PHYSICAL REVIEW D **86,** 121703(R) (2012)

Possible higher order correction to the chiral vortical conductivity in a gauge field plasma

De-fu Hou

Institute of Particle Physics, Key Laboratory of QLP of MOE, Huazhong Normal University, Wuhan 430079, China

Hui Liu

Physics Department, Jinan University, Guangzhou 510632, China

Hai-cang Ren

Physics Department, The Rockefeller University, 1230 York Avenue, New York, New York 10021-6399, USA
Institute of Particle Physics, Key Laboratory of QLP of MOE, Huazhong Normal University, Wuhan 430079, China

(Received 23 October 2012; published 18 December 2012)

The two-loop contributions to the chiral vortical conductivity are considered. The Kubo formula together with the anomalous Ward identity of the axial vector current suggest that there may be a nonzero correction to the coefficient of the T^2 term of the conductivity.

DOI: 10.1103/PhysRevD.86.121703 PACS numbers: 12.38.Aw, 24.85.+p

The chiral magnetic effect and the chiral vortical effect (CVE) have been actively investigated in recent years. Because of the triangle anomaly, an external magnetic field and/or a fluid vorticity will induce an electric current, a baryon current, and an axial vector current in a relativistic plasma. These currents will lead to separations of electric charges, the baryon numbers, and chirality, which may be observed in the quark-gluon plasma created through heavy ion collisions [1,2]. To the order of the linear response, we have

$$\begin{aligned} \vec{J}_{\rm em} &= \sigma^B_{\rm em}\vec{B} + \sigma^V_{\rm em}\vec{\omega}, \\ \vec{J}_{\rm b} &= \sigma^B_{\rm b}\vec{B} + \sigma^V_{\rm b}\vec{\omega}, \\ \vec{J}_5 &= \sigma^B_5\vec{B} + \sigma^V_5\vec{\omega}, \end{aligned} \tag{1}$$

for the currents driven by the magnetic field and the fluid vorticity. The anomalous transport coefficients (the σ's) above have been explored from a field theoretic point of view and by the holographic method [1,3–10]. An important question regarding the former approach is whether these coefficients—like their origin, the triangle anomaly—are free from the higher-order corrections of coupling constants. In case of the chiral magnetic effect, the nonrenormalization of $\sigma^B_{\rm em}$ in the homogeneous limit of a static magnetic field has been established [11–13] and the classical expression [1]

$$\sigma^B_{\rm em} = N_c\sum_f q_f^2\frac{e^2\mu_5}{2\pi^2}, \tag{2}$$

holds to all orders of electromagnetic and SU(N_c) gauge coupling, where N_c is the number of colors, q_f is the charge number of each flavor, and μ_5 is the chemical potential of the axial charge. The same conclusion for chiral magnetic effect can also be reached following the argument in Ref. [14]. In this paper, we shall address the parallel issue for the chiral vortical conductivity σ^V_5 to see whether it is subject to higher-order corrections.

The anomalous transport coefficient σ^V_5 was first introduced in Ref. [7] where the anomalous Ward identity together with the second law of thermodynamics yields for a relativistic plasma with an axial charge chemical potential μ_5 the expression [15]

$$\sigma^V_5 = \frac{\mu_5^2}{2\pi^2}. \tag{3}$$

It was soon realized in Ref. [17] that the general solution to the thermodynamic condition employed in Ref. [7] is given by

$$\sigma^V_5 = \frac{\mu_5^2}{2\pi^2} + cT^2, \tag{4}$$

where c is an undetermined constant. Then came the Kubo formula [16] and the one-loop calculation in Ref. [18] which confirms the general structure (4) and yields $c = \frac{1}{12}$. This result is also confirmed by kinetic theories [19]. The authors of Refs. [18,20,21] related the T^2 term to the gravity anomaly and a recent analysis [22] from a geometric point of view within a general hydrodynamical framework suggests the nonrenormalization of the T^2 term. But a field theoretic aspect regarding the higher corrections remains murky.

