

## On the metric operator for the imaginary cubic oscillator

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We show that the eigenvectors of the  $\mathcal{PT}$ -symmetric imaginary cubic oscillator are complete, but do not form a Riesz basis. This results in the existence of a bounded metric operator having intrinsic singularity reflected in the inevitable unboundedness of the inverse. Moreover, the existence of nontrivial pseudospectrum is observed. In other words, there is no quantum-mechanical Hamiltonian associated with it via bounded and boundedly invertible similarity transformations. These results open new directions in physical interpretation of  $\mathcal{PT}$ -symmetric models with intrinsically singular metric, since their properties are essentially different with respect to self-adjoint Hamiltonians, for instance, due to spectral instabilities.

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### I. INTRODUCTION

At the turn of the millennium, Bender *et al.* came up with the idea to extend quantum mechanics by considering Hamiltonians that are invariant under a space-time reflection  $\mathcal{PT}$  rather than being Hermitian [1,2]. The development of the so-called  $\mathcal{PT}$ -symmetric quantum mechanics was in fact initiated in these papers by considering a prominent Hamiltonian

$$H = -\frac{d^2}{dx^2} + ix^3. \quad (1)$$

While this operator is manifestly non-Hermitian, it is invariant under a simultaneous space reflection  $\mathcal{P}$  ( $x \mapsto -x$ ) and time reversal  $\mathcal{T}$  (complex conjugation). Moreover, numerical studies suggested that the spectrum of  $H$  is *real*, which was later proved in Ref. [3]. The Hamiltonian (1) can be considered as a prototype of many other examples of  $\mathcal{PT}$ -symmetric Hamiltonians that have been so far studied in the still growing literature (see Ref. [4] and references therein).  $\mathcal{PT}$ -symmetric models found applications in various domains of physics—namely in optics [5], solid state [6], Bose-Einstein condensates [7], LRC circuits [8], superconductivity [9], electromagnetism [10], and reflectionless scattering [11].

It is commonly accepted that a *quantum-mechanical* interpretation of  $\mathcal{PT}$  symmetry must be implemented through a similarity transformation  $\Omega$ , i.e.,

$$h := \Omega H \Omega^{-1}, \quad (2)$$

where  $h$  is a self-adjoint operator, i.e.,  $h = h^\dagger$ . This intertwining relation is closely related to the *quasi-Hermiticity* [12,13]

$$\Theta H = H^\dagger \Theta, \quad (3)$$

where  $\Theta$  is a positive operator often called a *metric operator* (its special variant  $\mathcal{PC}$  was suggested in Refs. [2,14]). Hamiltonian  $H$  with property (3) is called *quasi-Hermitian* because it is actually Hermitian with respect to the

modified inner product  $\langle \cdot, \Theta \cdot \rangle$ . The relation between  $\Omega$  and  $\Theta$  is the decomposition of a positive operator  $\Theta = \Omega^\dagger \Omega$ . The essential idea is that a non-Hermitian  $H$  can be viewed as an alternative representation of a Hermitian operator  $h$ .

The advantage of the above-described representation (2) stems from the observation that the Hermitian counterpart  $h$  for a differential albeit non-Hermitian operator  $H$  has typically a nonlocal and very complicated structure. This was demonstrated for a class of operators with non-Hermitian (not necessarily  $\mathcal{PT}$ -symmetric) point interactions in Ref. [15], where, in addition, explicit formulas for the similarity transformation  $\Omega$ , metric operator  $\Theta$ ,  $\mathcal{C}$  operator, and similar self-adjoint operator  $h$  were presented in a closed form. Nevertheless, the non-Hermiticity and nonlocality are not always equivalent in the described sense [16,17].

Partly motivated by the relevance of the cubic interaction in quantum field theory, the problem of similarity of the Hamiltonian (1) to a self-adjoint operator was investigated in several works [14,18]. However, due to the complexity of the task, the approach used in these papers was necessarily formal, based on developing the metric into an infinite series composed of unbounded operators. There has been no proof of the quasi-Hermiticity of the imaginary cubic oscillator so far. The objective of the present paper is to establish the following intrinsic facts about the metric of (1):

- (1) *There exists a bounded metric.* That is, operator (1) is quasi-Hermitian in the sense of (3) with bounded  $\Theta$ .
- (2) *No bounded metric with bounded inverse exists.* That is, any metric operator for (1) necessarily possesses an inevitable singularity.

