Higher order equations of motion and gravity

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Standard fundamental equations of motion for point particles are of second order in the time derivative. Here we are exploring the consequences of fundamental equations of motion with an additional small even higher order term to the standard formulation. This is related to two issues: (i) higher order equations of motion will have influence on the definition of the structure of possible interactions and in particular of the gravitational interaction, and (ii) such equations of motion provide a framework to test the validity of Newton's second law which is the basis for the definition of forces but which assumes from the very beginning that the fundamental equations of motion are of second order. We will show that starting with our generalized equations of motions it is possible to introduce the space-time metric describing the gravitational interaction by means of a standard gauge principle. Another main result within our model of even higher order derivatives is that for slowly varying and smooth fields the higher order derivatives either lead to runaway solutions or induces a *zitterbewegung*. We confront this higher order scheme with experimental data.

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I. INTRODUCTION

The most basic approach to the mathematical description of nature is provided by Newton axioms [1]. The most important of them state that (i) there are inertial systems, (ii) the acceleration of a body with respect to an inertial system is given by

$$m\ddot{\boldsymbol{x}} = \boldsymbol{F}(\boldsymbol{x}, \dot{\boldsymbol{x}}, t), \tag{1}$$

where m is the (inertial) mass of the body, $\ddot{\boldsymbol{x}}$ its acceleration, and $\boldsymbol{F}$ the force acting on the body which may depend on the position and the velocity of the particle, and that (iii) *actio* equals *reactio* [2].

Leaving aside the fundamental and still unresolved problem of how to really define an inertial system (see, e.g., Ref. [3]), the second Newton axiom is a tool to explore the forces and, thus, to measure the fields acting on particles. The electromagnetic field, for example, can be explored and measured through the observation of the acceleration of charged particles under different conditions (different initial conditions, different charges, etc.). Thus, by means of the dynamics of the form (1) the electromagnetic interaction and, in principle, all other interactions are defined. In many cases one uses quantum equations of motion like the Schrödinger or Dirac equation which, via the path integral approach or the Ehrenfest theorem, for example, are also based on (1). All these considerations extend to relativistic equations of motion. And all this is also the basis of the introduction of interactions through a gauge formalism which is basic for the theoretical description of the standard model.

From this observation it is clear that what one defines as interaction or as the corresponding force field depends on the basic structure of the equation of motion. If, for example, the equation of motion is of higher than second order in the time derivative, then interactions could be introduced in a different way and, thus, can have a different structure as we will show below. As a consequence, it is very important (i) to explore the structure of interactions and in particular of the gravitational interaction in the case of higher order fundamental equations of motion, and (ii) to develop a theoretical framework for the description of experiments testing the order of the equations of motion.

Equations of motion of higher order are related to a different initial value problem: one needs more initial values beyond the initial position and initial velocity. In physical terms this means that with respect to the time coordinate the equation of motion is more nonlocal than the ordinary second order equations. The extreme case that (formally) an infinite number of initial values are needed is related to equations of motion with memory as, e.g., generalized Langevin equations where the force equation possesses an additional term of the form $\int_0^t C(t - t')\dot{x}(t')dt'$, see, e.g., Ref. [4].

Also within quantum gravity scenarios it might be expected that the effective equation of point particles may contain small higher order time derivatives. In fact, since space-time fluctuations in the sense of, e.g., fluctuations of the metrical tensor, yield Langevin-like equations of motion also quantum gravity scenarios naturally are expected to lead to effective higher order equations of motion where the additional higher derivatives probably scale with, e.g., the Planck length.

Higher order derivatives in equations of motion occur in effective equations which take backreaction into account. One example for that is the Abraham-Lorentz equation for charged particles taking into account the electromagnetic

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waves radiated away [5] or the radiation damping equation in gravity where the emitted gravitational waves are taken into account, [6]. In the electromagnetic case the leading term is a third time derivative which leads to unphysical runaway solutions which still is an unresolved problem. However, these equations discussed in relation with radiation damping are no fundamental equations, they are effective equations emerging from the fact that one no longer regards the particles as test particles. Here we are only interested in the fundamental equations of motion.

Since spin is some element of nonlocality it is not astonishing that also the dynamics of spinning particles effectively can be described by means of a higher order theory [7].

Pais and Uhlenbeck [8] studied higher order mechanical models as a toy model for discussing properties of higher order field theories where higher order derivatives came in naturally in noncommutative models [9] or are introduced in order to eliminate divergences; see, e.g., Ref. [10]. Recently, it has been shown in Ref. [11] that the energy of the Pais-Uhlenbeck oscillator is bounded from below, is unitary and is free of ghosts; see also Ref. [12] for further studies in this direction.

In the following we will consider *fundamental* equations of motion of even higher order which can be derived from a variational principle. In order to be able to confront these modified equations of motion with experimental data, one first has to investigate the structure of interactions. This will be done by using a gauge principle for a second order Lagrange formalism (which in principle can be applied to Lagrangians of all orders). The solutions of the equations of motion coupled to these generalized gauge fields give—according to the chosen sign of the additional even higher derivative term—only runaway solutions or a *zitterbewegung* showing that in the latter case the standard equations of motion are rather robust against addition of higher order terms. We also discuss the experimental possibilities to search for effects related to such higher order time derivatives. As an interesting by-product of the corresponding higher order gauge formalism we obtain the standard space-time metric as an ordinary gauge field.

II. LAGRANGE FORMALISM

First we will use the Lagrange formalism in order to introduce interactions into a theory containing higher order derivatives. We introduce interactions by means of a gauge principle. We set up our notation by repeating shortly the standard first order formalism and then apply the gauge principle to a second order Lagrange formalism.

A. First order formalism

A Lagrange function of first order,

$$L = L_0(t, \boldsymbol{x}, \dot{\boldsymbol{x}}), \tag{2}$$

yields the Euler-Lagrange equation of motion,

$$0 = \boldsymbol{\nabla} L - \frac{d}{dt}\boldsymbol{\nabla}_{\dot{x}} L. \tag{3}$$

We obtain the same equation of motion from two Lagrange functions if they differ by a total time derivative only,

$$L(t,\boldsymbol{x},\dot{\boldsymbol{x}}) \to L'(t,\boldsymbol{x},\dot{\boldsymbol{x}}) = L_0(t,\boldsymbol{x},\dot{\boldsymbol{x}}) + \frac{d}{dt} f(t,\boldsymbol{x}), \tag{4}$$

where the function f is allowed to depend on t and $\boldsymbol{x}$ only. We can expand the total time derivative,

$$L'(t,\boldsymbol{x},\dot{\boldsymbol{x}}) = L_0(t,\boldsymbol{x},\dot{\boldsymbol{x}}) + \partial_t f(t,\boldsymbol{x}) + \dot{\boldsymbol{x}}\cdot\boldsymbol{\nabla} f(t,\boldsymbol{x}). \tag{5}$$

We are now invoking the gauge principle which prescribes the replacement,

$$\partial_t f(t,\boldsymbol{x}) \to -q\phi(t,\boldsymbol{x}), \qquad \boldsymbol{\nabla} f \to q\boldsymbol{A}(t,\boldsymbol{x}), \tag{6}$$

where q is the coupling parameter (charge). The new functions $\phi(t,\boldsymbol{x})$ and $\boldsymbol{A}(t,\boldsymbol{x})$ are the scalar and vector potential of the Maxwell theory. Then the Lagrangian coupled to these potentials reads

$$L'(t,\boldsymbol{x},\dot{\boldsymbol{x}}) = L_0(t,\boldsymbol{x},\dot{\boldsymbol{x}}) - q\phi(t,\boldsymbol{x}) + q\dot{\boldsymbol{x}}\cdot\boldsymbol{A}(t,\boldsymbol{x}). \tag{7}$$

With the choice $L_0(t,\boldsymbol{x},\dot{\boldsymbol{x}}) = \frac{1}{2}m\dot{\boldsymbol{x}}^2$ we obtain the standard Lorentz force equation of a charged particle moving in an electromagnetic field.