In a recent work [14], the authors addressed the issue based on diagrammatic analysis. They generalized the Coleman-Hill theorem [23] to the stress tensor insertion and proved the nonrenormalization of σ^V_5 for a σ model. As with gauge theories, they argued that the nonrenormalization remains valid in the large N_c limit because of the structure of the anomaly. Upon a close examination of their argument for a gauge theory plasma at the two-loop level, we found a diagram that is not covered. We shall point out this diagram and compute its contribution to σ^V_5 below.

For the sake of clarity, we shall consider a QED plasma with the Lagrangian density

$$\mathcal{L} = -\frac{1}{4e_0^2}V^{\mu\nu}V_{\mu\nu} - i\bar{\psi}\gamma^\mu D_\mu\psi + \frac{1}{2}h^{\mu\nu}T_{\mu\nu} + A^\mu J_{5\mu}, \tag{5}$$

where $V_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$ is the electromagnetic field tensor with the gauge potential V_μ, the covariant derivative

$$D_\mu = \partial_\mu - iV_\mu, \tag{6}$$

and we have added couplings to an external axial vector field A^μ and a metric perturbation $h^{\mu\nu}$ with the axial vector current

$$J_{5\mu} = i\bar{\psi}\gamma_\mu\gamma_5\psi \tag{7}$$

and the stress tensor

$$\begin{aligned} T_{\mu\nu} = {} & V_\mu^\rho V_{\nu\rho} - \frac{1}{4}\eta_{\mu\nu}V^{\rho\lambda}V_{\rho\lambda} \\ & + \frac{1}{4}(-D_\mu\bar{\psi}\gamma_\nu\psi - D_\nu\bar{\psi}\gamma_\mu\psi) \\ & + \frac{1}{4}(\bar{\psi}\gamma_\mu D_\nu\psi + \bar{\psi}\gamma_\nu D_\mu\psi). \end{aligned} \tag{8}$$

We have set $A_\mu = h_{\mu\nu} = 0$ in the expression of $T_{\mu\nu}$ above. The anomalous Ward identity of $J_{5\mu}$ reads

$$\partial_\mu J_5^\mu = \frac{e_0^2}{16\pi^2\sqrt{-g}}\epsilon^{\mu\nu\rho\lambda}V_{\mu\nu}V_{\rho\lambda}, \tag{9}$$

with g being the determinant of the metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$.

Following Ref. [16], the chiral vortical conductivity σ_5^V is given by the correlators between the axial current density and the energy flux density as $\mathcal{G}_{ij}(Q) = \sigma_5^V\epsilon_{ijk}q_k$ in the limit $Q = (0, \vec{q}) \to 0$, where

$$\mathcal{G}_{ij}(Q) = -\int_0^\infty dt\int d\vec{r}e^{-i\vec{q}\cdot\vec{r}}\frac{\mathrm{Tr}\{e^{-\beta H}[J_{5i}(\vec{r},t), T_{0j}(0,0)]\}}{\mathrm{Tr}e^{-\beta H}}, \tag{10}$$

and can be evaluated perturbatively in terms of thermal diagrams, where H is the Hamiltonian corresponding to the Lagrangian density (5) at $A_\mu = h_{\mu\nu} = 0$. All two-loop diagrams are shown in Fig. 1. The one-particle reducible diagram [Fig. 1(g)] does not contribute since the loop attached to the axial vector vertex vanishes, as can be checked explicitly. We have the two-loop contribution to $\mathcal{G}_{ij}(Q)$,

$$\mathcal{G}_{ij}^{(2)}(Q) = \mathcal{G}_{ij}^{(a)}(Q) + \mathcal{G}_{ij}^{(b-f)}(Q), \tag{11}$$

with

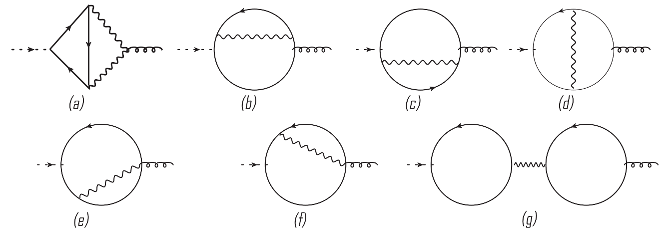

FIG. 1. The two-loop diagrams for the chiral vortical conductivity.

