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Modelling gravity on a hyper-cubic lattice

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We present a simple dynamical model of symmetric, nondegenerate $n \times n$ matrices of fixed signature defined on an n-dimensional hyper-cubic lattice with nearest-neighbour interactions. We show how this model is related to general relativity, and discuss multiple ways in which it can be useful for studying gravity, both classical and quantum. In particular, we show that the dynamics of the model when all matrices are close to the identity corresponds exactly to a finite-difference discretization of weak-field gravity in harmonic gauge. We also show that the action which defines the full dynamics of the model corresponds to the Einstein-Hilbert action to leading order in the lattice spacing, and use this observation to define a lattice analogue of the Ricci scalar and Einstein tensor. Finally, we perform a mean-field analysis of the statistical mechanics of this model.

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I. INTRODUCTION

Lattice models of gravity are typically defined using some discretization which is simultaneously coordinate and background independent. The most popular such discretization is the Regge calculus [1,2], used to study both classical general relativity (GR) [3,4], as well as to build models of quantum gravity based on dynamical simplices [5–17]. Other lattice discretizations of gravity include [18–23], but all of these formulations differ significantly from the current proposal.

In this paper we present a discrete model for gravity which is defined on a regular, in fact hyper-cubic, coordinate lattice. The implied background structure may be anathema to GR purists, however we will argue that this is still a useful thing to do, and can be usefully utilized to study GR.

That having a preferred background might not be entirely implausible may be inferred from a number of observations:

(1) Many interesting spacetimes can be put into Kerr-Schild form, which has a natural background [24].
(2) Many interesting spacetimes can be put into Painleve-Gullstrand [25] and/or de Donder (nonlinear harmonic) form [26], both of which possess natural background metrics.
(3) Many interesting spacetimes can be put into "relativistic acoustic" form, based on the "analogue spacetime" program, for which a natural background metric again exists [27].
(4) Physically interesting black holes can be put into horizon-penetrating coordinates, for which the metric components are finite at the horizon; the presence of a horizon does not *necessarily* imply "infinite deviations" from some assumed background metric [25,28].
(5) More exotically, recent speculations on ghost-free massive gravitons are most naturally phrased in terms of a combination of foreground and background metric [29].

In view of the above, we are willing to at least entertain the notion of background structure, to see how far we can get.

II. LATTICE ACTION

Consider an n-dimensional hyper-cubic lattice which has defined, at each site i, a $n \times n$ symmetric, nondegenerate matrix ${}^{i}g$, which is physically to be interpreted as the metric. Unless otherwise stated, we will assume in this article that the matrix is positive definite (and hence is a model for a Euclidean-signature Riemannian geometry); however, the model can easily be generalized to matrices of any fixed signature. The dynamics of the model is described by a particularly simple action defined as a sum over nearest-neighbor pairs.

First define an "average" metric linking the nearest-neighbor sites i and j:

$$ {}^{ij}\bar{g} = \frac{{}^{j}g + {}^{i}g}{2}. \tag{1} $$

Then set

$$ S = -\xi_n (s/L_P)^{n-2} \sum_{\langle ij \rangle} \left\{ \sqrt{\det({}^{ij}\bar{g})} - \frac{\sqrt{\det({}^{i}g)}}{2} - \frac{\sqrt{\det({}^{j}g)}}{2} \right\}. \tag{2} $$

Here $\langle ij \rangle$ denotes the link joining nearest-neighbor sites i and j. The lattice spacing is denoted s, and the n-dimensional Planck length is L_P, while ξ_n is some convenient dimension-dependent normalizing constant.

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Note that this action has a symmetry under both rigid (global) $SO(n)$ transformations $[{}^ig] \to O^T[{}^ig]O$, and under parity. As we shall see below, ultimately the reason this particularly simple action works is because of the intimate connection between hyper-volume excess/deficit and the Ricci scalar.

III. WEAK FIELD

Consider small fluctuations

$$ {}^ig = \mathbb{I} + {}^ih; \qquad |{}^ih| \ll 1. \tag{3} $$

To quadratic order in h we have

$$ \sqrt{\det({}^{ij}\bar{g})} = 1 + \frac{1}{4}\,\mathrm{tr}[{}^ih + {}^jh] + \frac{1}{32}\,\mathrm{tr}[{}^ih + {}^jh]^2 - \frac{1}{16}\,\mathrm{tr}[({}^ih + {}^jh)^2] + \mathcal{O}(h^3), \tag{4} $$

and

$$ \frac{1}{2}\sqrt{\det({}^ig)} = \frac{1}{2} + \frac{1}{4}\,\mathrm{tr}[{}^ih] + \frac{1}{16}\,\mathrm{tr}[{}^ih]^2 - \frac{1}{8}\,\mathrm{tr}[{}^ih^2] + \mathcal{O}(h^3). \tag{5} $$