Now we generalize this approach to higher order Lagrange functions.

B. Second order formalism

In the second order formalism we consider Lagrange functions of the form

$$L = L_0(t,\boldsymbol{x},\dot{\boldsymbol{x}},\ddot{\boldsymbol{x}}) \tag{8}$$

from which we obtain the fourth order equation of motion,

$$\begin{aligned} 0 &= \boldsymbol{\nabla} L - \frac{d}{dt}\boldsymbol{\nabla}_{\dot{x}} L + \frac{d^2}{dt^2}\boldsymbol{\nabla}_{\ddot{x}} L \\ &= \boldsymbol{\nabla} L - \frac{d}{dt}\left(\boldsymbol{\nabla}_{\dot{x}} L - \frac{d}{dt}\boldsymbol{\nabla}_{\ddot{x}} L\right), \end{aligned} \tag{9}$$

where $\boldsymbol{\nabla}_a$ denotes the gradient with respect to the variable a. Also in this case we obtain the same equation of motion from another Lagrange function if it differs from the original one by a total time derivative of a function f only. This function, however, now may depend on the velocities $\dot{\boldsymbol{x}}$:

$$L(t,\boldsymbol{x},\dot{\boldsymbol{x}},\ddot{\boldsymbol{x}}) \to L'(t,\boldsymbol{x},\dot{\boldsymbol{x}},\ddot{\boldsymbol{x}}) = L_0(t,\boldsymbol{x},\dot{\boldsymbol{x}},\ddot{\boldsymbol{x}}) + \frac{d}{dt} f(t,\boldsymbol{x},\dot{\boldsymbol{x}}). \tag{10}$$

The expansion of the total time derivative gives

$$L'(t, \boldsymbol{x}, \dot{\boldsymbol{x}}, \ddot{\boldsymbol{x}}) = L_0(t, \boldsymbol{x}, \dot{\boldsymbol{x}}, \ddot{\boldsymbol{x}}) + \partial_t f(t, \boldsymbol{x}, \dot{\boldsymbol{x}}) + \dot{\boldsymbol{x}} \cdot \boldsymbol{\nabla} f(t, \boldsymbol{x}, \dot{\boldsymbol{x}}) + \ddot{\boldsymbol{x}} \cdot \boldsymbol{\nabla}_{\dot{x}} f(t, \boldsymbol{x}, \dot{\boldsymbol{x}}). \tag{11}$$

The question now is how to employ the gauge principle. If we replace, e.g., $\partial_t f(t, \boldsymbol{x}, \dot{\boldsymbol{x}})$ by a function $\phi(t, \boldsymbol{x}, \dot{\boldsymbol{x}})$ then this function cannot describe an external field since it would depend on the velocity. A given external field should be given *per se* and should not depend on the state of motion of a particle. The properties of the external field should be independent of whether the particle is moving through it or not.

One way to introduce functions depending on time and position only is to assume that the function $f(t, \boldsymbol{x}, \dot{\boldsymbol{x}})$ is polynomial in the velocity. That means for some finite number N,

$$f(t, \boldsymbol{x}, \dot{\boldsymbol{x}}) = \sum_{n=0}^{N} f_{i_1 \dots i_n}(t, \boldsymbol{x}) \dot{x}^{i_1} \cdots \dot{x}^{i_n}. \tag{12}$$

In this case we can regard the functions $f_{i_1 \dots i_n}(t, \boldsymbol{x}) = f_{(i_1 \dots i_n)}(t, \boldsymbol{x})$ as gauge functions of an externally given interaction. For this setting we obtain for the new Lagrange function the same equations of motion

$$L'(t, \boldsymbol{x}, \dot{\boldsymbol{x}}, \ddot{\boldsymbol{x}}) = L_0(t, \boldsymbol{x}, \dot{\boldsymbol{x}}, \ddot{\boldsymbol{x}}) + \sum_{n=0}^{N} \partial_t f_{i_1 \dots i_n}(t, \boldsymbol{x}) \dot{x}^{i_1} \cdots \dot{x}^{i_n} + \sum_{n=0}^{N} \partial_i f_{i_1 \dots i_n}(t, \boldsymbol{x}) \dot{x}^i \dot{x}^{i_1} \cdots \dot{x}^{i_n} + \sum_{n=1}^{N} n f_{i_1 \dots i_n}(t, \boldsymbol{x}) \ddot{x}^{i_1} \dot{x}^{i_2} \cdots \dot{x}^{i_n}. \tag{13}$$

The gauge principle now allows one to replace these gauge functions by the gauge fields,

$$\partial_t f_{i_1 \dots i_n}(t, \boldsymbol{x}) \to -q_n \phi_{i_1 \dots i_n}(t, \boldsymbol{x}), \tag{14}$$

$$\partial_i f_{i_1 \dots i_n}(t, \boldsymbol{x}) \to q_n A_{ii_1 \dots i_n}(t, \boldsymbol{x}), \tag{15}$$

$$n f_{i_1 \dots i_n}(t, \boldsymbol{x}) \to q_n \psi_{i_1 \dots i_n}(t, \boldsymbol{x}), \tag{16}$$

where the q_n are the coupling parameters to these nth rank potentials. The symmetries of these gauge fields are

$$\phi_{i_1 \dots i_n}(t, \boldsymbol{x}) = \phi_{(i_1 \dots i_n)}(t, \boldsymbol{x}), \tag{17}$$

$$A_{ii_1 \dots i_n}(t, \boldsymbol{x}) = A_{(ii_1 \dots i_n)}(t, \boldsymbol{x}), \tag{18}$$

$$\psi_{i_1 \dots i_n}(t, \boldsymbol{x}) = \psi_{(i_1 \dots i_n)}(t, \boldsymbol{x}), \tag{19}$$

that is, all gauge fields are totally symmetric. These fields transform under the generalized gauge transformations as

$$\begin{aligned} q_n \phi_{i_1 \dots i_n}(t, \boldsymbol{x}) &\to q_n \phi'_{i_1 \dots i_n}(t, \boldsymbol{x}) \\ &= q_n \phi_{i_1 \dots i_n}(t, \boldsymbol{x}) - \partial_t f_{i_1 \dots i_n}(t, \boldsymbol{x}), \\ q_n A_{ii_1 \dots i_n}(t, \boldsymbol{x}) &\to q_n A'_{ii_1 \dots i_n}(t, \boldsymbol{x}) \\ &= q_n A_{ii_1 \dots i_n}(t, \boldsymbol{x}) + \partial_{(i} f_{i_1 \dots i_n)}(t, \boldsymbol{x}), \\ q_n \psi_{i_1 \dots i_n}(t, \boldsymbol{x}) &\to q_n \psi'_{i_1 \dots i_n}(t, \boldsymbol{x}) \\ &= q_n \psi_{i_1 \dots i_n}(t, \boldsymbol{x}) + n f_{i_1 \dots i_n}(t, \boldsymbol{x}). \end{aligned} \tag{20}$$

For $n = 0$ they reduce to the ordinary Maxwell gauge transformations.