$$\mathcal{G}_{ij}^{(a)}(Q) = T\int\frac{d^3\vec{p}}{(2\pi)^2}\sum_{p_0}\Lambda_{i\alpha\beta}(P,Q)D_{\alpha,0\gamma}(P_-)D_{j\gamma,\beta}(P_+), \tag{12}$$

with $P_- = P - \frac{Q}{2}$, $P_+ = P + \frac{Q}{2}$, and

$$\mathcal{G}_{ij}^{(b-f)}(Q) = T\int\frac{d^3\vec{p}}{(2\pi)^2}\sum_{p_0}\Lambda_{ij\alpha\beta}(P,Q)D_{\alpha\beta}(P), \tag{13}$$

where $p_0 = 2n\pi T$ with $n = 0, \pm1, \pm2, \ldots$ is the Matsubara energy and the photon propagators $D_{\mu\nu}(P)$, $D_{\rho,\mu\nu}(P)$, and $D_{\mu\nu,\rho}(P)$ are given by

$$\begin{aligned} D_{\mu\nu}(P) &= \frac{1}{P^2}\Big[\delta_{\mu\nu} + (\kappa-1)\frac{P_\mu P_\nu}{P^2}\Big], \\ D_{\rho,\mu\nu}(P) &= -\frac{1}{P^2}(P_\mu\delta_{\rho\nu} - P_\nu\delta_{\rho\mu}), \\ D_{\mu\nu,\rho}(P) &= D_{\rho,\mu\nu}(-P), \end{aligned} \tag{14}$$

with $P^2 = \vec{p}^2 + p_0^2$ and κ being the gauge parameter. The momenta in Eq. (11) are all Euclidean with the metric $\delta_{\mu\nu}$. The amplitudes $\Lambda_{i\alpha\beta}(P,Q)$ and $\Lambda_{ij\alpha\beta}(P,Q)$ of Eqs. (12) and (13) are related to the anomalous triangle diagram $\Pi_{\mu\alpha\beta}(K_1,K_2)$, and the kernel of its metric perturbation $\Pi_{\mu\alpha\beta,\rho\lambda}(Q,K_1,K_2)$, depicted in Fig. 2, via

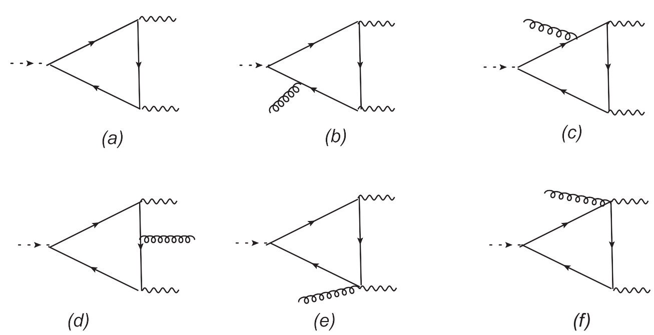

FIG. 2. The anomalous triangle and its metric perturbation. $\Pi_{\mu\alpha\beta}(K_1,K_2)$ in the text denotes the amputated part of (a) and $\Pi_{\mu\alpha\beta,\rho\lambda}(Q,K_1,K_2)$ denotes the sum of the amputated parts of (b)–(f), where (α, β) refer to the external photon lines with outgoing momenta (K_1, K_2), μ refers to the axial vector insertion with incoming momentum Q, and (ρ, λ) refers to the external graviton.

$$\Lambda_{i\alpha\beta}(P,Q)=\Pi_{i\alpha\beta}\left(P+\frac{Q}{2},-P+\frac{Q}{2}\right),$$
$$\Lambda_{ij\alpha\beta}(P,Q)=\Pi_{i\alpha\beta,0j}(Q,P,-P). \tag{15}$$

If there were no axial anomaly, the sum of all diagrams (a)–(f) in Fig. 1 would be of the order $O(q^2)$ in the limit $Q=(0,\vec{q})\to 0$ according to the Coleman-Hill-like argument employed in Ref. [14]. As to the contribution from the anomaly, following an elegant argument of Ref. [14], the sum of diagrams (b)–(f) in Fig. 2 couples only to the trace of the metric perturbation. Therefore the anomaly does not contribute to the diagrams (b)–(f) in Fig. 1 with the insertion of an off-diagonal component. The anomaly contribution to diagram (a) in Fig. 1, however, is not covered by the above argument and has to be examined separately.

