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Cosmology and the Korteweg-de Vries equation

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The Korteweg-de Vries equation is a nonlinear wave equation that has played a fundamental role in diverse branches of mathematical and theoretical physics. In the present paper, we consider its significance to cosmology. It is found that the Korteweg-de Vries equation arises in a number of important scenarios, including inflationary cosmology, the cyclic universe, loop quantum cosmology and braneworld models. Analogies can be drawn between cosmic dynamics and the propagation of the solitonic wave solution to the equation, whereby quantities such as the speed and amplitude profile of the wave can be identified with cosmological parameters such as the spectral index of the density perturbation spectrum and the energy density of the universe. The unique mathematical properties of the Schwarzian derivative operator are important to the analysis. A connection with dark solitons in Bose-Einstein condensates is briefly discussed.

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I. INTRODUCTION

The Korteweg–de Vries (KdV) equation [1] is the completely integrable, third-order, nonlinear partial differential equation:

$$\partial_t u + \partial_x^3 u + \frac{3}{u_0} u \partial_x u = 0, \tag{1}$$

where $u = u(x, t)$, $\partial_t = \partial/\partial t$, $\partial_x^3 = \partial^3/\partial x^3$, etc., u_0 is a constant and (x, t) represent space and time coordinates, respectively. This equation was originally derived within the context of small-amplitude, nonlinear water wave theory and it is well known that it admits a solitonic wave solution of the form

$$u = u_0 \lambda^2 \mathrm{sech}^2[\lambda(x - \lambda^2 t)/2], \tag{2}$$

where the constant $\lambda/2$ represents the wave number of the soliton [1]. The KdV soliton is characterized by the property that its speed and amplitude are proportional to the square of the wave number.

The KdV equation has played a central role in diverse branches of physics, including nonlinear optics, atomic and nuclear physics, Bose-Einstein condensates and astrophysical plasmas (see Ref. [2] for a review). As far as we are aware, however, it has not been discussed previously within a cosmological context (although see Ref. [3]). This is perhaps not surprising, given that the KdV equation is a third-order partial differential equation in two independent variables, whereas the field equations for spatially isotropic universes are second-order, ordinary differential equations (ODEs).

On the other hand, Eq. (1) can be reduced to the nonlinear ODE:

$$-\lambda^2 u' + u''' + \frac{3}{u_0} u u' = 0, \tag{3}$$

where a prime denotes $d/d\phi$ and $\phi \equiv x - \lambda^2 t$ represents a "wavelike" independent variable. The purpose of the present paper is to show that wave solutions to Eq. (3), and in particular the soliton solution (2), arise in a number of cosmological settings, including the inflationary paradigm, the cyclic universe scenario, loop quantum cosmology and braneworld models. We consider scenarios where the universe is dominated by a single self-interacting scalar field, ϕ. Interpreting the value of the scalar field in terms of a wavelike coordinate of the KdV equation then allows for a direct analogy to be drawn between cosmological dynamics on the one hand and solitonic behavior on the other. Cosmological parameters can then be identified with quantities such as the speed and amplitude of the corresponding wave.

The paper is organized as follows. In Sec. II, we discuss a connection between the KdV equation and the Schwarzian derivative operator that we employ in later sections. We proceed in Sec. III to consider the class of inflationary universes that generate density perturbation spectra with a constant spectral index. In Sec. IV, we address the same question within the context of the cyclic universe. In Sec. V, we find that the scaling solutions of various braneworld and loop quantum cosmological scenarios are analogous to the KdV soliton. We conclude with a discussion in Sec. VI on a connection with dark solitons in Bose-Einstein condensates. Unless otherwise stated, units are chosen such that $\hbar = c = 1$ and the Planck mass is normalized to $m_{\rm P} = \sqrt{8\pi}$.