As a consequence we obtain the Lagrangian of a particle coupled to the new interaction gauge fields:

$$L'(t, \boldsymbol{x}, \dot{\boldsymbol{x}}, \ddot{\boldsymbol{x}}) = L_0(t, \boldsymbol{x}, \dot{\boldsymbol{x}}, \ddot{\boldsymbol{x}}) - \sum_{n=0}^{N} q_n \phi_{i_1 \dots i_n}(t, \boldsymbol{x}) \dot{x}^{i_1} \cdots \dot{x}^{i_n} + \sum_{n=0}^{N} q_n A_{ii_1 \dots i_n}(t, \boldsymbol{x}) \dot{x}^i \dot{x}^{i_1} \cdots \dot{x}^{i_n} + \sum_{n=1}^{N} q_n \psi_{i_1 \dots i_n}(t, \boldsymbol{x}) \ddot{x}^{i_1} \dot{x}^{i_2} \cdots \dot{x}^{i_n}. \tag{21}$$

The last term can be rewritten as, modulo a total time derivative,

$$-\sum_{n=1}^{N} \frac{1}{n} q_n \frac{\partial}{\partial t} \psi_{i_1 \dots i_n}(t, \boldsymbol{x}) \dot{x}^{i_1} \dot{x}^{i_2} \cdots \dot{x}^{i_n} - \sum_{n=1}^{N} \frac{1}{n} q_n \partial_i \psi_{i_1 \dots i_n}(t, \boldsymbol{x}) \dot{x}^i \dot{x}^{i_1} \dot{x}^{i_2} \cdots \dot{x}^{i_n}, \tag{22}$$

so that we effectively have the Lagrangian

$$L'(t, \boldsymbol{x}, \dot{\boldsymbol{x}}, \ddot{\boldsymbol{x}}) = L_0(t, \boldsymbol{x}, \dot{\boldsymbol{x}}, \ddot{\boldsymbol{x}}) - \sum_{n=0}^{N} q_n \tilde{\phi}_{i_1 \dots i_n}(t, \boldsymbol{x}) \dot{x}^{i_1} \dot{x}^{i_2} \cdots \dot{x}^{i_n} + \sum_{n=0}^{N} q_n \tilde{A}_{ii_1 \dots i_n}(t, \boldsymbol{x}) \dot{x}^i \dot{x}^{i_1} \dot{x}^{i_2} \cdots \dot{x}^{i_n}, \tag{23}$$

where

$$\tilde{A}_{ii_1 \dots i_n}(t, \boldsymbol{x}) := A_{ii_1 \dots i_n}(t, \boldsymbol{x}) - \frac{1}{n} \partial_{(i} \psi_{i_1 \dots i_n)}(t, \boldsymbol{x}), \tag{24}$$

$$\tilde{\phi}_{i_1 \dots i_n}(t, \boldsymbol{x}) := \phi_{i_1 \dots i_n}(t, \boldsymbol{x}) + \frac{1}{n} \frac{\partial}{\partial t} \psi_{i_1 \dots i_n}(t, \boldsymbol{x}). \tag{25}$$

The equations of motion read

$$0 = \partial_j L_0 - \frac{d}{dt}\frac{\partial L_0}{\partial \dot{x}^j} + \frac{d^2}{dt^2}\frac{\partial L_0}{\partial \ddot{x}^j} - \sum_{n=0}^{N} q_n(\partial_j \tilde{\phi}_{i_1\ldots i_n}(t,\boldsymbol{x}) - n\partial_{i_1}\tilde{\phi}_{ji_2\ldots i_n}(t,\boldsymbol{x}) - (\partial_j \tilde{A}_{ii_1\ldots i_n}(t,\boldsymbol{x}) - (n+1)\partial_i \tilde{A}_{ji_1\ldots i_n}(t,\boldsymbol{x}))\dot{x}^i)\dot{x}^{i_1}\ldots\dot{x}^{i_n}$$
$$+ \sum_{n=0}^{N} q_n n \frac{\partial}{\partial t}\tilde{\phi}_{ji_2\ldots i_n}(t,\boldsymbol{x})\dot{x}^{i_2}\ldots\dot{x}^{i_n} - \sum_{n=0}^{N}(n+1)q_n\frac{\partial}{\partial t}\tilde{A}_{ji_1\ldots i_n}(t,\boldsymbol{x})\dot{x}^{i_1}\ldots\dot{x}^{i_n}$$
$$+ \sum_{n=0}^{N} q_n n(n-1)\tilde{\phi}_{ji_2\ldots i_n}(t,\boldsymbol{x})\ddot{x}^{i_2}\dot{x}^{i_3}\ldots\dot{x}^{i_n} - \sum_{n=0}^{N}(n+1)nq_n\tilde{A}_{ji_1\ldots i_n}(t,\boldsymbol{x})\ddot{x}^{i_1}\dot{x}^{i_2}\ldots\dot{x}^{i_n}. \qquad (26)$$

The additional gauge interaction adds a time derivative of second order. The principal part of the differential equation remains unaffected.

Below we use the most simple second order Lagrangian without interaction

$$L_0(t,\boldsymbol{x},\dot{\boldsymbol{x}},\ddot{\boldsymbol{x}}) = \frac{m}{2}\dot{\boldsymbol{x}}^2 - \frac{\epsilon}{2}\ddot{\boldsymbol{x}}^2, \qquad (27)$$

which leads to a fourth order equation of motion. The parameter ϵ has the dimension mass $\times$ time^2. If one assumes that this additional term has emerged from influences of quantum gravity then it might be natural to identify $\epsilon \sim M_{\rm Pl}T_{\rm Pl}^2 \sim 10^{-95}$ kg s^2 which is extremely small (the product of the mass and the square of the Compton time gives for an electron $\sim 10^{-71}$ kg s^2, for a proton we get $\sim 10^{-74}$ kg s^2, and for a typical atom $\sim 10^{-77}$ kg s^2).