The anomalous Ward identity (9) implies that

$$q_i\Lambda_{i\alpha\beta}(P,Q)=-\frac{e_0^2}{2\pi^2}\epsilon_{\alpha\beta\rho i}P^\rho q_i. \tag{16}$$

Taking the derivative with respect to momentum $\vec{q}$ on both sides, we derive

$$\Lambda_{i\alpha\beta}(P,Q)=-\frac{e_0^2}{2\pi^2}\epsilon_{\alpha\beta\rho i}P_\rho-q_j\frac{\partial}{\partial q_i}\Lambda_{j\alpha\beta}(P,Q). \tag{17}$$

In the absence of infrared divergence, we end up with a nonzero limit as $\vec{q}\to 0$,

$$\Lambda_{i\alpha\beta}(P,Q)\to-\frac{e_0^2}{2\pi^2}\epsilon_{\alpha\beta\rho i}P_\rho. \tag{18}$$

Inserting this nonzero limit into Eq. (12), we find the anomaly contribution

$$\mathcal{G}_{ij}^{\text{anom}}(Q)=-\frac{e_0^2T}{2\pi^2}\epsilon_{\alpha\beta\nu i}\int\frac{d^3\vec{p}}{(2\pi)^2}\times\sum_{p_0}P_\nu D_{\alpha,0\gamma}(P_-)D_{j\gamma,\beta}(P_+). \tag{19}$$

Dropping the terms beyond linear order in $\vec{q}$, we obtain

$$\begin{aligned}\mathcal{G}_{ij}^{\text{anom}}(Q)&=\frac{e_0^2T}{2\pi^2}\int\frac{d^3\vec{p}}{(2\pi)^2}\sum_{p_0}\frac{1}{(\vec{p}^2+p_0^2)^2}\\&\quad\cdot\left[-\frac{1}{2}\epsilon_{ikl}p_lq_kp_j+(\vec{p}^2+p_0^2)\epsilon_{ijk}p_k-\frac{1}{2}p_0^2\epsilon_{ijk}q_k\right]\\&=\sigma_5^{V(2)}\epsilon_{ijk}q_k,\end{aligned} \tag{20}$$

with $\sigma_5^{V(2)}$ being the two-loop contribution to the CVE coefficient, given by

$$\sigma_5^{V(2)}=\frac{e_0^2T}{4\pi^2}\sum_{p_0}\int\frac{d^3\vec{p}}{(2\pi)^3}\frac{\frac{1}{3}\vec{p}^2-p_0^2}{(\vec{p}^2+p_0^2)^2}. \tag{21}$$

In the last step, we have dropped the second term in the numerator of the integrand because it is odd in P, and have replaced p_lp_j by $\frac{1}{3}\vec{p}^2\delta_{lj}$. The integral of Eq. (21) can be calculated by dimensional regularization. We have $\sigma_5^{V(2)}=\lim_{d\to3}\sigma_{5,d}^{V(2)}$, with

$$\begin{aligned}\sigma_{5,d}^{V(2)}&=\frac{e_0^2T}{4\pi^2}\sum_{p_0}\int\frac{d^d\vec{p}}{(2\pi)^d}\frac{\frac{1}{d}p^2-p_0^2}{(p^2+p_0^2)^2}\\&=\frac{e_0^2T}{16\pi^2}\frac{(d-1)\omega_d}{2^d\pi^{d-1}\sin\frac{\pi d}{2}}\sum_{p_0}|p_0|^{d-2}\\&=\frac{e_0^2T^{d-1}}{32\pi^3}\frac{(d-1)\omega_d}{\sin\frac{\pi d}{2}}\zeta(2-d),\end{aligned} \tag{22}$$

where ω_d is the solid angle in d dimensions. Therefore,

$$\sigma_5^{V(2)}=\frac{e_0^2}{48\pi^2}T^2, \tag{23}$$

and the coefficient c of Eq. (4) takes the form

$$c=\frac{1}{12}+\frac{e_0^2}{48\pi^2}. \tag{24}$$

Because of the universality of the axial anomaly, the second term above is intact if the fermion number and the axial charge chemical potentials are switched on. In other words, the μ_5^2 of Eq. (4) is not renormalized by higher-order terms and our result is not in contradiction with the thermodynamic argument of Ref. [7].

