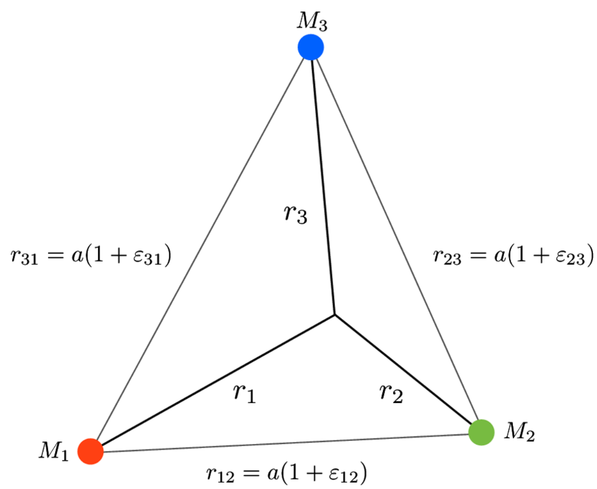

FIG. 1 (color online). PN triangular configuration. Each mass is located at one of the apexes. $r_I \equiv |\boldsymbol{r}_I| (I = 1, 2, 3)$ denotes the orbital radius of each body. Each ε_{IJ} denotes the relativistic correction to each side length at the 1PN order. In the equilateral case, $\varepsilon_{12} = \varepsilon_{23} = \varepsilon_{31} = 0$, namely, $r_{12} = r_{23} = r_{31} = a$.

Equation (3) gives

$$\omega_{\mathrm{N}}^2 = \frac{M}{a^3}, \tag{4}$$

where ω_{N} denotes the Newtonian angular velocity. The orbital radius $r_I \equiv |\boldsymbol{r}_I|$ of each body with respect to the common center of mass is obtained as [2]

$$r_1 = a\sqrt{\nu_2^2 + \nu_2\nu_3 + \nu_3^2}, \tag{5}$$

$$r_2 = a\sqrt{\nu_1^2 + \nu_1\nu_3 + \nu_3^2}, \tag{6}$$

$$r_3 = a\sqrt{\nu_1^2 + \nu_1\nu_2 + \nu_2^2}, \tag{7}$$

where we define the mass ratio as $\nu_I \equiv M_I/M$.

III. POST-NEWTONIAN EQUILATERAL TRIANGULAR SOLUTION

Next, let us study the dominant part of general relativistic effects on this solution. Namely, we take account of the term at the 1PN order by employing the Einstein-Infeld-Hoffman (EIH) equation of motion in the standard PN coordinate as [13–15]

$$\begin{aligned} \frac{d\boldsymbol{v}_K}{dt} = &\sum_{A\neq K} \boldsymbol{r}_{AK}\frac{M_A}{r_{AK}^3}\left[1 - 4\sum_{B\neq K}\frac{M_B}{r_{BK}} - \sum_{C\neq A}\frac{M_C}{r_{CA}}\left(1 - \frac{\boldsymbol{r}_{AK}\cdot\boldsymbol{r}_{CA}}{2r_{CA}^2}\right)\right.\\ &\left. + \boldsymbol{v}_K^2 + 2\boldsymbol{v}_A^2 - 4\boldsymbol{v}_A\cdot\boldsymbol{v}_K - \frac{3}{2}(\boldsymbol{v}_A\cdot\boldsymbol{n}_{AK})^2\right]\\ &- \sum_{A\neq K}(\boldsymbol{v}_A - \boldsymbol{v}_K)\frac{M_A\boldsymbol{n}_{AK}\cdot(3\boldsymbol{v}_A - 4\boldsymbol{v}_K)}{r_{AK}^2}\\ &+ \frac{7}{2}\sum_{A\neq K}\sum_{C\neq A}\boldsymbol{r}_{CA}\frac{M_A M_C}{r_{AK}r_{CA}^3}, \end{aligned} \tag{8}$$

where $\boldsymbol{v}_I$ denotes the velocity of each mass in an inertial frame and we define

$$\boldsymbol{n}_{IJ} \equiv \frac{\boldsymbol{r}_{IJ}}{r_{IJ}}. \tag{9}$$

Note that Eq. (8) for the EIH equation expresses the acceleration of each mass, where the force exerted on one mass is divided by the mass. The PN force includes a product of three masses, whereas the acceleration by Eq. (8) does that of two masses.