We have chosen the prominent Hamiltonian (1) to prove the negative result 2 just because the  $ix^3$  potential is considered as the *fons et origo* of  $\mathcal{PT}$ -symmetric quantum mechanics [1,2]. However, the absence of a bounded or boundedly invertible metric is by far not restricted to the Hamiltonian (1) only. For instance, the method of the

present paper also applies to an equally extensively studied  $x^2 + ix^3$  potential and many others, see Eq. (17) and the surrounding text.

Our results have important consequences for the physical interpretation of the  $\mathcal{PT}$ -symmetric Hamiltonians. If the metric happens to be singular (i.e., unbounded, not invertible or unboundedly invertible), the quantum-mechanical interpretation using the similarity transformation is lost. Indeed, the eigenvectors, despite possibly being complete, do not form a “good” basis, i.e., an unconditional (Riesz) basis. The spectrum of such highly non-self-adjoint operators does not contain sufficient information about the system, and in addition to the reality and (algebraic) simplicity of the spectrum, more involved spectral-theoretic properties (such as basicity, pseudospectrum, etc.) must be taken into account.

Our result about the singularity of any metric may seem negative at first glance. However, we believe that in the same way as the exceptional points represent one of the most interesting configurations, where important physical phenomena arise, the established intrinsic singularities in the metric operator are precisely the point where new developments of the physics of  $\mathcal{PT}$ -symmetric models may originate.

This paper is organized as follows. In Sec. II we emphasize some aspects of unbounded operators and defects of quasi-Hermiticity based on singular metrics. In Sec. III we recall known facts about the imaginary cubic oscillator and perform our proofs of the new properties regarding the metric operator. Finally, in Sec. IV we refer to some open problems and comment on possible extensions of our results.

## II. INFINITE-DIMENSIONAL SUBTLETIES

While the concepts of similarity to a self-adjoint operator and quasi-Hermiticity work smoothly if the dimension of the underlying Hilbert space is finite, i.e., for matrices, essential difficulties may appear in the infinite-dimensional spaces. The reason is obviously in the unboundedness of operators, which unavoidably restricts their domains of definitions to a nontrivial subset of the Hilbert space. Therefore, the sense in which equalities (2) and (3) hold must be carefully explained. We focus on the metric operator further; nonetheless, the similarity transformation may be discussed along the same lines.

Relation (3) is an operator equality and as such it requires that the operator domains  $\text{Dom}(\Theta H)$  and  $\text{Dom}(H^\dagger \Theta)$  are equal in addition to the validity of the corresponding vector identity  $\Theta H \psi = H^\dagger \Theta \psi$  for every  $\psi \in \text{Dom}(\Theta H) \cap \text{Dom}(H^\dagger \Theta)$ . Problems arise if the involved operators are unbounded, since one of the operator domains of the products or their intersection might be reduced to a single element 0. To avoid such pathological situations, it is usually assumed that the metric operator  $\Theta$  is bounded. Then the above requirements reduce to the mapping property  $\Theta[\text{Dom}(H^\dagger)] \subset \text{Dom}(H)$  and the quasi-Hermitian identity should hold for every  $\psi \in \text{Dom}(H)$ .

If, in addition to the boundedness,  $\Theta$  is boundedly invertible, then some fundamental and extremely useful properties of self-adjoint operators are valid for  $H$  as well: real spectrum, spectral decomposition, spectral stability with respect to perturbations, unitary evolution (in a topologically equivalent Hilbert space), etc. However, if the metric becomes singular, none of the mentioned properties is guaranteed by the validity of (3). As a matter of fact, as we demonstrate in this paper, the imaginary cubic oscillator and many other  $\mathcal{PT}$ -symmetric Hamiltonians, despite possessing real spectra, exhibit pathological features with respect to self-adjoint behavior due to the intrinsic singularities of the metric (and therefore also in  $\mathcal{C}$  operators and similarity transformations). Let us demonstrate the defects of the theories with singular metrics in the following subsections.