II. THE KDV EQUATION AND THE SCHWARZIAN DERIVATIVE OPERATOR

The KdV equation (3) admits an auto-Bäcklund transformation, whereby a solution $u = u(\phi)$ can be derived from a given seed solution $\bar{u} = \bar{u}(\phi)$ [4]. For the special

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case where the seed is the trivial solution $\bar{u} = 0$, such a transformation reduces to the condition that a solution to Eq. (3) is given by

$$u = u_0(\lambda^2 - 4y^2), \tag{4}$$

where λ^2 is a constant and the function $y = y(\phi)$ is a particular solution to the first-order Riccati equation:

$$y' = \frac{u}{4u_0} = \frac{\lambda^2}{4} - y^2. \tag{5}$$

For example, the solution

$$y = \frac{\lambda}{2}\tanh(\lambda\phi/2), \tag{6}$$

generates the KdV soliton (2).

Solutions to Eq. (5) satisfy a third-order ODE given by

$$S[y(\phi)] \equiv \frac{y'''}{y'} - \frac{3}{2}\left(\frac{y''}{y'}\right)^2 = -\frac{\lambda^2}{2}, \tag{7}$$

as may be verified by direct differentiation of Eq. (5). The left-hand side of Eq. (7) is the Schwarzian derivative operator (often referred to simply as the *Schwarzian*) [5]. This operator exhibits a number of remarkable properties, one of which we exploit in the present work: it is the unique combination of derivatives that is invariant under a homographic transformation corresponding to the group of fractional linear transformations. This follows since, for a function $y = y(\phi)$, the composition $N(y) \equiv [\ln(y')]' = y''/y'$ transforms under the inversion $y \to 1/y$ such that $N(1/y) = N(y) - 2y'/y$. Some straightforward algebra then implies that the operator $S(y) \equiv N' - N^2/2$ is invariant under inversion. Moreover, since $S(y) = S(my + n)$ for any m, $n \in \Re$, $S(y)$ is invariant under the full group of fractional linear transformations.

This implies that if $\bar{y}(\phi)$ is a particular "seed" solution to the differential equation $S[y] = f(\phi)$ for some function $f(\phi)$, the *general solution* to such an equation is given by

$$y(\phi) = \frac{a\bar{y} + b}{c\bar{y} + d}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathrm{R}), \tag{8}$$

where a, b, c, d are constants such that $ad - bc = 1$.

To summarize, therefore, if a particular solution to the Schwarzian equation (7) for constant λ can be found, the general solution (8) can be written down immediately. Restricting the general solution to satisfy the first-order constraint (5) then generates a solution (4) to the KdV equation (3). We employ this result in the following sections for a number of cosmological models.

III. INFLATIONARY DENSITY PERTURBATIONS AND THE KDV SOLITON

The inflationary scenario remains the cornerstone of modern, early universe cosmology. While there is currently considerable interest in multiple-field versions of the paradigm, our aim in this section is to revisit the simplest version of the scenario, namely inflation driven by a single, minimally coupled, slowly rolling scalar inflaton field, ϕ. (For reviews, see, e.g., Refs. [6,7].) We consider the general class of models that generate a density perturbation spectrum with a *constant* spectral index to lowest order in the slow-roll approximation. Our main aim is to highlight some interesting mathematical features of the underlying differential equations that have not been previously discussed.

The cosmological Friedmann equations in Hamilton-Jacobi form are given by

$$3H^2 - 2H'^2 = V(\phi), \qquad \dot{\phi} = -2H', \tag{9}$$

where the Hubble parameter $H = H(\phi)$ is viewed as a function of ϕ, $V(\phi)$ denotes the inflaton potential and a prime and dot denote differentiation with respect to the inflaton field and cosmic time, respectively. The energy density of the universe is $\rho(\phi) = 3H^2(\phi)$. It is assumed implicitly and without loss of generality that the inflaton varies monotonically with cosmic time such that $\dot{\phi} > 0$ ($H' < 0$).