We consider three masses in a circular motion with the angular velocity ω, so that each r_I can be a constant. In addition, the common center of mass remains unchanged for the equilateral triangular configuration as shown in Ref. [12]. Hence, the PN location $\boldsymbol{r}_I$ and orbital radius r_I of each body are unchanged from the Newtonian ones. As a consequence, the equation of motion for M_1 can be written as [12]

$$-\omega^2\boldsymbol{r}_1 = -\frac{M}{a^3}\boldsymbol{r}_1 + \boldsymbol{\delta}_{\mathrm{EIH1}}, \tag{10}$$

where $\boldsymbol{\delta}_{\mathrm{EIH1}}$ denotes the PN terms defined as

$$\begin{aligned} \boldsymbol{\delta}_{\mathrm{EIH1}} = &\frac{1}{16}\frac{M^2}{a^3}\frac{1}{\sqrt{\nu_2^2 + \nu_2\nu_3 + \nu_3^2}}\{\{16(\nu_2^2 + \nu_2\nu_3 + \nu_3^2)\\ &\times[3 - (\nu_1\nu_2 + \nu_2\nu_3 + \nu_3\nu_1)] + 9\nu_2\nu_3[2(\nu_2 + \nu_3)\\ &+ \nu_2^2 + 4\nu_2\nu_3 + \nu_3^2]\}\boldsymbol{n}_1 + 3\sqrt{3}\nu_2\nu_3(\nu_2 - \nu_3)\\ &\times(5 - 3\nu_1)\boldsymbol{n}_{\perp 1}\}, \end{aligned} \tag{11}$$

by using $\boldsymbol{n}_1 = \boldsymbol{r}_1/r_1$ and $\boldsymbol{n}_{\perp 1} = \boldsymbol{v}_1/r_1\omega$ defined as the unit normal vector to $\boldsymbol{r}_1$.

Equations (10) and (11) seem to disagree with Eqs. (31) and (32) in Ref. [12]. However, it is not the case. This equality can be shown by noting that the angular velocity in the PN term, which appears at Eqs. (31) and (32) in Ref. [12], is equal to M/a^3.

One can obtain the equation of motion for M_2 and M_3 by cyclic manipulations as $1 \to 2 \to 3 \to 1$. These expressions show that the equilateral triangular solution is present at 1PN order only for two cases: (i) three finite masses are equal and (ii) one mass is finite and the other two are zero [16].

IV. POST-NEWTONIAN TRIANGULAR SOLUTION FOR THREE FINITE MASSES

For the restricted three-body problem, an inequilateral triangular solution was investigated [5,6]. Hence, for three finite masses, we study a PN triangular configuration.

Let us denote each side length of a PN triangle as

$$r_{IJ} = a(1 + \varepsilon_{IJ}), \tag{12}$$

where ε_{IJ} denotes the nondimensional correction at the 1PN order (see Fig. 1). Here, if all three corrections are equal (i.e., $\varepsilon_{12} = \varepsilon_{23} = \varepsilon_{31} = \varepsilon$), a PN configuration is still an equilateral triangle, though each side length is changed by a scale transformation as $a \to a(1 + \varepsilon)$. Namely, one of the degrees of freedom for $(\varepsilon_{12}, \varepsilon_{23}, \varepsilon_{31})$ corresponds to a scale transformation, and this is unphysical. In other words, we take account of only the corrections which keep the size of the system but change its shape. For its simplicity, we adopt the arithmetic mean of three side lengths in order to characterize the size of the system as

$$\frac{r_{12} + r_{23} + r_{31}}{3} = a\left[1 + \frac{1}{3}(\varepsilon_{12} + \varepsilon_{23} + \varepsilon_{31})\right] \tag{13}$$