Thus to quadratic order in h we have

$$ S \propto \sum_i \sum_{j:\langle ij\rangle} \left(\mathrm{tr}[({}^jh - {}^ih)^2] - \frac{1}{2}\,\mathrm{tr}[{}^jh - {}^ih]^2 \right) + \mathcal{O}(h^3). \tag{6} $$

Taking the lattice spacing to zero the finite differences become derivatives, and the action (up to an arbitrary multiplicative constant) is given by

$$ S \propto \int_{\mathbb{R}^n} \left(\partial_\sigma h_{\mu\nu} \partial^\sigma h^{\mu\nu} - \frac{1}{2}\,\partial^\mu h^\nu{}_\nu \partial_\mu h^\sigma{}_\sigma \right) d^nx + \mathcal{O}(h^3). \tag{7} $$

This is precisely the action for linearized GR in (globally defined and linearized) harmonic gauge

$$ \partial^\nu h_{\mu\nu} - \frac{1}{2}\,\partial_\mu h^\sigma{}_\sigma = 0. \tag{8} $$

Thus we see, in a very clear and convincing manner, that the lattice action (2) can be used to model weak-field GR provided the very commonly used (linearized) harmonic coordinates are adopted.

IV. CONTINUUM LIMIT

We now show that the discrete lattice action (2) has a continuum limit, which is given to leading order by the Einstein-Hilbert action. Let us first use permutation symmetry to rewrite the action as

$$ S \propto \sum_{\langle ij\rangle} \left\{ \sqrt{\det({}^{ij}\bar{g})} - \sqrt{\det({}^ig)} \right\}. \tag{9} $$

We can explicitly factor out a $\sqrt{\det({}^ig)}$ to obtain

$$ S = \sum_i \sqrt{\det({}^ig)}\ {}^iS. \tag{10} $$

Here we define the site-specific contribution to the action for site i in terms of a sum over its nearest neighbors,

$$ {}^iS \propto \sum_{j:\langle ij\rangle} \left(2^{-n/2}\sqrt{\det(\mathbb{I} + [{}^ig]^{-1}[{}^jg])} - 1 \right). \tag{11} $$

Let us now work on a continuum manifold with metric $g_{\mu\nu}(x)$, and choose a (site-specific) Riemann normal coordinate system—such that site i is taken to be the origin, and the metric at site i is ${}^ig_{\mu\nu} = \delta_{\mu\nu}$. Let ℓ^μ_{ij} denote the unit vector pointing from site i to site j. The coordinate system is such that the geodesics generated by these coordinate unit vectors are straight lines:

$$ {}^{ij}x^\mu(\lambda) = \ell^\mu_{ij}\lambda. \tag{12} $$

Then in the immediate neighborhood of the origin we can construct the vertices of the hyper-cubic lattice such that the nearest neighbors are connected by geodesics of length s and have coordinate locations ${}^jx^\mu = \ell^\mu_{ij}s$.

A very well-known and quite standard result for the Riemann normal coordinate system is that to quadratic order

$$ {}^jg_{\mu\nu} = g_{\mu\nu}({}^{ij}x) = \delta_{\mu\nu} - \frac{1}{3}\mathcal{R}_{\mu\alpha\nu\beta}\ell^\alpha_{ij}\ell^\beta_{ij}s^2 + \mathcal{O}(s^3). \tag{13} $$

But then

$$ \{[{}^ig]^{-1}[{}^jg]\}^\mu{}_\nu = \delta^\mu{}_\nu - \frac{1}{3}\mathcal{R}^\mu{}_{\alpha\nu\beta}\ell^\alpha_{ij}\ell^\beta_{ij}s^2 + \mathcal{O}(s^3), \tag{14} $$

so

$$ 2^{-n/2}\sqrt{\det(\mathbb{I} + [{}^ig]^{-1}[{}^jg])} = 1 - \frac{1}{12}\mathcal{R}_{\alpha\beta}\ell^\alpha_{ij}\ell^\beta_{ij}s^2 + \mathcal{O}(s^3). \tag{15} $$

Hence the site-specific contribution to the action is

$$ {}^iS \propto \sum_{j:\langle ij\rangle} \mathcal{R}_{\alpha\beta}\ell^\alpha_{ij}\ell^\beta_{ij}s^2 + \mathcal{O}(s^3). \tag{16} $$

Now using the easily established result

$$ \sum_{j:\langle ij\rangle} \ell^\alpha_{ij}\ell^\beta_{ij} = 2\delta^{\alpha\beta}, \tag{17} $$

we see that to quadratic order in the lattice spacing (which is also the geodesic distance), the continuum analogue of Eq. (11) is given by

$$ {}^iS \propto \mathcal{R}s^2 + \mathcal{O}(s^3). \tag{18} $$

Thus to leading order in the lattice spacing our lattice action corresponds to the Einstein-Hilbert action.