III. NOETHER THEOREM

Also for higher order Lagrangians conservation laws can be obtained from the variational principle if we allow nonvanishing variations at the initial and final points. The variations are as usual,

$$\bar{t} = t + \tau(t), \qquad (28)$$

$$\bar{\boldsymbol{x}}(\bar{t}) = \boldsymbol{x}(t) + \Delta\boldsymbol{x}(t). \qquad (29)$$

One should bear in mind that here the $\boldsymbol{x}$ need not be variables of the configuration space. For these general variations, the variation of the action is

$$\delta S = \bar{S} - S = \int_{\bar{t}_1}^{\bar{t}_2} L(\bar{\boldsymbol{x}}(\bar{t}),\dot{\bar{\boldsymbol{x}}}(\bar{t}),\ddot{\bar{\boldsymbol{x}}}(\bar{t}),\bar{t})d\bar{t} - \int_{t_1}^{t_2} L(\boldsymbol{x}(t),\dot{\boldsymbol{x}}(t),\ddot{\boldsymbol{x}}(t),t)dt. \qquad (30)$$

Proceeding as in the first order case described in textbooks we arrive at

$$\delta S = \int_{t_1}^{t_2}\left(\boldsymbol{\nabla}L - \frac{d}{dt}\boldsymbol{\nabla}_{\dot{x}}L + \frac{d^2}{dt^2}\boldsymbol{\nabla}_{\ddot{x}}L\right)\delta\boldsymbol{x}dt + \left(\boldsymbol{\nabla}_{\ddot{x}}L\cdot\frac{d}{dt}(\Delta\boldsymbol{x} - \tau\dot{\boldsymbol{x}})\right. + \left.\left(\boldsymbol{\nabla}_{\dot{x}}L - \frac{d}{dt}\boldsymbol{\nabla}_{\ddot{x}}L\right)\cdot(\Delta\boldsymbol{x} - \tau\dot{\boldsymbol{x}}) + \tau L\right)_{t_1}^{t_2}. \qquad (31)$$

If the equations of motion are fulfilled, and if the action is invariant under the variations (27) and (28), then we obtain the conserved quantity

$$\boldsymbol{p}_2\cdot\frac{d}{dt}(\Delta\boldsymbol{x} - \tau\dot{\boldsymbol{x}}) + (\boldsymbol{p}_1 - \dot{\boldsymbol{p}}_2)\cdot(\Delta\boldsymbol{x} - \tau\dot{\boldsymbol{x}}) + \tau L = \text{const}, \qquad (32)$$

where we defined the momenta

$$\boldsymbol{p}_1 = \boldsymbol{\nabla}_{\dot{x}}L \quad \text{and} \quad \boldsymbol{p}_2 = \boldsymbol{\nabla}_{\ddot{x}}L. \qquad (33)$$

If the action does not vanish but, instead, changes with a total time derivative of a function $F(\boldsymbol{x}(t),\dot{\boldsymbol{x}}(t),t)$, then the equations of motion in terms of the Euler-Lagrange equations do not change and we have a modified conserved quantity,

$$\boldsymbol{p}_2\cdot\frac{d}{dt}(\Delta\boldsymbol{x} - \tau\dot{\boldsymbol{x}}) + (\boldsymbol{p}_1 - \dot{\boldsymbol{p}}_2)\cdot(\Delta\boldsymbol{x} - \tau\dot{\boldsymbol{x}}) + \tau L - F = \text{const}. \qquad (34)$$

From this general Noether theorem we derive the following conserved quantities.

A. Momentum conservation

At first we consider the transformations $\tau(t) = 0$ and $\Delta\boldsymbol{x}(t) = \text{const}$. We obtain the conserved momentum

$$\boldsymbol{P} = \boldsymbol{p}_1 - \dot{\boldsymbol{p}}_2 = \text{const}. \qquad (35)$$

For the Lagrangian (26) we get

$$\boldsymbol{P} = m\dot{\boldsymbol{x}} + \epsilon\dddot{\boldsymbol{x}} = \text{const}. \qquad (36)$$

This may also directly be inferred from the Euler-Lagrange equations of motion (9).

B. Energy conservation

Next we consider the transformations $\tau(t) = \tau_0$ and $\Delta\boldsymbol{x}(t) = 0$. The corresponding conserved energy is

$$E = \boldsymbol{p}\cdot\ddot{\boldsymbol{x}} + (\boldsymbol{p}_1 - \dot{\boldsymbol{p}}_2)\cdot\dot{\boldsymbol{x}} - L = \boldsymbol{p}_2\cdot\ddot{\boldsymbol{x}} + \boldsymbol{P}\cdot\dot{\boldsymbol{x}} - L = \text{const}. \qquad (37)$$

For the Lagrangian (26) we obtain

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}\epsilon(2\dddot{x}\cdot\dot{x} - \ddot{x}^2). \tag{38}$$

C. Angular momentum conservation

In the next example we assume that the action is invariant under the transformations $\tau(t) = 0$ and $\Delta\boldsymbol{x} = \Delta\boldsymbol{\varphi}\times\boldsymbol{x}$. This corresponds to a rotation and the corresponding conserved angular momentum is

$$\boldsymbol{L} = \boldsymbol{x}\times(\boldsymbol{p}_1 - \dot{\boldsymbol{p}}_2) + \dot{\boldsymbol{x}}\times\boldsymbol{p}_2 = \boldsymbol{x}\times\boldsymbol{P} + \dot{\boldsymbol{x}}\times\boldsymbol{p}_2 = \text{const.} \tag{39}$$

For the Lagrangian (26) we obtain the conserved angular momentum

$$\boldsymbol{L} = \boldsymbol{x}\times(m\dot{\boldsymbol{x}}) + \epsilon(\boldsymbol{x}\times\dddot{\boldsymbol{x}} - \dot{\boldsymbol{x}}\times\ddot{\boldsymbol{x}}) = \boldsymbol{x}\times\boldsymbol{P} - \epsilon\dot{\boldsymbol{x}}\times\ddot{\boldsymbol{x}} = \text{const.} \tag{40}$$

D. Proper Galileo transformation

At last we consider the Galileo transformations $\tau(t) = 0$ and $\Delta\boldsymbol{x} = \Delta\boldsymbol{v}t$ for the Lagrangian (26), where $\Delta\boldsymbol{v}$ is assumed to be very small. The term $\frac{1}{2}m\dot{\boldsymbol{x}}^2$ changes by a total differential of $m\boldsymbol{x}\cdot\Delta\boldsymbol{v}$ so that we obtain the conserved quantity

$$\boldsymbol{C} = \boldsymbol{p}_2 + (\boldsymbol{p} - \dot{\boldsymbol{p}}_2)t - m\boldsymbol{x} = \text{const} \tag{41}$$

from which we deduce a uniform motion

$$\boldsymbol{x} = \frac{\boldsymbol{p}_2}{m} + \frac{\boldsymbol{p} - \dot{\boldsymbol{p}}_2}{m}t + \boldsymbol{x}_0. \tag{42}$$

IV. THE MOST SIMPLE GAUGE MODEL, $N = 0$

A. Equation of motion

The most simple case with nontrivial dynamics is given for the special case that f in (12) is a function of t and $\boldsymbol{x}$ only, which means $N = 0$. Then we have

$$L'(t,\boldsymbol{x},\dot{\boldsymbol{x}},\ddot{\boldsymbol{x}}) = L_0(t,\boldsymbol{x},\dot{\boldsymbol{x}},\ddot{\boldsymbol{x}}) - q\phi(t,\boldsymbol{x}) + q\dot{x}^iA_i(t,\boldsymbol{x}). \tag{43}$$

For L_0 from (26) the equations of motion read

$$\epsilon\ddddot{\boldsymbol{x}} + m\ddot{\boldsymbol{x}} = q\boldsymbol{E}(t,\boldsymbol{x}) + q\dot{\boldsymbol{x}}\times\boldsymbol{B}(t,\boldsymbol{x}), \tag{44}$$

where $\boldsymbol{E}$ and $\boldsymbol{B}$ are the electric and magnetic field derived as usual from the scalar and vector potentials ϕ and $\boldsymbol{A}$. This equation of motion is the standard one with a small additional fourth order term.