(see the Appendix for possible choices of fixing the unphysical degree of freedom). This arithmetic mean of the PN triangle is chosen to be the same as a side length of the Newtonian equilateral triangle as

$$a\left[1 + \frac{1}{3}(\varepsilon_{12} + \varepsilon_{23} + \varepsilon_{31})\right] = a, \tag{14}$$

so that the degree of freedom for a scale transformation can be fixed. Otherwise, a degree of freedom for a scale transformation would remain so that an ambiguity due to the similarity could enter our results. Thus, we obtain a constraint on $(\varepsilon_{12}, \varepsilon_{23}, \varepsilon_{31})$ as

$$\varepsilon_{12} + \varepsilon_{23} + \varepsilon_{31} = 0. \tag{15}$$

Hence, we look for the remaining two conditions for determining $(\varepsilon_{12}, \varepsilon_{23}, \varepsilon_{31})$ in the following.

We assume a circular motion of each body, where the angular velocity of each mass is denoted as ω_I $(I = 1, 2, 3)$. At the 1PN order, the equation of motion for M_1 becomes

$$\begin{aligned} -\omega_1^2 \boldsymbol{r}_1 &= M_2 \frac{\boldsymbol{r}_{21}}{r_{21}^3} + M_3 \frac{\boldsymbol{r}_{31}}{r_{31}^3} + \boldsymbol{\delta}_{\rm EIH1} \\ &= -\frac{M}{a^3}\boldsymbol{r}_1 - \frac{3}{2}\frac{M}{a^2}\frac{1}{\sqrt{\nu_2^2 + \nu_2\nu_3 + \nu_3^2}} \\ &\quad \times \{[\nu_2(\nu_1 - \nu_2 - 1)\varepsilon_{12} + \nu_3(\nu_1 - \nu_3 - 1)\varepsilon_{31}]\boldsymbol{n}_1 \\ &\quad + \sqrt{3}\nu_2\nu_3(\varepsilon_{12} - \varepsilon_{31})\boldsymbol{n}_{\perp 1}\} + \boldsymbol{\delta}_{\rm EIH1} + {\rm O(2PN)}. \end{aligned} \tag{16}$$

Note that each mass location $\boldsymbol{r}_I$ may be different from the Newtonian one, because the origin of the coordinates is chosen as the common center of mass in the 1PN approximation. However, we can replace $\boldsymbol{r}_I$ with the Newtonian location of M_1 because of the following reason. The two terms including $\boldsymbol{r}_I$ of Eq. (16) are expanded as

$$-\omega_1^2 \boldsymbol{r}_1 = -\omega_1^2 \boldsymbol{r}_{\rm N1} - \omega_{\rm N}^2 \boldsymbol{r}_{\rm PN1} + {\rm O(2PN)}, \tag{17}$$

$$-\frac{M}{a^3}\boldsymbol{r}_1 = -\frac{M}{a^3}\boldsymbol{r}_{\rm N1} - \frac{M}{a^3}\boldsymbol{r}_{\rm PN1}, \tag{18}$$

respectively, where $\boldsymbol{r}_{\rm N1}$ and $\boldsymbol{r}_{\rm PN1}$ denote the Newtonian location and the 1PN correction, respectively. By using Eqs. (4), (17), and (18) imply that the 1PN corrections to $\boldsymbol{r}_I$ cancel out in Eq. (16).

Furthermore, $\boldsymbol{n}_1$ and $\boldsymbol{n}_{\perp 1}$ also have PN corrections. However, these corrections multiplied by ε_{12} (or ε_{31}) make 2PN (or higher-order) contributions in Eq. (16), and hence they can be neglected. Also in $\boldsymbol{\delta}_{\rm EIH1}$, 1PN corrections to $\boldsymbol{n}_1$ and $\boldsymbol{n}_{\perp 1}$ lead to 2PN, since they are multiplied by 1PN term as M^2/a^3. We obtain the equation of motion for M_2 and M_3 by cyclic manipulations as $1 \to 2 \to 3 \to 1$.