To convince ourselves of the robustness of this result, we have also evaluated $\sigma_5^{V(2)}$ *à la* a Pauli-Villars-like regularization, which amounts to $\sigma_5^{V(2)}=\lim_{M_s\to\infty}\sigma_{5,M}^{V(2)}$, with

$$\sigma_{5,M}^{V(2)}=\frac{e_0^2T}{4\pi^2}\sum_{p_0}\int\frac{d^3\vec{p}}{(2\pi)^3}\left[\frac{\frac{1}{3}\vec{p}^2-p_0^2}{(\vec{p}^2+p_0^2)^2}-\sum_sC_s\frac{\frac{1}{3}\vec{p}^2-p_0^2}{(\vec{p}^2+p_0^2+M_s^2)^2}\right], \tag{25}$$

where the coefficients C_s are chosen to make the integral and the summation divergence-free. On writing

$$\sigma_{5,M}^{V(2)}=\frac{e_0^2}{4\pi^2}\left\{\int\frac{d^4\vec{P}}{(2\pi)^4}[\cdots]+\left(T\sum_{p_0}-\int_{-\infty}^{\infty}\frac{dp_0}{2\pi}\right)\int\frac{d^3\vec{p}}{(2\pi)^3}[\cdots]\right\}, \tag{26}$$

we have for the first term inside the bracket

$$\begin{aligned}\int\frac{d^4\vec{P}}{(2\pi)^4}[\cdots]&=\int\frac{d^4\vec{P}}{(2\pi)^4}\left(\frac{1}{3}\vec{P}^2-\frac{4}{3}p_0^2\right)\cdot\left[\frac{1}{(\vec{p}^2+p_0^2)^2}-\sum_sC_s\frac{1}{(\vec{p}^2+p_0^2+M_s^2)^2}\right]\\&=0,\end{aligned} \tag{27}$$

with $P^2=\vec{p}^2+p_0^2$ because of the four-dimensional rotational symmetry once the integral is made convergent by the regulators. As for the rest of the terms, following the standard treatment of the summation over p_0 in terms of a contour integral, we find

$$\lim_{M_s\to\infty}\left(T\sum_{p_0}-\int_{-\infty}^{\infty}\frac{dp_0}{2\pi}\right)\int\frac{d^3\vec{p}}{(2\pi)^3}[\cdots]$$

$$=\int\frac{d^3\vec{p}}{(2\pi)^3}\left[\frac{2}{3}\frac{e^{\frac{p}{T}}}{(e^{\frac{p}{T}}-1)^2}-\frac{1}{3p}\frac{1}{e^{\frac{p}{T}}+1}\right]=\frac{T^2}{12},\qquad(28)$$

which confirms Eq. (23).

Our analysis can be trivially generalized to a QCD-like non-Abelian gauge theory with N_c colors and N_f flavors. This amounts to replacing e_0^2 of Eq. (18) by $N_f\mathrm{tr}T^lT^{l'}g_0^2=\frac{1}{2}\delta^{ll'}N_fg_0^2$, with T^l being the SU(N_c) generator for quarks in the fundamental representation and g_0 the Yang-Mills coupling, and to summing Eq. (19) over adjoint gluons. On writing

$$\sigma_5^V=N_cN_f\left(\frac{\mu_5^2}{2\pi^2}+cT^2\right),\qquad(29)$$

we have

$$c=\frac{1}{12}+\frac{N_c^2-1}{2N_c}\frac{g_0^2}{48\pi^2},\qquad(30)$$

and the second term is not suppressed in the large N_c limit for a fixed 't Hooft coupling $N_cg_0^2$. This makes the strong 't Hooft coupling limit nontrivial, an issue that may be addressed by the holographic principle.