The Hubble slow-roll parameters are defined by

$$\epsilon \equiv 2\frac{H'^2}{H^2}, \qquad \eta \equiv 2\frac{H''}{H}, \tag{10}$$

and the spectral index is given by

$$1 - n_s = \frac{2}{(1-\epsilon)^2}\left[\epsilon - \frac{(1-\epsilon^2)}{2}\left(\frac{d\ln\epsilon}{d\mathcal{N}}\right)\right], \tag{11}$$

where higher-order derivative terms in $\mathcal{N} \equiv -\ln(aH)$ have been neglected. To lowest order in the slow-roll approximation, $1 - n_s = 4\epsilon - 2\eta$. This condition is a second-order, nonlinear ODE for the dependent variable $H = H(\phi)$:

$$4\frac{H''}{H} - 8\frac{H'^2}{H^2} = -(1 - n_s). \tag{12}$$

Considerable insight into the nature of the general solution to Eq. (12) may be gained by expressing the energy density of the universe in terms of the gradient of a "potential" function, $W = W(\phi)$, such that

$$H^2(\phi) \equiv 4H_0^2 W'(\phi), \tag{13}$$

where H_0 is an arbitrary constant. Substituting this definition into Eq. (12) yields

$$S[W(\phi)] \equiv \frac{W'''}{W'} - \frac{3}{2}\left(\frac{W''}{W'}\right)^2 = -\frac{\lambda^2}{2}, \tag{14}$$

where $\lambda^2 \equiv 1 - n_s$.

The left-hand side of Eq. (14) is the Schwarzian derivative of the function $W(\phi)$ and it is interesting that it arises in this cosmological context. Consequently, we may now determine from Eqs. (13) and (14) the general forms of the

Hubble parameters for the full family of inflationary cosmologies that generate a constant spectral index.

Since current observational bounds on the spectral index inferred from the WMAP7 + H0 data set are $0.939 < n_s < 0.987$ at the 2σ confidence limit [8], we focus on the red perturbation spectrum, $\lambda^2 > 0$. A particular solution to Eq. (14) is $\bar{W}(\phi) = \exp(\lambda\phi)$ and the general solution is therefore given directly by

$$W(\phi) = \frac{ae^{\lambda\phi} + b}{ce^{\lambda\phi} + d}, \quad H^2(\phi) = 4\lambda H_0^2 \frac{e^{\lambda\phi}}{(ce^{\lambda\phi} + d)^2}. \tag{15}$$

Moreover, comparing Eq. (14) with Eq. (7) and (13) with Eq. (5) immediately implies that $H^2(\phi)$ satisfies the KdV equation

$$-(1 - n_s)H^{2\prime} + H^{2\prime\prime\prime} + \frac{3}{H_0^2} H^2 H^{2\prime} = 0, \tag{16}$$

if the general solution to Eq. (14) is restricted to satisfy the Riccati equation

$$W' = \frac{\lambda^2}{4} - W^2. \tag{17}$$

It may be verified that condition (17) is satisfied if

$$ad = -bc = 1/2, \qquad \lambda = \frac{1}{cd} = \frac{2a}{c}. \tag{18}$$

As a result, the solution (15) can be expressed as

$$H^2(\phi) = H_0^2 \lambda^2 \mathrm{sech}^2\left(\frac{\lambda}{2}\frac{\sqrt{8\pi}}{m_{\mathrm{P}}}\phi\right), \tag{19}$$

for $cd > 0$ and

$$H^2(\phi) = -H_0^2 \lambda^2 \mathrm{cosech}^2\left(\frac{\lambda}{2}\frac{\sqrt{8\pi}}{m_{\mathrm{P}}}\phi\right), \tag{20}$$

if $cd < 0$, where we have specified $c = |d|$ without loss of generality and have restored the dependence on the Planck mass for future reference. [If $c \neq |d|$, Eqs. (19) and (20) can be recovered by performing a linear translation $\lambda^{-1}\ln|d/c|$ on the value of the inflaton field.]

Equations (19) and (20) are both wave solutions to the KdV equation (16) and the former has precisely the form of the nonsingular KdV soliton (2). This suggests a direct analogy can be drawn between such a wave and inflationary cosmology. In such an analogy, the inflaton field plays the role of a characteristic, wavelike coordinate on a two-dimensional spacetime $\{x, t\}$, the speed of the soliton is determined by the deviation of the spectral index away from the scale-invariant, Harrison-Zel'dovich spectrum $n_s = 1$, and the amplitude profile of the soliton is parametrized by the energy density of the universe (in appropriate units).