The PN equilibrium configurations can be present if and only if the following conditions (a) and (b) hold. (a) Each mass has to satisfy the EIH equation of motion, and (b) a triangular configuration does not change with time. Condition (a) is equivalent to (a′) the coefficients of $\boldsymbol{n}_{\perp I}$ in the equation of motion for each mass are zero:

$$\varepsilon_{12} - \varepsilon_{31} - \frac{1}{8}\frac{M}{a}(\nu_2 - \nu_3)(5 - 3\nu_1) = 0, \tag{19}$$

$$\varepsilon_{23} - \varepsilon_{12} - \frac{1}{8}\frac{M}{a}(\nu_3 - \nu_1)(5 - 3\nu_2) = 0, \tag{20}$$

$$\varepsilon_{31} - \varepsilon_{23} - \frac{1}{8}\frac{M}{a}(\nu_1 - \nu_2)(5 - 3\nu_3) = 0. \tag{21}$$

Condition (b) is restated as (b′) the angular velocity for each mass is the same in order to keep the distance between masses unchanged:

$$\omega_1^2 - \omega_2^2 = 0, \tag{22}$$

$$\omega_1^2 - \omega_3^2 = 0. \tag{23}$$

Equations (22) and (23) are rewritten as

$$
\begin{aligned}
&\frac{3}{2}\frac{M}{a^3}\frac{1}{\nu_2^2+\nu_2\nu_3+\nu_3^2}[\nu_2(\nu_1-\nu_2-1)\varepsilon_{12}+\nu_3(\nu_1-\nu_3-1)\varepsilon_{31}]\\
&\quad-\frac{3}{2}\frac{M}{a^3}\frac{1}{\nu_3^2+\nu_3\nu_1+\nu_1^2}[\nu_3(\nu_2-\nu_3-1)\varepsilon_{23}+\nu_1(\nu_2-\nu_1-1)\varepsilon_{12}]\\
&\quad-\frac{M^2}{a^4}\left\{\frac{9}{16}\frac{1}{\nu_2^2+\nu_2\nu_3+\nu_3^2}\nu_2\nu_3[2(\nu_2+\nu_3)+\nu_2^2+4\nu_2\nu_3+\nu_3^2]\right\}\\
&\quad+\frac{M^2}{a^4}\left\{\frac{9}{16}\frac{1}{\nu_3^2+\nu_3\nu_1+\nu_1^2}\nu_3\nu_1[2(\nu_3+\nu_1)+\nu_3^2+4\nu_3\nu_1+\nu_1^2]\right\}=0,
\end{aligned}
\tag{24}
$$

$$
\begin{aligned}
&\frac{3}{2}\frac{M}{a^3}\frac{1}{\nu_2^2+\nu_2\nu_3+\nu_3^2}[\nu_2(\nu_1-\nu_2-1)\varepsilon_{12}+\nu_3(\nu_1-\nu_3-1)\varepsilon_{31}]\\
&\quad-\frac{3}{2}\frac{M}{a^3}\frac{1}{\nu_1^2+\nu_1\nu_2+\nu_2^2}[\nu_1(\nu_3-\nu_1-1)\varepsilon_{31}+\nu_2(\nu_3-\nu_2-1)\varepsilon_{23}]\\
&\quad-\frac{M^2}{a^4}\left\{\frac{9}{16}\frac{1}{\nu_2^2+\nu_2\nu_3+\nu_3^2}\nu_2\nu_3[2(\nu_2+\nu_3)+\nu_2^2+4\nu_2\nu_3+\nu_3^2]\right\}\\
&\quad+\frac{M^2}{a^4}\left\{\frac{9}{16}\frac{1}{\nu_1^2+\nu_1\nu_2+\nu_2^2}\nu_1\nu_2[2(\nu_1+\nu_2)+\nu_1^2+4\nu_1\nu_2+\nu_2^2]\right\}=0,
\end{aligned}
\tag{25}
$$

respectively. It seems that $(\varepsilon_{12}, \varepsilon_{23}, \varepsilon_{31})$ do not always satisfy the above five conditions Eqs. (19)–(23) simultaneously. However, the number of independent conditions turns out to be two.