V. LATTICE RICCI SCALAR

With suitable normalization,

$$ {}^iR = -6\sum_{j:\langle ij\rangle}\left(2^{-n/2}\sqrt{\det(\mathbb{I} + [{}^ig]^{-1}[{}^jg])} - 1\right). \tag{19}$$

Then trivially adapting the discussion above,

$$ {}^iR = \mathcal{R}s^2 + \mathcal{O}(s^3). \tag{20}$$

To lowest nontrivial order in the lattice spacing, the discrete quantity ${}^iR$ matches its continuum analogue $\mathcal{R}$.

VI. STRONG FIELD EQUATIONS OF MOTION

The discrete version of the Einstein tensor is easily obtained by computing

$$ {}^iG \propto \frac{1}{\sqrt{\det({}^ig)}}\frac{\delta S}{\delta[{}^ig]}. \tag{21}$$

We find

$$ \frac{\delta S}{\delta[{}^ig]} = \frac{1}{2}\sum_{j:\langle ij\rangle}(\sqrt{\det({}^{ij}\bar{g})}({}^{ij}\bar{g})^{-1} - \sqrt{\det({}^ig)}({}^ig)^{-1}). \tag{22}$$

After picking a suitable normalization, we set

$$ {}^iG = -6\sum_{j:\langle ij\rangle}\left[\sqrt{\frac{\det({}^{ij}\bar{g})}{\det({}^ig)}}({}^{ij}\bar{g})^{-1} - ({}^ig)^{-1}\right]. \tag{23}$$

But now (again adopting site-specific Riemann normal coordinates at the site i, so ${}^ig \to \mathbb{I}$), we have already seen (to quadratic order) the equivalent of

$$ [{}^{ij}\bar{g}]^{-1}_{\mu\nu} = \delta_{\mu\nu} + \frac{1}{6}\mathcal{R}_{\mu\alpha\nu\beta}\ell^\alpha_{ij}\ell^\beta_{ij}s^2 + \mathcal{O}(s^3), \tag{24}$$

and

$$ \det[{}^{ij}\bar{g}] = 1 - \frac{1}{6}\mathcal{R}_{\alpha\beta}\ell^\alpha_{ij}\ell^\beta_{ij}s^2 + \mathcal{O}(s^3), \tag{25}$$

whence, to quadratic order

$$ {}^iG_{\mu\nu} = \frac{1}{2}\sum_{j:\langle ij\rangle}\left[\left(\mathcal{R}_{\mu\alpha\nu\beta} - \frac{1}{2}\delta_{\mu\nu}\mathcal{R}_{\alpha\beta}\right)\ell^\alpha_{ij}\ell^\beta_{ij}s^2\right] + \mathcal{O}(s^3). \tag{26}$$

Summing over the nearest-neighbor sites j, the discrete Einstein tensor is related to the continuum Einstein tensor by

$$ \begin{aligned} {}^iG_{\mu\nu} &= \left[\mathcal{R}_{\mu\nu} - \frac{1}{2}\delta_{\mu\nu}\mathcal{R}\right]s^2 + \mathcal{O}(s^3) \\ &= \mathcal{G}_{\mu\nu}s^2 + \mathcal{O}(s^3). \end{aligned} \tag{27}$$

VII. EXTERNAL STRESS-ENERGY TENSOR

As is usual in lattice models, we can also add an external current ${}^iJ$ to probe the dynamics. The most natural object to add is

$$ S_J = \sum_i \sqrt{\det({}^ig)}\,\mathrm{tr}[{}^ig\,{}^iJ]. \tag{28}$$

The external current ${}^iJ$ is interpretable in terms of the discrete stress-energy tensor ${}^iT$ via

$$ {}^iT = \frac{1}{\sqrt{\det({}^ig)}}\frac{\delta S_J}{\delta[{}^ig]} = {}^iJ + \frac{1}{2}[{}^ig]^{-1}\mathrm{tr}[{}^ig\,{}^iJ]. \tag{29}$$

The strong field discrete Einstein equations in the presence of external stress energy are then quite simply

$$ {}^iG_{\mu\nu} \propto {}^iT_{\mu\nu}. \tag{30}$$

So formally at least, the discrete lattice model contains all the correct ingredients for adequately dealing with large swathes of standard GR. This procedure clearly generalizes to placing some matter model on the lattice.