The first slow-roll parameter for the "soliton" solution (19) is given by

$$\epsilon(\phi) = \frac{\lambda^2}{2}\tanh^2(\lambda\phi/2), \tag{21}$$

and determines the tensor-scalar ratio, $r \equiv \mathcal{P}_T^2/\mathcal{P}_S^2 = 16\epsilon$, where $\mathcal{P}_S^2 = H^2/(64\pi^4\epsilon)$ and $\mathcal{P}_T^2 = H^2/(4\pi^4 m_{\mathrm{P}}^2)$ are the amplitudes of the density and gravitational wave perturbations [6,7]. This parameter is bounded from above such that $r < 8(1 - n_s)$, which is consistent with the current observational 2σ bound $r < 0.24$ [8].

We therefore focus on the nonsingular KdV wave. Given the general form of the Hubble parameter, the inflationary potential can be deduced directly from the Hamilton-Jacobi equation (9), $V = H^2(3 - \epsilon)$. Substituting Eqs. (19) and (21) and expanding the brackets up to linear order in ϵ to be consistent with slow roll implies that

$$V(\phi) = H_0^2\lambda^2\left[3 - \left(3 + \frac{\lambda^2}{2}\right)\tanh^2\left(\frac{\lambda}{2}\phi\right)\right]. \tag{22}$$

It is interesting to remark that after some straightforward algebra, Eq. (22) can be expressed in the form

$$\left(\frac{2}{6 + \lambda^2}\right)\left(V + \frac{H_0^2\lambda^4}{2}\right) = \lambda^2 H_0^2 \mathrm{sech}^2\left(\frac{\lambda}{2}\phi\right). \tag{23}$$

Hence, a rescaled and linearly shifted version of the potential also satisfies a KdV equation.

It is straightforward to verify that the potential (22) corresponds precisely to that derived in Ref. [9] by a different method. As discussed in Ref. [9], from the cosmic dynamical systems point of view, the late-time attractor of this model is the power-law solution $\epsilon = \lambda^2$. There are two different solutions, Eqs. (19) and (20), depending on whether the initial value of ϵ is greater or less than λ^2 [9]. Within the context of the present discussion, the power-law model is the seed solution $\bar{W}(\phi) = \exp(\lambda\phi)$ and the two models are characterized by $\mathrm{sgn}(cd)$, i.e., by $\mathrm{sgn}(W')$.

On the other hand, it is important to emphasize that the solution for n_s is derived from an approximate ODE (12) and must necessarily break down as the speed of the inflaton increases. Indeed, to next-to-leading order in slow roll, the equation for the spectral index becomes the nonlinear, third-order ODE [10]

$$4\epsilon - 2\eta + 8(C+1)\epsilon^2 - (6 + 10C)\epsilon\eta + 2C\xi^2 = 1 - n_s, \tag{24}$$

where $\xi^2 = 4H'H'''/H^2$ is the third slow-roll parameter and $C \simeq -0.73$. Consequently, the error introduced by restricting the analysis to slow roll, whereby the above correspondence with the KdV soliton arises is associated with neglecting the third derivative of the Hubble parameter, for example. It is unlikely an exact solution to such an ODE can be found. Nonetheless, the approximation to slow roll improves as $\phi \to 0$. In other words, there is a value of the inflaton below which the correspondence is accurate.

Nonetheless, the solution (19) does represent an exact cosmological background driven by an effective potential of the form:

$$V(\phi) = H_0^2 \lambda^2 \mathrm{sech}^2\left(\frac{\lambda}{2}\phi\right)\left[3 - \frac{\lambda^2}{2}\tanh^2\left(\frac{\lambda}{2}\phi\right)\right], \qquad (25)$$

and the late-time attractor of this potential as $\phi \to \infty$ is the power-law solution, as discussed above. Equation (25) further illustrates how the solitonic interpretation of the background cosmology becomes more accurate at early times ($\phi \to 0$).