The reason is as follows. By eliminating ε_{12} from Eqs. (19) and (20), we obtain Eq. (21). Moreover, the left-hand sides of Eqs. (24) and (25) always vanish if and only if Eqs. (19) and (20) are satisfied. These can be seen by direct calculations.

Thus, we obtain the expressions for $(\varepsilon_{12}, \varepsilon_{23}, \varepsilon_{31})$ as

$$\varepsilon_{12}=\frac{1}{24}\frac{M}{a}[(\nu_2-\nu_3)(5-3\nu_1)-(\nu_3-\nu_1)(5-3\nu_2)], \tag{26}$$

$$\varepsilon_{23}=\frac{1}{24}\frac{M}{a}[(\nu_3-\nu_1)(5-3\nu_2)-(\nu_1-\nu_2)(5-3\nu_3)], \tag{27}$$

$$\varepsilon_{31}=\frac{1}{24}\frac{M}{a}[(\nu_1-\nu_2)(5-3\nu_3)-(\nu_2-\nu_3)(5-3\nu_1)], \tag{28}$$

which recover previous results for the restricted three-body problem [5].

Substituting Eqs. (26) and (28) into Eq. (16), we obtain the angular velocity as

$$\omega_1=\omega_{\rm N}\left(1+\frac{M}{a}\omega_{\rm PN}\right), \tag{29}$$

where

$$\omega_{\rm PN}=-\frac{1}{16}[29-14(\nu_1\nu_2+\nu_2\nu_3+\nu_3\nu_1)]. \tag{30}$$

By using $\nu_1+\nu_2+\nu_3=1$, one can immediately show $\omega_{\rm PN}<0$, so that we find $\omega<\omega_{\rm N}$ for the same masses and a. In other words, for the same masses and angular velocity, the PN triangular configuration is always smaller than the Newtonian one.

Table I shows the relativistic corrections of the distance between each body for Lagrange point L_4 (L_5) of the Solar System. Here we choose M_1 and M_2 as the Sun and each planet, respectively. In the case of the restricted three-body problem ($\nu_3\to 0$), it is convenient to use Eqs. (19) and (20) rather than Eqs. (26)–(28), because it is natural to change not r_{12} but location of M_3. Equation (20) implies that the correction of distance between each planet and Lagrange point L_4 (L_5) is approximately $5/16$ of the Schwarzschild radius of the Sun. Hence, we obtain the same values of this correction for Earth and Jupiter. Similar corrections are mentioned also in the previous paper [6].

TABLE I. The corrections for Lagrange point L_4 (L_5) of the Solar System. Equations (19) and (20) are used for the evaluation. Here, we choose M_1 and M_2 as the Sun and each planet, respectively. Thus, $r_{12}=a(1+\varepsilon_{12})$ is the distance between the Sun and each planet.

Planet	Sun-L_4 (L_5) [m]	Planet-L_4 (L_5) [m]
Jupiter	−0.353	−923
Earth	−0.00111	−923

The above PN effects, however, are so tiny that they could be neglected in the near-future measurements [17].

It is interesting to extend this 1PN work to higher PN orders for gravitational wave physics (see Refs. [18–21] for the equation of motion and compact binaries).

V. CONCLUSION

We reexamined the post-Newtonian effects on Lagrange's equilateral triangular solution for the three-body problem. For three finite masses, it was found that a general triangular configuration satisfies the post-Newtonian equation of motion in general relativity if and only if it has the relativistic corrections to each side length. It was shown also that the post-Newtonian triangular configuration is always smaller than the Newtonian one for the same masses and angular velocity. Studying the correction to stability of this configuration is left as future work.

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APPENDIX: CHOICES OF FIXING THE UNPHYSICAL DEGREE OF FREEDOM

Instead of the arithmetic mean of the three side lengths, one might wish to use the geometric mean of them or the triangular area. Hence, let us mention briefly these cases.