### A. Spectrum

Let  $\mathcal{H}$  be a complex Hilbert space. The *spectrum* is meaningfully defined only for closed operators, i.e., those operators  $H$  for which the elements  $\{\psi, H\psi\}$  with  $\psi \in \text{Dom}(H)$  form a closed linear subspace of  $\mathcal{H} \times \mathcal{H}$ . If  $\mathcal{H}$  were finite dimensional then the spectrum of  $H$ ,  $\sigma(H)$ , would be exhausted by eigenvalues, i.e., those complex numbers  $\lambda$  for which  $H - \lambda$  is not injective. In general, however, there are additional parts of spectra composed by those  $\lambda$  which are not eigenvalues but  $H - \lambda: \text{Dom}(H) \rightarrow \mathcal{H}$  is not bijective: depending on whether the range  $\text{Ran}(H - \lambda)$  is dense in  $\mathcal{H}$  or not, one speaks about the *continuous* or *residual* spectrum, respectively. In other words, the complement of the spectrum of  $H$ , called the *resolvent set* of  $H$ ,  $\rho(H)$ , is composed of all the complex numbers  $z$  for which the resolvent operator  $(H - z)^{-1}: \mathcal{H} \rightarrow \mathcal{H}$  exists and is bounded.

It is an important property of self-adjoint operators that their (total) spectrum is always nonempty, real, and that the residual spectrum is empty. For non-self-adjoint operators, however, the spectrum can be empty or cover the whole complex plane, see, e.g., Refs. [16,19].

Let us demonstrate how singular metrics lead to pathological situations as regards spectral properties. Let  $H$  be an operator with purely discrete spectrum (i.e., just isolated eigenvalues with finite multiplicities) and assume that the similarity relation (2) holds with unbounded  $\Omega^{-1}$ . Then  $\text{Ran}(h - \lambda) \subset \text{Ran}(\Omega) \neq \mathcal{H}$  for every  $\lambda \in \mathbb{C}$ . Consequently, the whole complex plane except for the set of eigenvalues of  $H$  belongs to the continuous spectrum of  $h$ . Summing up, *the continuous spectrum is not preserved by unbounded similarity transformations*. It is a striking phenomenon since the continuous part of spectrum contains physical energies corresponding to scattering/propagating states.

A similar argument shows that unbounded  $\Theta$  satisfying (3) with  $\text{Dom}(\Theta) \supset \text{Dom}(H)$  and  $\Theta[\text{Dom}(H)] \subset \text{Dom}(H^\dagger)$  cannot exist for closed operators  $H$  with a physically

reasonable property  $\sigma(H) \neq \mathbb{C}$ . In this way, one can also show that the  $\mathcal{C}$  operator of Kuzhel [20] for (1) cannot exist.

### B. Eigenbasis

Eigenfunctions of self-adjoint operators corresponding to different eigenvalues are mutually orthogonal. Furthermore, the set of all eigenfunctions  $\{\psi_n\}_{n=1}^{\infty}$  of a self-adjoint operator with purely discrete spectrum can be normalized in such a way that it forms a *complete orthonormal family* in the Hilbert space  $\mathcal{H}$ . Recall that the completeness means that the orthogonal complement in  $\mathcal{H}$  of the linear span of the family consists only of the zero function. A necessary and sufficient condition for completeness of an orthonormal family  $\{\psi_n\}_{n=1}^{\infty}$  is the validity of the Parseval equality

$$\forall \psi \in \mathcal{H}, \quad \sum_{n=1}^{\infty} |\langle \psi_n, \psi \rangle|^2 = \|\psi\|^2. \quad (4)$$

In this case we also have the unique expansion

$$\forall \psi \in \mathcal{H}, \quad \psi = \sum_{n=1}^{\infty} c_n \psi_n. \quad (5)$$

That is,  $\{\psi_n\}_{n=1}^{\infty}$  is a *basis* in  $\mathcal{H}$ .