IV. DENSITY PERTURBATIONS IN THE CYCLIC UNIVERSE AND THE KDV SOLITON

During inflation, quantum fluctuations in the inflaton field become frozen on super-Hubble radius scales because the comoving Hubble scale decreases with time due to the rapid, accelerated expansion of the universe. However, the comoving Hubble radius can also decrease if the universe undergoes a phase of slow, decelerated *contraction* driven by a negative scalar field potential. This is the basis of the cyclic universe scenario. As shown in Ref. [11], the expression (11) for the spectral index is invariant under the duality $\epsilon \to 1/\epsilon$ and this implies there exists a one-to-one correspondence between inflationary and cyclic models that generate identical spectral indices.

This suggests that a similar analogy between gravitational and solitonic physics may be established for the cyclic universe scenario. Indeed, in the Hamilton-Jacobi formalism of the cosmological Friedmann equations (9), the definition of the Hubble parameter $H = \dot{a}/a$ in terms of the scale factor a implies that

$$a'H' = -\frac{1}{2}aH. \qquad (26)$$

Integrating Eq. (26) then yields the dependence of the scale factor on the scalar field in terms of the quadrature

$$a(\phi) = a_0 \exp\left[-\frac{1}{2}\int^{\phi} d\phi \frac{H}{H'}\right], \qquad (27)$$

where a_0 is an arbitrary constant.

However, Eq. (26) is invariant under the simultaneous interchange $H(\phi) \leftrightarrow a(\phi)$. If we therefore consider a "dual" cosmology where the Hubble parameter is given by $\tilde{H}(\phi) = a(\phi)$, Eq. (27) implies that the dual scale factor is given by the quadrature

$$\tilde{a}(\phi) = \tilde{a}_0 \exp\left[-\frac{1}{2}\int^{\phi} d\phi \frac{a}{a'}\right]. \qquad (28)$$

Since the seed cosmology $\{H(\phi), a(\phi)\}$ itself satisfies Eq. (26), the duality between the two scenarios is given by the simultaneous interchange of the Hubble parameters and scale factors of the two scenarios when all parameters are expressed as functions of the scalar field [12]:

$$\tilde{a}(\phi) = H(\phi), \qquad \tilde{H}(\phi) = a(\phi). \qquad (29)$$

It is straightforward to verify that under this duality, the Hubble slow-roll parameter indeed transforms as $\tilde{\epsilon} = 1/\epsilon$.

As a result of this duality, the analysis of Sec. III applies directly to the cyclic universe. We conclude, therefore, that there exists a one-to-one correspondence between solutions to the KdV equation and the respective cyclic cosmological models when the spectral index is constant. For the cyclic model that is dual to Eq. (19), the speed of the soliton is once more determined by the spectral index, whereas the amplitude of the soliton is now proportional to the square of the cosmic scale factor.

V. MODIFIED COSMOLOGY

In recent years, considerable interest has focused on cosmological dynamics arising from modified gravity theories. This is motivated in part by open questions in early universe cosmology, such as the singularity problem, as well as providing alternative scenarios to dark energy models. At a phenomenological level, such modifications can be quantified by altering the standard form of the Friedmann equation such that

$$3H^2 = \rho L^2(\rho), \qquad (30)$$

where $L = L(\rho)$ is a given function of the energy density and is determined by the specific model in question.

Of particular interest in such scenarios are scaling (attractor) solutions, since these enable the generic asymptotic behavior of a cosmological model to be better understood. Scaling solutions are characterized by the property that the energy densities of the component matter fields scale at the same rate as the universe expands (contracts). Such solutions were classified in Ref. [13] under the assumption that the matter content of the universe is comprised of a self-interacting scalar field with potential $V(\phi)$ and a barotropic fluid with an adiabatic index γ. By introducing a parameter

$$\lambda \equiv -\frac{1}{L}\frac{V'}{V}, \qquad (31)$$

it was found that $\lambda =$ constant is a necessary condition for a scaling solution to exist. In that case, there exists an attractor solution for $\lambda^2 > 3\gamma$, where the relative contribution of the scalar field energy density to the total density of the universe is $\Omega_\phi = 3\gamma/\lambda^2$. There exists a second stable solution if $\lambda^2 < 6$ where the scalar field dominates the fluid. It was further shown that these solutions exist if the condition

$$\rho\frac{\rho''}{\rho'^2} - 1 = \rho\frac{d\ln[L(\rho)]}{d\rho}, \qquad (32)$$

is satisfied [13].