VIII. MEAN-FIELD ANALYSIS

In addition to using this lattice model to study classical GR, it can also be used as a discrete model for studying quantum gravity. A first step in this direction is to perform a mean-field analysis of the action (2) which we now rewrite in terms of the site-specific form as

$$ S = -\alpha_n\sum_i\sqrt{\det({}^ig)}\sum_{j:\langle ij\rangle}\left(2^{-n/2}\sqrt{\det(\mathbb{I} + [{}^ig]^{-1}[{}^jg])} - 1\right), \tag{31}$$

where $\alpha_n = \zeta_n(s/L_P)^{n-2}$ is the constant appearing in Eq. (2). This action is translationally invariant and thus we take the mean-field ansatz—assuming the physics is dominated by some translation invariant average $M = \langle g\rangle$, plus small fluctuations

$$ {}^ig = M + \delta[{}^ig]. \tag{32}$$

This allows us to replace ${}^jg$ with M in the coupling term of the action. We find

$$ \begin{aligned} &\sum_{j:\langle ij\rangle}\left(2^{-n/2}\sqrt{\det(\mathbb{I} + [{}^ig]^{-1}[{}^jg])} - 1\right) \\ &\quad\to 2n\left(2^{-n/2}\sqrt{\det(\mathbb{I} + [{}^ig]^{-1}M)} - 1\right). \end{aligned} \tag{33}$$

The total action then becomes

$$ S_{\mathrm{mf}} = -\alpha_n 2n\sum_i\sqrt{\det({}^ig)}\left(2^{-n/2}\sqrt{\det(\mathbb{I} + [{}^ig]^{-1}M)} - 1\right). \tag{34}$$

Thus the mean-field partition function,

$$ Z_{\mathrm{mf}} = \int\prod_i d^ig\, e^{-\beta S_{\mathrm{mf}}}, \tag{35}$$

is given by

$$Z_{\mathrm{mf}} = \left(\int dg e^{\gamma \sqrt{\det g} (2^{-n/2} \sqrt{\det(\mathbb{I} + g^{-1}M)} - 1)} \right)^N, \tag{36}$$

where $\gamma = 2n\alpha_n\beta$. Now performing a change of variables $g = \sqrt{M}\tilde{g}\sqrt{M}$, which has Jacobian $\det J = (\det M)^n$, after dropping the tilde the integral which determines the mean-field partition function is given by

$$\int dg e^{\gamma \sqrt{\det(M)} (\sqrt{\det(\frac{I+g}{2})} - \sqrt{\det g})}. \tag{37}$$

Thus, we see that our mean-field analysis results in a random matrix model which is invariant under $O(n)$ transformations. Because the matrix model has this symmetry group we can perform the diagonalization $g = O^T \Lambda O$ where Λ is the matrix of eigenvalues, all of which are by assumption positive. The Jacobian of this transformation is given by [30,31]

$$dg = dO \prod_A d\lambda_A \prod_{A<B} |\lambda_A - \lambda_B|. \tag{38}$$

After integrating over the orthogonal group, the integral appearing in the mean-field partition function becomes

$$\int \prod_A d\lambda_A \prod_{A<B} |\lambda_A - \lambda_B| e^{-\gamma \sqrt{\det(M)} V(\lambda)}, \tag{39}$$

where the function $V(\lambda)$ is given by

$$V(\lambda) = \prod_A \sqrt{\lambda_A} - \prod_A \sqrt{\frac{1 + \lambda_A}{2}}. \tag{40}$$

After a bit of work we obtain the (useful but suboptimal) constraints

$$\prod_{\lambda_A < 1} \sqrt{\lambda_A} - \prod_{\lambda_A > 1} \sqrt{\lambda_A} < V(\lambda) < \prod_A \sqrt{\lambda_A}. \tag{41}$$

Only for the trivial case $n = 1$ (a one-dimensional chain) is the function $V(\lambda)$ bounded from below; for $n \geq 2$ there are directions in eigenvalue space where $V(\lambda)$ becomes arbitrarily negative. Thus the mean-field truncation of the lattice action gives a random matrix model exhibiting pathology similar to that of the Einstein-Hilbert action. Further study of this model, using the techniques of random matrix theory, may shed light on how to deal with this feature.

IX. DISCUSSION

We have seen that with the particularly simple discrete action (2) one can successfully encode a very large fraction of standard GR. The action is gauge fixed, with only rigid (global) rotations and parity inversions as symmetries, and seems automatically to be in the de Donder (nonlinear harmonic) gauge; this is a feature, rather than a problem. (Gauge fixing is not a problem *per se* [26], since to make physical predictions one ultimately has to do so anyway).

Presumably there is some more general gauge-invariant action of which this is gauge-fixed version. We note also that this model does not exhibit the active diffeomorphism symmetry of the continuum theory and thus there are $n(n+1)/2$ dynamical degrees of freedom per lattice site; this is standard for discretizations of gravity [32]. What is perhaps a little surprising is just how far one can get with such a simple discrete action.

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