Eigenfunctions of non-Hermitian operators are typically not orthogonal. Even worse, they may not form a basis or even not a complete family. In this respect, it is absolutely essential to stress that the completeness of a nonorthonormal family  $\{\psi_n\}_{n=1}^{\infty}$  does not imply that any  $\psi \in \mathcal{H}$  admits a unique expansion (5); see, e.g., Ref. [21] for further details.

The notion of ‘‘eigenbasis’’ is so important in quantum mechanics that one needs to have a replacement for (4) in the case of eigenfunctions of non-Hermitian operators. This is provided by the notion of *Riesz basis*

$$\forall \psi \in \mathcal{H}, \quad C^{-1} \|\psi\|^2 \leq \sum_{n=1}^{\infty} |\langle \psi_n, \psi \rangle|^2 \leq C \|\psi\|^2 \quad (6)$$

with a positive constant  $C$  independent of  $\psi$ . Eigenfunctions of an operator  $H$  with purely discrete spectrum form a Riesz basis if and only if  $H$  is quasi-Hermitian (3) with bounded and boundedly invertible metric  $\Theta$ .

As in the case of spectrum, *Riesz-basicty property is not preserved by unbounded transformations*. As a matter of fact, it is the objective of the present paper to show that the eigenfunctions of (1) do not form a Riesz basis so that the metric  $\Theta$  is necessarily singular. Any claim of the type ‘‘(1) is similar to a self-adjoint operator’’ is thus necessarily of doubtful usefulness for physics, since  $H$  and  $h$  appearing in (2) would have very different basicity properties.

### C. Pseudospectrum

The notion of pseudospectra arose as a result of the realization that several pathological properties of highly non-Hermitian operators were closely related. We refer to the classical monographs by Trefethen and Embree [22]

and Davies [21] for more information on the subject, physical and numerical applications, and many references.

Given a positive number  $\varepsilon$ , we define the *pseudo-spectrum* of  $H$  by

$$\sigma_{\varepsilon}(H) := \{z \in \mathbb{C} \mid \|(H - z)^{-1}\| > \varepsilon^{-1}\}, \quad (7)$$

with the convention that  $\|(H - z)^{-1}\| = \infty$  for  $z \in \sigma(H)$ . The pseudospectrum always contains an  $\varepsilon$ -neighborhood of the spectrum:

$$\{z \in \mathbb{C} \mid \text{dist}(z, \sigma(H)) < \varepsilon\} \subset \sigma_{\varepsilon}(H). \quad (8)$$

Since equality holds here if  $H$  is self-adjoint (or more generally normal), it follows that the notion of pseudospectra becomes trivial for such operators. On the other hand, if  $H$  is ‘‘highly non-self-adjoint,’’ the pseudospectrum  $\sigma_{\varepsilon}(H)$  is typically ‘‘much larger’’ than the  $\varepsilon$ -neighborhood of the spectrum.

For non-Hermitian operators the pseudospectra are much more reliable objects than the spectrum itself. Probably the strongest support for this claim is due to the phenomenon of *spectral instability*: very small perturbations may drastically change the spectrum of a non-Hermitian operator. For instance, new complex eigenvalues can appear very far from the original ones. On the other hand, perturbations whose norm is less than  $\varepsilon$  still lie inside  $\sigma_{\varepsilon}(H)$ . These effects were extensively studied in numerics, hydrodynamics, optics, etc. (see Ref. [22] and references therein).

Of course, such pathological situations do not occur for self-adjoint operators whose spectrum is changed at most by the norm of the perturbation. It is also impossible for operators similar to self-adjoint operators by bounded and boundedly invertible similarity transformations. On the other hand, *the pseudospectrum is not preserved by unbounded transformations* (we refer to Ref. [23] for a warning discussion of the shifted harmonic oscillator in this context). The pseudospectrum thus represents a useful test whether a given non-Hermitian operator can be similar to a self-adjoint one via a physically reasonable transformation. In this paper we show that the pseudospectrum of (1) is highly nontrivial.

### D. Singular metric?

The observations made in previous subsections constitute strong support for our belief that the singular metrics are not relevant objects for physical interpretation of non-Hermitian Hamiltonians, since they yield only singular similarity transformations. However, putting it differently, singular metrics necessarily lead to fundamentally new physics, since the transformed operators exhibit completely different properties.