In this section, we investigate the conditions that allow for scaling solutions in modified gravity to be interpreted as KdV-type solitons. We proceed by rewriting Eq. (32) in the form

$$\frac{\rho''}{\rho} - \frac{3}{2}\frac{\rho'^2}{\rho^2} = \frac{\rho'^2}{\rho^2}\left[\rho\frac{d\ln L}{d\rho} - \frac{1}{2}\right]. \quad (33)$$

Defining a new variable $\sigma = \sigma(\phi)$:

$$\rho = 4\rho_0\sigma', \quad (34)$$

where ρ_0 is an arbitrary constant, then transforms Eq. (33) into a Schwarzian differential equation:

$$S[\sigma(\phi)] = \left(\frac{\sigma''}{\sigma'}\right)^2\left[\rho\frac{d\ln L}{d\rho} - \frac{1}{2}\right], \quad (35)$$

where the dependence of the square bracket on σ is implicit.

We now look for particular solutions to Eq. (35) that satisfy the condition that the Schwarzian of σ is constant:

$$S[\sigma(\phi)] = -\frac{\lambda^2}{2}. \quad (36)$$

The discussion of Sec. III implies that the energy density of the universe will satisfy the KdV equation if $\sigma' = (\lambda^2/4) - \sigma^2$. As we saw above, there are two possible solutions when $\lambda^2 > 0$, depending on whether $\sigma' > 0$ or $\sigma' < 0$. (The product $\rho_0\sigma'$ is assumed implicitly to be positive definite.)

Equations (35) and (36) imply that

$$\left(\frac{\sigma''}{\sigma'}\right)^2\left[\rho\frac{d\ln L}{d\rho} - \frac{1}{2}\right] = -\frac{\lambda^2}{2}. \quad (37)$$

Since $\sigma'' = -2\sigma\sigma'$, it follows that

$$\left(\frac{\sigma''}{\sigma'}\right)^2 = \lambda^2 - \frac{\rho}{\rho_0}. \quad (38)$$

Consequently, Eq. (37) simplifies to the first-order ODE

$$\frac{d\ln L}{d\rho} = -\frac{1}{2}\frac{1}{\rho_0\lambda^2 - \rho}, \quad (39)$$

and integrating yields the solution

$$L = \left(1 - \frac{\rho}{\rho_0\lambda^2}\right)^{1/2}, \quad (40)$$

where we have chosen the appropriate branch of the general solution and the constant of integration to ensure that the standard relativistic cosmology is recovered in the low-energy limit $\rho \ll \rho_0\lambda^2$. The modified Friedmann equation (30) is therefore given by

$$3H^2 = \rho\left(1 - \frac{\rho}{\rho_0\lambda^2}\right). \quad (41)$$

Finally, the results of Sec. III can be carried over to deduce that the corresponding scaling solutions when expressed in terms of the energy density are given by the KdV wave solutions

$$\rho = \rho_0\lambda^2\text{sech}^2(\lambda\phi/2), \quad (42)$$

when $\sigma' > 0$ ($\rho_0 > 0$) and

$$\rho = -\rho_0\lambda^2\text{cosech}^2(\lambda\phi/2), \quad (43)$$

when $\sigma' < 0$ ($\rho_0 < 0$).