For a fist discussion we may simplify further by taking a vanishing magnetic field, $\boldsymbol{B} = 0$ and a constant electric field $\boldsymbol{E}(\boldsymbol{x}) = \boldsymbol{E}_0 = \text{const}$,

$$\epsilon\ddddot{\boldsymbol{x}} + m\ddot{\boldsymbol{x}} = q\boldsymbol{E}_0. \tag{45}$$

This is the equation we would like to solve.

B. Particle motion

The first two time integrations are easily performed and give

$$\epsilon\ddot{\boldsymbol{x}} + m\boldsymbol{x} = \frac{q}{2}\boldsymbol{E}_0(t - t_0)^2 + \boldsymbol{a}(t - t_0) + \boldsymbol{b}, \tag{46}$$

where $\boldsymbol{a}$ and $\boldsymbol{b}$ are integration constants. Next we introduce a new function $\bar{\boldsymbol{x}}$ through $\boldsymbol{x}_0 = \boldsymbol{x} + \epsilon\bar{\boldsymbol{x}}$ with $\boldsymbol{x}_0 = \frac{q}{2m}\boldsymbol{E}_0(t - t_0)^2 + \frac{1}{m}\boldsymbol{a}(t - t_0) + \frac{1}{m}\boldsymbol{b}$ which represents the solution of the corresponding equation of motion without the fourth order term. Here we assume that ϵ is very small, since no deviation from Newton's second law has been observed, so far. The differential equation for $\bar{\boldsymbol{x}}$ then reads

$$\ddot{\bar{\boldsymbol{x}}} + \frac{m}{\epsilon}\bar{\boldsymbol{x}} = -\frac{q}{m\epsilon}\boldsymbol{E}_0. \tag{47}$$

A further substitution $\hat{\boldsymbol{x}} = \bar{\boldsymbol{x}} + \frac{\epsilon}{m}\frac{q}{m\epsilon}\boldsymbol{E}_0$ yields the equation for a particle in a harmonic potential,

$$\ddot{\hat{\boldsymbol{x}}} + \frac{m}{\epsilon}\hat{\boldsymbol{x}} = 0, \tag{48}$$

which, according to the sign of $\frac{m}{\epsilon}$, possesses the solution

$$\hat{\boldsymbol{x}} = \boldsymbol{A}\cos(\omega t) + \boldsymbol{B}\sin(\omega t) \quad \text{for } \frac{m}{\epsilon} > 0, \tag{49}$$

$$\hat{\boldsymbol{x}} = \boldsymbol{A}_1\cosh(\omega t) + \boldsymbol{A}_2\sinh(\omega t) \quad \text{for } \frac{m}{\epsilon} < 0, \tag{50}$$

for some amplitudes $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{A}_1$, and $\boldsymbol{A}_2$ and with $\omega = \sqrt{m/\epsilon}$.

As a consequence, for $m/\epsilon > 0$ we arrived at the solution

$$\boldsymbol{x}(t) = \frac{q}{2m}\boldsymbol{E}_0(t - t_0)^2 + \frac{1}{m}\boldsymbol{a}(t - t_0) + \frac{1}{m}\boldsymbol{b} + \epsilon\boldsymbol{A}\cos(\omega t) + \epsilon\boldsymbol{B}\sin(\omega t) - \frac{\epsilon}{m}\frac{q}{m}\boldsymbol{E}_0, \tag{51}$$

that turns to the standard solution in the limit $\epsilon \to 0$. For $m/\epsilon < 0$ the solutions become exponentially growing runaway solutions which are in contradiction to physical observations. Accordingly, we have to choose $m/\epsilon > 0$. Since, by convention, $m > 0$ we then have $\epsilon \geq 0$. (In principle it is possible to choose the inertial mass to be negative by definition. This then may induce modified sign conventions for coupling constants.)

The velocity is

$$\dot{\boldsymbol{x}}(t) = \frac{q}{m}\boldsymbol{E}_0(t - t_0) + \frac{1}{m}\boldsymbol{a} - \sqrt{m\epsilon}\boldsymbol{A}\sin(\omega t) + \sqrt{m\epsilon}\boldsymbol{B}\cos(\omega t), \tag{52}$$

which also approaches the standard expression for $\epsilon \to 0$. The acceleration

$$\ddot{\boldsymbol{x}}(t) = \frac{q}{m}\boldsymbol{E}_0 - m\boldsymbol{A}\cos(\omega t) - m\boldsymbol{B}\sin(\omega t), \quad (53)$$

however, has a large fluctuating term of order 0 which does not disappear for $\epsilon \longrightarrow 0$.

For very small positive ϵ the additional term in the path (50) is a very small but fast oscillating *zitterbewegung*. One may turn the above result into a positive statement: At least within a Lagrangian approach and assuming that higher order terms are small, the paths originating from a corresponding higher order modification are rather inert against these modifications. The mean orbits behave like orbits given by second order equations of motion. We will present some ways to experimentally search for these fundamental oscillations in Sec. VI.

This result may be extended to the case of slowly varying arbitrary electromagnetic fields. The equation of motion for an arbitrary electromagnetic field (43) may be attacked through the substitution $\boldsymbol{x} = \epsilon\bar{\boldsymbol{x}} + \boldsymbol{x}_0$, where $\boldsymbol{x}_0$ is assumed to solve the equation of motion without the fourth order term. If we assume that the force $\boldsymbol{F}(\boldsymbol{x}) = q\boldsymbol{E}(\boldsymbol{x}) + q\boldsymbol{v}\times\boldsymbol{B}(\boldsymbol{x})$ is very smooth and that the deviation $\epsilon\bar{\boldsymbol{x}}$ is very small (that is, if $\bar{\boldsymbol{x}}\cdot\boldsymbol{\nabla}\boldsymbol{F} \ll m\ddot{\boldsymbol{x}}_0$ and can be neglected), then we obtain

$$\ddddot{\boldsymbol{x}}_0 + \epsilon\ddddot{\bar{\boldsymbol{x}}} + m\ddot{\bar{\boldsymbol{x}}} = 0. \quad (54)$$

This can be integrated twice,

$$\ddot{\boldsymbol{x}}_0 + \epsilon\ddot{\bar{\boldsymbol{x}}} + m\bar{\boldsymbol{x}} = \boldsymbol{a}t + \boldsymbol{b}, \quad (55)$$

where $\boldsymbol{a}$ and $\boldsymbol{b}$ are two integration constants. Inserting the equation for $\ddot{\boldsymbol{x}}_0$ yields

$$\ddot{\bar{\boldsymbol{x}}} + \frac{m}{\epsilon}\bar{\boldsymbol{x}} = -\frac{1}{m\epsilon}\boldsymbol{F}(\boldsymbol{x}_0) + \frac{1}{\epsilon}\boldsymbol{a}t + \frac{1}{\epsilon}\boldsymbol{b}. \quad (56)$$

With a new variable $\hat{\boldsymbol{x}} = \bar{\boldsymbol{x}} - \frac{1}{m}\boldsymbol{a}t + \frac{1}{m}\boldsymbol{b} - \frac{1}{m^2}\boldsymbol{F}(\boldsymbol{x}_0)$ we have

$$\ddot{\hat{\boldsymbol{x}}} + \frac{m}{\epsilon}\hat{\boldsymbol{x}} = 0. \quad (57)$$

Then again $\hat{\boldsymbol{x}}$ is a fast oscillating term which adds to the standard solution. The total solution then is

$$\boldsymbol{x}(t) = \boldsymbol{x}_0(t) + \epsilon\left(\hat{\boldsymbol{x}}(t) + \frac{1}{m}\boldsymbol{a}t - \frac{1}{m}\boldsymbol{b} + \frac{1}{m^2}\boldsymbol{F}(\boldsymbol{x}_0(t))\right). \quad (58)$$

This solution consists of the standard solution $\boldsymbol{x}_0(t)$ which is the main motion and additional small terms: a small fast oscillating term, a kind of *zitterbewegung* and a small position-dependent displacement, a small offset term, and a small linearly growing term. In order to be compatible with observations we choose $\boldsymbol{a} = 0$ and $\boldsymbol{b} = 0$ so that the last two mentioned terms disappear.