In this context we feel it necessary to mention that there exists a recent attempt of Mostafazadeh [24] reproducing equivalently the original idea of Kretschmer and Szymanowski [25] to include singular metric operators into the notion of quasi-Hermiticity. It involves a construction of a self-adjoint operator to which the original non-Hermitian

operator with purely discrete real spectrum is similar “at any cost.” Analogous ideas for unbounded  $\mathcal{C}$  operators can be found in Ref. [26]. However, any such strategy has important drawbacks that cannot be avoided. The problem with a singular metric is mentioned already in Ref. [13], where an example of an operator possessing a bounded metric operator without a bounded inverse and having *non-real spectrum* was constructed. As a corollary, Diedonné states, “in spite of the quasi-Hermiticity (without bounded inverse of  $\Theta$ ), there is for instance no hope of building functional calculus that would follow more or less the same pattern as the functional calculus of self-adjoint operators.”

The drawbacks are that the aforementioned non-self-adjoint pathologies of  $H$  are completely ignored when analyzing the “similar” self-adjoint operator  $h$  instead. This can be illustrated already for two-by-two matrices: a Jordan-block matrix  $H$  and a diagonal matrix  $h$  with the same real eigenvalues. Although the matrices possess the same (real) spectrum, their respective properties are very different, particularly the basicity properties of eigenvectors and spectral stability with respect to small perturbations. But the construction of Refs. [24,25], when used in finite dimension, simply means that the authors disregard the Jordan-block structure of the non-Hermitian matrix  $H$  and associate to it just the diagonal matrix  $h$  with the same eigenvalues. The metric operator and “similarity transformation” are noninvertible in this case. However, equality (3) and a weaker variant of (2), i.e.,  $\Omega H = h\Omega$ , do hold. Stating that  $h$  should in any reasonable sense represent  $H$  is obviously very doubtful, since, for instance, all the physics of exceptional points would be omitted.

In infinite-dimensional spaces the situation is even more complex, since another possibility of a singularity of metric exists, namely, the unboundedness of the inverse. Although this may seem to be a minor issue or only a technical problem of infinite dimension, such an interpretation is very misleading. The pathological properties of non-self-adjoint  $H$  with only unboundedly invertible metric may be much more serious than the existence of finite-dimensional Jordan blocks, i.e., usual exceptional points. In the latter case, although the metric cannot be invertible, the eigenvectors together with generalized ones may form a Riesz basis. In other words, except for a finite-dimensional subspace,  $H$  is similar to a self-adjoint operator. Therefore, a version of the spectral decomposition (generalized Jordan form) may be available and the spectrum of  $H$  may be stable with respect to small perturbations. This is not the case of the imaginary cubic oscillator where there is no Riesz basis of eigenvectors and no spectral stability: complex eigenvalues may appear very far from the unperturbed real ones even though the norm of the perturbation is arbitrarily small.

### III. IMAGINARY CUBIC OSCILLATOR

Let us begin by properly introducing the Hamiltonian (1) as a closed realization in the Hilbert space  $L^2(\mathbb{R})$ . We

consider the *maximal realization* of the differential expression (1) by taking for the operator domain of  $H$  the maximal domain

$$\text{Dom}(H) := \{\psi \in L^2(\mathbb{R}) \mid -\psi'' + ix^3\psi \in L^2(\mathbb{R})\}. \quad (9)$$

By an approach of Edmunds and Evans [27], Sec. VII.2, based on a distributional Kato’s inequality, it follows that such a defined operator  $H$  is *m-accretive* and that it coincides with the closure of (1) initially defined on infinitely smooth functions of compact support. (The difficulties with the existence of different closed extensions, cf. Ref. [28], do not arise here since  $\text{Re } V$  is trivially bounded from below.)