One possible loophole with the above analysis concerns the generalization of the Coleman-Hill theorem to the stress tensor insertion. In the case of the vector or axial current insertion to a diagram with only external gauge boson lines, the transversality of the diagram post-insertion can be established algebraically prior to integrating the loop momenta [24] (formally for the axial current case). We find that this is not obvious with the stress tensor insertion corresponding to one-loop diagrams with one boson line of each diagram of Fig. 1 cut open. This may be attributed to the fact that the external lines of these diagrams do not carry the conserved charges but do carry energies and momenta. If there is no complication with the generalization of the Coleman-Hill theorem, we do find a two-loop term of the chiral vortical coefficient given by Eq. (23). In any case, it would be interesting to verify or disprove this result through an explicit calculation of all two-loop diagrams of Fig. 1. Alternatively, the Matsubara formulation of the correlator in Eq. (10) can also be evaluated nonperturbatively on a lattice at $\mu_5=0$ without running into sign problems.

We are grateful to S. Golkar for email communications on their paper [14] and valuable comments. We are also indebted to him for informing us that they reached a similar conclusion in their forthcoming version of Ref. [14] when this work is finished. D. Hou would like to thank S. Lin for valuable discussions. An email communication from K. Landsteiner is warmly acknowledged. This work is supported partly by NSFC under Grants No. 10975060, No. 11135011, No. 11221504 and No. 10947002.

[1] K. Fukushima, D. E. Khareev, and H. J. Warringa, Phys. Rev. D **78**, 074033 (2008).
[2] D. E. Khareev and D. T. Son, Phys. Rev. Lett. **106**, 062301 (2011).
[3] H. U. Yee, J. High Energy Phys. 11 (2009) 085.
[4] J. Erdmenger, M. Haack, M. Kaminski, and A. Yarom, J. High Energy Phys. 01 (2009) 055.
[5] N. Banerjee, J. Bhattacharya, S. Bhattacharyya, S. Dutta, R. Loganayagam, and P. Surowka, J. High Energy Phys. 01 (2011) 094.
[6] M. Torabian and H. U. Yee, J. High Energy Phys. 08 (2009) 020.
[7] D. T. Son and P. Surowka, Phys. Rev. Lett. **103**, 191601 (2009).
[8] T. Kalaydzhyan and I Kirsch, Phys. Rev. Lett. **106**, 211601 (2011).
[9] S. Lin, Phys. Rev. D **85**, 045015 (2012).
[10] V. P. Nair, R. Ray, and S. Roy, arXiv:1112.4022.
[11] V. Rubakov, arXiv:1005.1888.
[12] D. K. Hong, Phys. Lett. B **699**, 305 (2011).
[13] D. Hou, H. Liu, and H.-c. Ren, J. High Energy Phys. 05 (2011) 046.
[14] S. Golkar and D. T. Son, arXiv:1207.5806.
[15] The vertical conductivity in Ref. [7] is defined in the Landau frame, given by $\frac{\mu_5^2}{2\pi^2}(1-\frac{2}{3}\frac{n_5\mu_5}{u+p})$ with n_5, u, and p being the axial-charge density, the energy density and the pressure of the fluid, respectively. The second term inside the parentheses represents the contribution from the energy flow and is subtracted out in this paper. See Ref. [16] for clarifications.
[16] I. Amado, K. Landsteiner, and F. Pena-Benitez, J. High Energy Phys. 05 (2011) 081.
[17] Y. Neiman and Y. Oz, J. High Energy Phys. 03 (2011) 023.
[18] K. Landsteiner, E. Megias, and F. Pena-Benitez, Phys. Rev. Lett. **107**, 021601 (2011).
[19] J. H. Gao, Z. T. Liang, S. Pu, Q. Wang, and X. N. Wang, arXiv:1203.0725.
[20] K. Landsteiner, E. Megias, L. Melgar, and F. Pena-Benitez, J. High Energy Phys. 09 (2011) 121.
[21] S. Chapman, Y. Neiman, and Y. Oz, J. High Energy Phys. 07 (2012) 128.
[22] K. Jensen, R. Loganayagam, and A. Yarom, arXiv:1207.5824.
[23] S. Coleman and B. Hill, Phys. Lett. **159B**, 184 (1985).
[24] J. Bjorken and S. Drell, *Relativistic Quantum Fields* (McGraw-Hill, New York, 1965), Sec. 17.9, p. 197.