C. Conserved energy

The conserved energy in this most simple model reads

$$E = \frac{m}{2}\dot{\boldsymbol{x}}^2 + \frac{\epsilon}{2}(2\dddot{\boldsymbol{x}}\cdot\dot{\boldsymbol{x}} - \ddot{\boldsymbol{x}}^2) + q\phi, \quad (59)$$

which also can be obtained by multiplying the equation of motion by $\dot{\boldsymbol{x}}$ and partial integration. Insertion of the solution $\boldsymbol{x}(t)$ verifies the constancy of this expression. The energy is not definite.

V. THE NEXT GAUGE MODEL, $N = 1$

A. The equation of motion

Now we would now like to explore the model where f is a function which is polynomial of first order in the velocities, that is, $f(t, \boldsymbol{x}, \dot{\boldsymbol{x}}) = f^{(0)}(t, \boldsymbol{x}) + f^{(1)}_i(t, \boldsymbol{x})\dot{x}^i$. Then the corresponding gauged Lagrange function reads

$$L(t, \boldsymbol{x}, \dot{\boldsymbol{x}}, \ddot{\boldsymbol{x}}) = L_0(t, \boldsymbol{x}, \dot{\boldsymbol{x}}, \ddot{\boldsymbol{x}}) - q_0\phi + q_0\dot{x}^iA_i - q_1\dot{x}^i\phi_i + q_1\dot{x}^i\dot{x}^jA_{ij} + q_1\ddot{x}^i\psi_i, \quad (60)$$

where all functions depend on t and $\boldsymbol{x}$. Here q_0 is the coupling related to $f^{(0)}$, and q_1 the coupling related to f_i. Beside the usual scalar and vector potential, ϕ and A_i we have in addition two vector potentials ϕ_i and ψ_i and a tensorial potential A_{ij}. The gauge transformations are given by

$$q_0\phi' = q_0\phi - \partial_t f^{(0)}, \qquad q_0A'_i = q_0A_i + \partial_i f^{(0)},$$
$$q_1\phi'_i = q_1\phi - \partial_t f^{(1)}_i, \qquad q_1\psi'_i = q_1\psi_i + f^{(1)}_i, \quad (61)$$
$$q_1A'_{ij} = q_1A_{ij} + \partial_{(i}f^{(1)}_{j)}.$$

The corresponding Euler-Lagrange equation is

$$0 = \frac{\partial L_0}{\partial x^i} - \frac{d}{dt}\frac{\partial L_0}{\partial \dot{x}^i} + \frac{d^2}{dt^2}\frac{\partial L_0}{\partial \ddot{x}^i} \quad (62)$$

$$- q_0\partial_i\phi - q_0\partial_tA_i + q_1\partial_t\phi_i + q_1\partial_t^2\psi_i \quad (63)$$

$$+ \dot{x}^j(\partial_i(q_0A_j - q_1\phi_j) - \partial_j(q_0A_i - q_1\phi_i) + 2q_1\partial_t\partial_{[j}\psi_{i]}) \quad (64)$$

$$- 2q_1\left(\ddot{x}^j\tilde{A}_{ij} + \frac{1}{2}(\partial_k\tilde{A}_{ij} + \partial_j\tilde{A}_{ik} - \partial_i\tilde{A}_{jk})\dot{x}^j\dot{x}^k\right) - 2q_1\partial_t\tilde{A}_{ij}\dot{x}^j, \quad (65)$$

where we defined $\tilde{A}_{ij} := A_{ij} - \partial_{(i}\psi_{j)}$. The first line (61) is the (still unspecified) equation of motion of the free particle, the second line (62) describes a force due to two electric fields $\boldsymbol{E}^{(0)} := -\boldsymbol{\nabla}\phi - \partial_t\boldsymbol{A}$ and $\boldsymbol{E}^{(1)} := \partial_t\boldsymbol{\phi} + \partial_t^2\boldsymbol{\psi}$, the third line (63) describes the standard magnetic field $\boldsymbol{B}^{(0)} := \boldsymbol{\nabla}\times\boldsymbol{A}$ as well as an additional magnetic field $\boldsymbol{B}^{(1)} := \boldsymbol{\nabla}\times(-\boldsymbol{\phi} + \partial_t\boldsymbol{\psi})$, hence these last two lines together look like a generalized Lorentz force, where the fields are all gauge invariant. The fourth line (64) resembles the form of covariant derivative with a connection based on a second rank tensor $\tilde{A}_{ij}$.

For the Lagrangian L_0 from (26) we obtain the fourth order equation of motion,

$$\epsilon \dddot{x}^j \delta_{ij} + m g_{ij}(D_{\dot{x}}\dot{x})^j + m\partial_t g_{ij}\dot{x}^j = q_0 E_i^{(0)} + q_1 E_i^{(1)} + (\dot{\boldsymbol{x}} \times (q_0 \boldsymbol{B}^{(0)} + q_1 \boldsymbol{B}^{(1)}))_i, \tag{66}$$

where we introduced an effective 3-metric

$$g_{ij} = \delta_{ij} + 2\frac{q_1}{m}\tilde{A}_{ij}, \tag{67}$$

and where the covariant derivative $D_{\dot{x}}$ is formulated with a Christoffel symbol based on g_{ij}. This metric is invariant under the gauge transformations (60). The second and third term on the left-hand side can be regarded as equation of motion arising from the variation of $\int g_{ij}(t, \boldsymbol{x})\dot{x}^i\dot{x}^j dt$. If in (65) we let $\epsilon \to 0$ then only the fourth order derivative term will vanish; the other terms, in particular those containing the metric, remain.

One should note that by means of our second order gauge principle we were able to introduce the gravitational interaction in terms of a space metric by means of an ordinary gauge field. Together with this metric gauge field we also introduced an interaction with an electromagnetic gauge field. Therefore, by means of *one* gauge procedure we introduced the interaction with a gravitational field as well as with an electromagnetic field. By omitting the $n = 0$ part of this gauge formalism, or equivalently, by setting $q_0 = 0$, we have a combined gauge formalism for the metric and an electromagnetic field. We also can choose $\boldsymbol{E}^{(1)} = 0$ and $\boldsymbol{B}^{(1)} = 0$ while keeping $\tilde{A}_{ij} \neq 0$, $\boldsymbol{E}^{(0)} \neq 0$, and $\boldsymbol{B}^{(0)} \neq 0$. This latter choice introduces the electromagnetic field and the space metric independently.