Now it can be rigorously verified that  $H$  is *PT-symmetric*, i.e.,  $[H, \mathcal{PT}] = 0$ , where the commutator should be interpreted as  $\mathcal{PT}H\psi = H\mathcal{PT}\psi$  for all  $\psi \in \text{Dom}(H)$ , with  $(\mathcal{P}\psi)(x) := \psi(-x)$  and  $(\mathcal{T}\psi)(x) := \overline{\psi(x)}$ . Moreover, since the adjoint  $H^\dagger$  of  $H$  is simply obtained by taking  $-i$  instead of  $i$  in the definition of the operator (including the operator domain), it can be also verified that  $H$  is *P-self-adjoint*,  $H^\dagger = \mathcal{P}H\mathcal{P}$ , and *T-self-adjoint*,  $H^\dagger = \mathcal{T}H\mathcal{T}$ . The latter is a particularly useful property for non-self-adjoint operators since it implies that the residual spectrum of  $H$  is empty [29].

As an immediate consequence of the fact that  $H$  is *m-accretive*, we know that the spectrum of  $H$  is located in the right complex half-plane. Furthermore, it has been shown in Ref. [3] that *all eigenvalues of  $H$  are real* and simple (in the sense of geometric multiplicity). The algebraic simplicity has been established in Ref. [30]. The fact that the *spectrum of  $H$  is purely discrete* follows from the compactness of its resolvent. The latter can be deduced from the identity

$$\text{Dom}(H) = \{\psi \in H^2(\mathbb{R}) \mid x^3\psi \in L^2(\mathbb{R})\} \quad (10)$$

established in Ref. [31] and the compact embedding of this set into  $L^2(\mathbb{R})$ . Furthermore, the authors of Ref. [31] show that the resolvent of  $H$  is a Hilbert-Schmidt operator. The key ingredient in the proof is the explicit knowledge of the resolvent kernel of  $H^{-1}$  that can be written in terms of Hankel functions with known asymptotics. A deeper analysis of the resolvent of  $H$  reveals that it actually belongs to the trace class [32]; alternatively, one can use a general result of Robert [33].

#### A. Completeness of eigenfunctions

Let us show that the eigenfunctions of  $H$  form a complete set in  $L^2(\mathbb{R})$ . Recall that the completeness of  $\{\psi_n\}_{n=1}^\infty$  means that the span of  $\psi_n$  is dense in  $L^2(\mathbb{R})$ , or equivalently  $(\text{span}\{\psi_n\}_{n=1}^\infty)^\perp = \{0\}$ . Nevertheless, we stress that the result on completeness does not imply that any  $\psi$  admits the unique expansion (5).

The *m-accretivity* of  $H$  implies  $\text{Re}\langle\psi, H\psi\rangle \geq 0$  for all  $\psi \in \text{Dom}(H)$ . Consequently,  $-iH$  is *dissipative*; i.e.,  $\text{Im}\langle\psi, H\psi\rangle \leq 0$  for all  $\psi \in \text{Dom}(H)$ . It is then easy to

check that the imaginary part of the resolvent of  $-iH$  at  $\xi < 0$  is non-negative, i.e.,

$$\frac{1}{2i}((-iH - \xi)^{-1} - (iH^\dagger - \xi)^{-1}) \geq 0 \quad (11)$$

in the sense of forms. Since the resolvent is trace class, it is enough to apply the completeness theorem [[34], Thm. VII.8.1] to the operator  $(-iH - \xi)^{-1}$ .

More specifically, it follows by this result that  $H$  has a complete system of eigenvectors and generalized eigenvectors. The latter, however, do not appear in our situation since all the eigenvalues are algebraically simple (see above).

### B. Existence of a bounded metric

Already at this stage we can show that there exists a bounded metric for  $H$ . We would like to emphasize that this follows actually in general from the reality and simplicity of eigenvalues and completeness of eigenfunctions for  $H$ . We remark that  $H^\dagger$  shares these properties due to the simplicity of eigenvalues and  $\mathcal{T}$ - or  $\mathcal{P}$ -self-adjointness of  $H$ .