The Friedmann equation (41) arises in a number of cosmological models that are directly motivated by quantum gravity considerations. When $\rho_0 < 0$, the model corresponds to the Randall-Sundrum braneworld scenario [14], where our observable universe is interpreted as a codimension one-brane embedded in five-dimensional, Z_2 symmetric anti–de Sitter space. The coefficient $\rho_0\lambda^2$ is determined by the tension of the brane [14]. On the other hand, the case where $\rho_0 > 0$ results in the Friedmann equation for the Shtanov-Sahni bouncing braneworld [15]. In this model, the universe is again interpreted as a one-brane embedded in a five-dimensional spacetime sourced by a bulk cosmological constant, but the extra fifth dimension is now assumed to be timelike.

Furthermore, the Friedmann equation (41) arises generically in loop quantum cosmological (LQC) scenarios when $\rho_0 > 0$ [16]. In LQC models, the quadratic corrections in the energy density originate from nonperturbative quantum geometric corrections and become important at high energy scales. Indeed, in such a framework, $\rho_0\lambda^2$ determines a critical density

$$\rho_0\lambda^2 = \rho_{\text{crit}} = \frac{\sqrt{3}}{16\pi^2\gamma^3}\rho_{\text{Pl}}, \quad (44)$$

where ρ_{Pl} is the Planck density and $\gamma \approx 0.2375$ is the Immirzi parameter [16].

In effect, therefore, by focusing on the KdV equation we have arrived at three different cosmological models that are all inspired by quantum gravity effects. The corresponding scaling solutions (42) and (43) are those found previously in Refs. [13,17]. We may now interpret these solutions as wave solutions of the KdV equation. The nonsingular soliton solution reflects the nonsingular nature of the Shtanov-Sahni and LQC scaling solutions. Due to the nature of the quantum corrections, the universe collapses from infinity ($\phi \to -\infty$), undergoes a nonsingular bounce at $\rho = \rho_{\text{crit}}$ ($\phi = 0$) and then expands to infinity ($\phi \to +\infty$). From the point of view of a stationary laboratory "observer," such dynamics would be equivalent to the time dependence of the soliton amplitude as the wave propagates (modulo the appropriate relationship between cosmic and laboratory times). The cosmic bounce corresponds to the passing of the peak of the wave. In the LQC scenario, the "speed" of the wave is parametrized by the fractional energy density of the scalar field and the barotropic index of the fluid, $\lambda^2 = 3\gamma/\Omega_\phi$.

VI. DISCUSSION

In the present work, it has been shown for the first time that the KdV equation arises in a number of important cosmological scenarios, including the inflationary universe, the cyclic universe, loop quantum cosmology and braneworld cosmology. In each model, cosmological solutions can be reinterpreted as wavelike solutions to the KdV equation, and this allows for analogies to be drawn between cosmic dynamics and wave propagation.

For example, in the inflationary scenario, we have found that the ODE determining the spectral index of the density perturbation spectrum generated during single-field, slow-roll inflation is closely connected to both the Schwarzian derivative operator and the KdV wave equation. In principle, this allows for the full family of inflationary models that generate a constant spectral index to be classified in a very straightforward manner in terms of solutions to the KdV equation.

In the region of observational parameter space $r < 8(1 - n_s)$, a formal analogy was established between the nonsingular KdV soliton and the inflating universe. In such a correspondence, the scalar field plays the role of a wavelike coordinate, the speed of the soliton is determined by the value of the spectral index and the amplitude of the soliton is parametrized by the energy density of the universe. Due to the duality between the inflationary and cyclic universes, similar conclusions hold for the simplest version of the cyclic universe scenario, although in this case the amplitude of the soliton is related to the scale factor of the universe.

The general conditions for scaling solutions in a class of modified cosmological models sourced by a scalar field and a perfect fluid were considered. Requiring that the integral of the cosmic energy density (with respect to the scalar field) has a constant Schwarzian derivative led naturally to a modified Friedmann equation that arises generically in the Randall-Sundrum and Shtanov-Sahni braneworld models [14,15] and loop quantum cosmology scenarios [16]. In all cases, the corrections to the Friedmann equation are quadratic in the energy density. For such models, the scaling solutions may be interpreted as wave solutions to the KdV equation, where the cosmic energy density is again analogous to the soliton wave amplitude.