The structure of the above equation of motion (65) is

$$\epsilon \dddot{\boldsymbol{x}} + m\ddot{\boldsymbol{x}} = \boldsymbol{K}(t, \boldsymbol{x}, \dot{\boldsymbol{x}}), \tag{68}$$

where $\boldsymbol{K}(t, \boldsymbol{x}, \dot{\boldsymbol{x}})$ is a polynomial of order two in the velocities which slightly generalizes (43). As a consequence, for ordinary situations (smooth forces) and small ϵ we again obtain some *zitterbewegung* resulting from the inclusion of higher order derivatives. Again, the main (mean) motion is described by the second order part of the equation of motion.

B. The conserved energy

We assume that all fields do not depend on time and determine the conserved energy. Then again we obtain an expression where the space metric appears at the right place. Applying (36) yields

$$E = \frac{1}{2} m g_{ij}\dot{x}^i\dot{x}^j + q_0\phi + \frac{1}{2}\epsilon(2\dddot{\boldsymbol{x}}\cdot\dot{\boldsymbol{x}} - \ddot{\boldsymbol{x}}^2). \tag{69}$$

As expected, the effective metric (66) enters the conserved energy.

C. Relativistic formulation

It is clear from $\ddot{x}^i\psi_i = -\dot{x}^i\frac{d}{dt}\psi_i = -\dot{x}^i\partial_t\psi_i - \dot{x}^j\dot{x}^j\partial_j\psi_i$ which holds modulo a total time derivative, that

$$-q_1\dot{x}^i\phi_i + q_1\dot{x}^i\dot{x}^j A_{ij} + q_1\ddot{x}^i\psi_i = -q_1\dot{x}^i(\phi_i + \partial_t\psi_i) + q_1\dot{x}^i\dot{x}^j(A_{ij} - \partial_j\psi_i). \tag{70}$$

This explains the definitions of $\boldsymbol{E}^{(1)}$ and $\boldsymbol{B}^{(1)}$. Furthermore, with $x^a = (t, x^i)$, $a = 0, \ldots, 3$, and the 3 + 1 splitting

$$q_0 A_a\dot{x}^a + q_1 A_{ab}\dot{x}^a\dot{x}^b = q_0 A_0 + q_1 A_{00} + (q_0 A_i + 2q_1 A_{i0})\dot{x}^i + q_1 A_{ij}\dot{x}^i\dot{x}^j, \tag{71}$$

we can reproduce, with redefinitions, the gauge ansatz (59) above. Therefore, the gauge field part in the Lagrangian (59) can be written in 4-covariant form:

$$L'(t, \boldsymbol{x}, \dot{\boldsymbol{x}}, \ddot{\boldsymbol{x}}) = L_0(t, \boldsymbol{x}, \dot{\boldsymbol{x}}, \ddot{\boldsymbol{x}}) - q_0 A_a\dot{x}^a + q_1 A_{ab}\dot{x}^a\dot{x}^b. \tag{72}$$

It is the last term quadratic in the velocities which, when interpreted as gauge field, requires the second order Lagrangian.

VI. COMPARISON WITH EXPERIMENTS

We would like to describe possible experiments which may be sensitive to the higher order modifications in the equations of motion. We take situations where a charged particle is placed in a constant electric field, e.g., within a capacitor. In standard theory the charged particle will be accelerated and the final velocity depends on the voltage only, not on the spacing between the plates of the capacitor. This will be different in our model. To make the situation as simple as possible we consider a one-dimensional problem and take as an initial condition that the particle is at absolute rest at $t_0 = 0$.

We take our general solution (50) (in one dimension and with $t_0 = 0$)

$$x(t) = \frac{q}{2m}E_0 t^2 + \frac{1}{m}at + \frac{1}{m}b + \epsilon A\cos(\omega t) + \epsilon B\sin(\omega t) - \frac{\epsilon}{m}\frac{q}{m}E_0, \tag{73}$$

$$\dot{x}(t) = \frac{q}{m}E_0 t + \frac{1}{m}a - A\sqrt{m\epsilon}\sin(\omega t) + B\sqrt{m\epsilon}\cos(\omega t), \tag{74}$$

$$\ddot{x}(t) = \frac{q}{m}E_0 - Am\cos(\omega t) - Bm\sin(\omega t), \tag{75}$$

$$\dddot{x}(t) = \epsilon A\left(\frac{m}{\epsilon}\right)^{\frac{3}{2}}\sin(\omega t) - \epsilon B\left(\frac{m}{\epsilon}\right)^{\frac{3}{2}}\cos(\omega t), \tag{76}$$

and determine the parameters in terms of the initial conditions $x(0) = 0$, $\dot{x}(0) = 0$, $\ddot{x}(0) = 0$, and $\dddot{x}(0) = 0$. We obtain $B = 0$, $a = 0$, $A = \frac{q}{m^2}E_0$, and $b = 0$. The solution then reads

$$x(t) = \frac{q}{m}E_0\left(\frac{1}{2}t^2 + \frac{\epsilon}{m}(\cos(\omega t) - 1)\right), \quad (77)$$

and we also have

$$\dot{x}(t) = \frac{q}{m}E_0\left(t - \sqrt{\frac{\epsilon}{m}}\sin(\omega t)\right), \quad (78)$$

$$\ddot{x}(t) = \frac{q}{m}E_0(1 - \cos(\omega t)), \quad (79)$$

$$\dddot{x}(t) = \frac{q}{m}E_0\sqrt{\frac{m}{\epsilon}}\sin(\omega t). \quad (80)$$

The appearance of the *zitterbewegung* is connected with the external field E_0. Free particles do not perform such a motion.

Now we discuss three different measurements whose outcome depend on ϵ: (i) the time of flight in an accelerator, (ii) measurement of the acceleration by atomic interferometry, (iii) noise in electronic circuits. We always calculate first order effects.

A. Determination of time of flight

In standard theory ($\epsilon = 0$) we have the solution $x(t) = \frac{q}{2m}E_0 t^2$. For a charged particle being accelerated through its motion through an accelerator of length L we have $L = \frac{q}{2m}\frac{\Delta\phi}{L}t^2$, where $\Delta\phi$ is the potential difference the particle traverses. Then $t^2 = \frac{2m}{q\Delta\phi}L^2$ so that t is proportional to the spacing L and the final velocity does not depend on the distance L. This will be different for our higher order theory.

We determine the time the particle needs to traverse the length L. For that we first have to calculate t from $L = x(t)$:

$$L = \frac{q}{m}E_0\left(\frac{1}{2}t^2 + \frac{\epsilon}{m}(\cos(\omega t) - 1)\right), \quad (81)$$

which is a transcendental equation. For $\epsilon = 0$ we have $t = t_0 = \sqrt{\frac{2mL}{qE_0}}$. Therefore we make the approximative ansatz $t = t_0 + t_1$, where $t_1 \ll t_0$ and t_1 is of the order ϵ. Solving for t_1 gives the time of flight to first order correction in ϵ so that

$$t = \sqrt{\frac{2mL}{qE_0}} - \sqrt{\frac{qE_0}{2mL}}\frac{\epsilon}{m}\left(\cos\left(\omega\sqrt{\frac{2mL}{qE_0}}\right) - 1\right). \quad (82)$$

We use this time in order to calculate the velocity at $x(t) = L$ and obtain

$$\dot{x}(L) = \dot{x}_0\left(1 + \frac{\epsilon}{4m}\frac{\dot{x}_0^2}{L^2}\left(1 - \cos\left(\omega\sqrt{\frac{mL^2}{2q\Delta\phi}}\right)\right) + \sqrt{\frac{\epsilon}{4m}}\frac{\dot{x}_0}{L}\sin\left(\omega\sqrt{\frac{2mL^2}{q\Delta\phi}}\right)\right), \quad (83)$$

where we substituted $E_0 = \frac{\Delta\phi}{L}$ and also inserted the velocity of the standard theory $\dot{x}_0 = \sqrt{-\frac{2q\Delta\phi}{m}}$. In the standard theory $\dot{x}(L)$ does not depend on L. Here, by varying L we get oscillations in the velocity due to the sin and cos terms, and also a small offset $\frac{\epsilon}{4m}\frac{\dot{x}_0^3}{L^2}$.