The geometric mean of three side lengths by Eq. (12) is written up to the 1PN order as

$$(r_{12}r_{23}r_{31})^{1/3} = a\left[1 + \frac{1}{3}(\varepsilon_{12} + \varepsilon_{23} + \varepsilon_{31})\right] + O(\varepsilon^2), \tag{A1}$$

where $O(\varepsilon^2)$ denotes the second order of $(\varepsilon_{12}, \varepsilon_{23}, \varepsilon_{31})$, namely, at the 2PN order. This expression is identical with the arithmetic mean by Eq. (13). In addition, we can obtain the condition that these means are equal to a side length of the Newtonian equilateral triangle as

$$\varepsilon_{12} + \varepsilon_{23} + \varepsilon_{31} = 0. \tag{A2}$$

Next, we consider a PN triangular area. A triangular area S is given by Heron's formula as

$$S = \sqrt{s(s - r_{12})(s - r_{23})(s - r_{31})}, \tag{A3}$$

where

$$s = \frac{r_{12} + r_{23} + r_{31}}{2}. \tag{A4}$$

Hence, substitution of Eq. (12) into Eq. (A3) leads to

$$S = \frac{\sqrt{3}}{4}a^2\left[1 + \frac{2}{3}(\varepsilon_{12} + \varepsilon_{23} + \varepsilon_{31})\right] + O(\varepsilon^2), \tag{A5}$$

which corresponds to the triangular area of each side length [Eq. (A1)]. Therefore, a PN triangular area is equal to a Newtonian equilateral triangular one if and only if

$$\varepsilon_{12} + \varepsilon_{23} + \varepsilon_{31} = 0. \tag{A6}$$

This is identical with the condition in Eq. (A2).

As a consequence, at the 1PN level, the arithmetic mean of the three side lengths, the geometric mean of them, and the triangular area lead to the same characterization of the size of the system.

[1] H. Goldstein, *Classical Mechanics* (Addison-Wesley, Reading, MA, 1980).

[2] J. M. A. Danby, *Fundamentals of Celestial Mechanics* (William-Bell, Richmond, VA, 1988).

[3] C. Marchal, *The Three-Body Problem* (Elsevier, Amsterdam, 1990).

[4] M. Connors, P. Wiegert, and C. Veillet, Nature (London) **475**, 481 (2011).

[5] E. Krefetz, Astron. J. **72**, 471 (1967).

[6] T. I. Maindl, *Completing the Inventory of the Solar System*, edited by T. W. Rettig and J. M. Hahn, Astron. Soc. Pac. Conf. Ser. Vol. 107 (Astronomical Society of the Pacific, San Francisco, 1996), p. 147.

[7] H. Asada, Phys. Rev. D **80**, 064021 (2009).

[8] N. Seto and T. Muto, Phys. Rev. D **81**, 103004 (2010).

[9] J. D. Schnittman, Astrophys. J. **724**, 39 (2010).

[10] Y. Torigoe, K. Hattori, and H. Asada, Phys. Rev. Lett. **102**, 251101 (2009).

[11] K. Nordtvedt, Phys. Rev. **169**, 1014 (1968).

[12] T. Ichita, K. Yamada, and H. Asada, Phys. Rev. D **83**, 084026 (2011).

[13] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, New York, 1973).

[14] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Pergamon, New York, 1962).

[15] H. Asada, T. Futamase, and P. Hogan, *Equations of Motion in General Relativity* (Oxford University, New York, 2011).

[16] Reference [12] suggests that if two of the masses are finite and equal, and the third one is zero, an equilateral configuration satisfies the EIH equation of motion. However, the angular velocity of each mass is not the same in this case. Note that this case is not an equilibrium configuration.

[17] K. Yamada and H. Asada, Mon. Not. R. Astron. Soc. **83**, 084026 (2011).

[18] T. Damour, P. Jaranowski, and G. Schäfer, Phys. Rev. D **63**, 044021 (2001).

[19] Y. Itoh, T. Futamase, and H. Asada, Phys. Rev. D **63**, 064038 (2001).

[20] K. G. Arun, L. Blanchet, B. R. Iyer, and M. S. S. Qusailah, Phys. Rev. D **77**, 064035 (2008).

[21] Y. Itoh, Phys. Rev. D **80**, 124003 (2009).