Finally, it is worth remarking that the KdV equation is closely related to the nonlinear Schrödinger equation. This equation admits solutions that determine the propagation of solitons in Bose-Einstein condensates [18,19]. A Bose-Einstein condensate is the ground state of a gas of N interacting bosons trapped by an external potential. In the limit where the interaction between the atoms is sufficiently weak, the mean-field approximation may be employed. In this case, the macroscopic wave function for the condensate, ψ, satisfies the Gross-Pitaevskii equation [20]:

$$i\hbar\partial_t\psi = -\frac{\hbar^2}{2m}\nabla^2\psi + V(\mathbf{x}, t)\psi + g|\psi|^2\psi, \tag{45}$$

where $V(\mathbf{x}, t)$ is the trapping potential and m is the mass of the atoms forming the condensate. The scattering coefficient is given by $g = 4\pi\hbar^2 Na/m$, where a is the (s-wave) scattering length.

By employing sufficiently anisotropic trapping potentials, it is possible to reduce the condensate to a quasi–one-dimensional configuration. Typically, the potential is given by $V(x) = \Omega^2 x^2/2$, where the trap strength $\Omega \ll 1$. To a first approximation, therefore, the potential can be ignored. In this limit, the condensate becomes homogeneous and the Gross-Pitaevskii equation is identical to the integrable nonlinear Schrödinger equation:

$$i\hbar\partial_t\psi = -\frac{\hbar^2}{2m}\partial_x^2\psi + g|\psi|^2\psi. \tag{46}$$

The defocusing nonlinear Schrödinger equation (46), where $g > 0$, supports dark soliton solutions [18]. A dark soliton is an envelope excitation characterized by a dip in the ambient density and a phase jump across the density minimum. Such solitons have been observed in a variety of Bose-Einstein condensates in recent years (for a recent review, see Ref. [21]). The solution is given by [19]

$$\begin{aligned} |\psi|^2 &= (1 - |\psi_{\rm ds}|^2), \\ |\psi_{\rm ds}|^2 &= \left(1 - \frac{v^2}{c^2}\right)\mathrm{sech}^2\left[\sqrt{1 - \left(\frac{v}{c}\right)^2}\frac{(x - vt)}{\xi}\right], \end{aligned} \tag{47}$$

where n is the background density, v is the speed of the soliton, $(x - vt)$ is its position and $c = \sqrt{ng/m}$ is the sound speed in the condensate. The spatial extent of the soliton is characterized by the healing length $\xi = \hbar/\sqrt{mng} = 1/\sqrt{4\pi na}$ In general, the speed of the soliton is bounded from above by the sound speed, $v < c$.

The density profile of the dark soliton (47) corresponds precisely to the energy density of the inflationary universe (19) and the LQC scaling solution (42). On dimensional grounds, we can make the identification

$$\frac{\phi}{m_{\rm p}} \leftrightarrow (x - vt)\sqrt{na}, \tag{48}$$

and view the scalar field as a wavelike coordinate. Modulo a constant of proportionality, the spectral index may then be identified with the speed of the soliton:

$$1 - n_s \leftrightarrow \left(1 - \frac{v^2}{c^2}\right), \tag{49}$$

whereas in the LQC model, the speed is proportional to the kinemetic parameters:

$$\frac{\gamma}{\Omega_\phi} \leftrightarrow \left(1 - \frac{v^2}{c^2}\right). \tag{50}$$

The deviation of the spectral index away from the scale-invariant perturbation spectrum is proportional to the speed of the soliton relative to the condensate sound speed. The maximal speed of the soliton is attained in the limit of the scale-invariant spectrum, $n_s \rightarrow 1$. For the LQC scaling solution, the maximal speed corresponds to an equation of state $p = -\rho$ for the fluid, which is the limit of a cosmological constant.

The above analogies are not intended to be precise, but they do nonetheless suggest that a new link between gravitational and nongravitational systems might be established through the KdV equation. It would be interesting to formalize such analogies further to establish kinematic correspondences between cosmology and condensed matter physics.

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