Since ϵ is assumed to be small, then a change in L will result in fast oscillations which probably cannot be resolved. Therefore, averaging over a small L interval yields

$$\langle\dot{x}(L)\rangle = \dot{x}_0\left(1 + \frac{\epsilon}{4m}\frac{\dot{x}_0^2}{L^2}\right). \quad (84)$$

Therefore in the mean the velocity after traversing the capacitor is a bit larger.

Rewriting the above result as relative velocity deviation

$$\frac{\langle\dot{x}(L)\rangle - \dot{x}_0}{\dot{x}_0} = \frac{\epsilon}{4m}\frac{\dot{x}_0^2}{L^2}, \quad (85)$$

it is clear that one obtains good estimates for ϵ for large velocities $\dot{x}_0$, short L and small m. Taking, e.g., an electron with final energy of 10 MeV, a traversed distance of $L = 1$ m, an accuracy to measure the relative velocity of 1%, and assuming that no effect is observed, then we arrive at an estimate $\epsilon \leq 10^{-50}$ kg s^2.

B. Interferometry

Acceleration can be measured directly with, e.g., atomic interferometry. This has been proposed first by Bordé [13] and today's best performance gives an uncertainty of the measured acceleration of $\Delta\ddot{x} \approx 10^{-8}$ m/s^2 [14]. However, while this accuracy is valid for a constant acceleration, in our case we have a fast varying acceleration.

We use the phase shift

$$\Delta\phi = A(\omega)k\ddot{x}T^2, \quad (86)$$

where k is the wave vector of the laser beam acting as beam splitter and T is the time between the laser pulses. For the acceleration we take (78). Since that part of our acceleration we are interested in and which we like to detect this way is fluctuating, we have to amend the standard phase shift for a dc acceleration, $\Delta\phi = kgT^2$, by a transfer function $A(\omega)$ which has been determined in Ref. [15]. As an example, we use as charged particle ionized helium and take typical values for the laser wavelength $\lambda = 780$ nm, a pulse spacing time of $T = 100$ ms, and an electric field strength of $E_0 = 10^{10}$ V/m. An experimentally reachable accuracy of the phase measurement is $\Delta\phi = 10^{-3}$ rad.

We are interested in the largest ω which we are able to detect. With the specifications given we can determine that ω from the condition $A(\omega) = 10^{-25}$. This gives $\omega = 10^{12}$ Hz which is the maximum frequency whose effect on the phase shift we are able to detect. If we assume that nothing is detected this gives an estimate of

$$\epsilon \leq \frac{m}{\omega^2} = 10^{-50}\ \mathrm{kg\,s^2}. \tag{87}$$

C. Electronic noise

Another situation where a kind of *zitterbewegung* of a charged particle may induce an effect is electronics. As a most simple model we may assume that the particle under consideration is located within a capacitor. An oscillation of this particle will induce an electronic noise in the electric circuit. This electronic noise can be estimated by

$$C\langle U^2\rangle = m\langle \dot{x}^2\rangle, \tag{88}$$

where U is the voltage. Using only the oscillating terms in the velocity we obtain

$$C\langle U^2\rangle = \frac{1}{2}\epsilon\left(\frac{q}{m}E_0\right)^2. \tag{89}$$

Therefore, a modified dynamics will induce an electronic noise—beside other noise like Nyquist noise or a shot noise. However, due to its different characteristics, it might be disentangled from the other noise sources. Our noise should be a fundamental noise not depending on temperature, the finite number of charged particles in the electric circuit, etc. A good cryogenic noise limit is of the order 1 nV/$\sqrt{\mathrm{Hz}}$ [16] in a wide frequency range, that is 1 nV for a measurement of duration 1 s. Taking as granted that no fundamental noise of this kind has been seen under the conditions of a molecular vacuum and cryogenic temperature, we may get a first estimate $\epsilon \leq 10^{-68}$ kg s^2, where we took a capacitance of 0.48 pF, a distance between the capacitor plates of 15 μm, a voltage of 1000 V. Taking into account a bandwidth of 1 GHz, we get a more realistic estimate $\epsilon \leq 10^{-50}$ kg s^2.

VII. CONCLUSION AND OUTLOOK

We studied the physics resulting from a hypothetically given higher order equation of motion substituting standard Newton's law. In order (i) to investigate the consequences of such models and (ii) to confront such a model with experiments we employed a gauge principle to couple these equations to external forces. This procedure we carried through in the framework of higher order Lagrangian formalism. The resulting gauge fields have a richer structure than in the ordinary first order Lagrange formalism. Beside the ordinary gauge fields usually obtained in the zeroth order formalism, we obtained for the first order gauge model the standard space-time metric carrying the gravitational interaction in terms of a gauge field.

Then we discussed physical consequences of equations of motion with an additional small even higher order time derivative, coupled to smooth and slowly varying gauge fields up to our first order gauge formalism. We solved the equation of motion for the simplest case and deduced observational consequences in three experimental situations. Leaving aside runaway solutions which are connected with a certain choice of the sign of the additional higher derivative term and which are contradicting any observation, a small even higher order term influences the time of flight of accelerated particles, yields an acceleration fluctuation accessible in atomic interferometry, and also induces a fundamental noise in electronic devices. Until now no deviation from Newton's second law has been observed. Very rough and preliminary estimates for the parameter characterizing the higher order term could be derived. However, a comparison with the Planck scale version of this parameter shows that the experimental estimates are far away from reaching the corresponding quantum gravity scale.

The next step is to implement a higher order theory for point particles in a relativistic context. A possible Lagrange function to start with might be $L_0 = m\sqrt{\eta_{ab}\dot{x}^a\dot{x}^b + \epsilon\eta_{ab}\ddot{x}^a\ddot{x}^b}$, where the dot is the derivative with respect to some parameter along the path. We would also like to implement our principle in higher order field theory, e.g., for scalar field Lagrange densities of the type $\mathcal{L} = \eta^{ab}\partial_a\phi^*\partial_b\phi + \mu\eta^{ab}\eta^{cd}\partial_a\partial_c\phi^*\partial_b\partial_d\phi$. A further step would be to set up some field equations for the new gauge fields. A first guess might be to have the usual Maxwell equations for the A_a and the Einstein field equations for the g_{ab}.

For gauge models with $N > 1$ one obtains equations of motion with an acceleration which is multiplied with coefficients which depend on the velocity; see Eq. (25). Also all other terms are polynomials of the velocity of degree $N + 1$. Altogether, the equation of motion takes a form reminding one of the equation of motion in Finsler geometry. For a recent discussion of a class of Finsler space-times and Solar system tests, see Ref. [17].

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