In detail, let  $H$  be a densely defined and closed operator such that  $\rho(H) \cap \rho(H^\dagger) \cap \mathbb{R} \neq \emptyset$  and let  $z_0$  be a number from this intersection. Then the existence of a bounded positive  $\Theta$  satisfying (3) is equivalent to the fact that the resolvent of  $H$  satisfies (3), i.e.,

$$\Theta(H - z_0)^{-1} = (H^\dagger - z_0)^{-1}\Theta. \quad (12)$$

Thus we can transfer the problem of finding the metric for an unbounded  $H$  to the same problem for its bounded resolvent. Using Ref. [13], Prop. 3, which is in fact the construction of a bounded metric using the well-known formula

$$\Theta := \sum_{n=1}^{\infty} c_n \phi_n \langle \phi_n, \cdot \rangle \quad (13)$$

with  $\phi_n$  being the eigenfunctions of  $H^\dagger$  and  $c_n > 0$  tending to zero sufficiently fast, yields the following: If all the eigenvalues of  $H^\dagger$  are real and the associated eigenfunctions  $\phi_n$  are complete, then a bounded metric for  $(H - z_0)^{-1}$ , and therefore for  $H$ , exists.

In our situation, we know that all the eigenvalues of (1) as well as its adjoint are simple and real; for  $z_0$  we can take any negative number due to m-accretivity of  $H$  and  $H^\dagger$ , and we have shown that the eigenfunctions of  $H$  and therefore also  $H^\dagger$  are complete. Hence, the existence of a bounded  $\Theta$  follows.

### C. Singularity of any metric

After the two preceding positive results, we show now that any metric for the imaginary cubic oscillator is singular, i.e., either unbounded or unboundedly invertible. We proceed by contradiction. Let there exist a bounded positive operator  $\Theta$  with bounded inverse satisfying (3). Then the following norm estimate for the resolvent holds

$$\|(H - z)^{-1}\| \leq \frac{C}{|\text{Im}z|} \quad (14)$$

for every  $z \in \mathbb{C}$  such that  $\text{Im}z \neq 0$ , where  $C$  is a positive constant bounded by  $\|\sqrt{\Theta}\| \|\sqrt{\Theta^{-1}}\|$ . By establishing a lower bound to the resolvent appearing in (14), we show that the inequality (14) cannot hold. The lower bound follows by a direct construction of a continuous family of approximate eigenstates of complex energies far from the spectrum due to Davies [35].

Using the strategy in Ref. [35], Thm. 2, we consider  $\|(H - \sigma z)^{-1}\|$  with  $\sigma > 0$  large and  $0 < \arg z < \pi/2$ . By a simple scaling argument in  $x$ , the problem can be transferred into a semiclassical one, namely,  $\|(H - \sigma z)^{-1}\| = \sigma^{-1} \|(H_h - z)^{-1}\|$ , where

$$H_h := -h^2 \frac{d^2}{dx^2} + ix^3, \quad (15)$$

with  $h := \sigma^{-5/6}$ . In order to apply [[35], Thm. 1], we have to verify that  $\text{Im}V'(a) \neq 0$ , where  $V(x) := ix^3$  and  $a$  is obtained from the relation  $z = \eta^2 + ia^3$  with  $\eta \in \mathbb{R} \setminus \{0\}$ . However, this can be easily checked for  $\text{Im}V'(a) = 3a^2$  and  $a \neq 0$  since  $\text{Im}z \neq 0$  by assumption. It then follows from Ref. [35], Thm. 1, that the norm of the resolvent of  $H_h$  diverges faster than any power of  $h^{-1}$  as  $h \rightarrow 0$ . More specifically, there exists positive  $h_0$  and for each  $n > 0$  a positive constant  $c_n$  such that if  $h \in (0, h_0)$  then

$$\|(H_h - z)^{-1}\| \geq \frac{c_n}{h^n}. \quad (16)$$

The relation between  $H$  and  $H_h$  provides an analogous claim for  $\|(H - \sigma z)^{-1}\|$  and therefore the resolvent bound (14) when combined with (16) cannot hold if  $n$  is chosen sufficiently large (namely,  $n > 6/5$ ).

## IV. CONCLUDING REMARKS

Although the imaginary cubic oscillator (1) is  $\mathcal{PT}$ -symmetric with purely real and discrete spectrum, it cannot be similar (via a bounded and boundedly invertible transformation) to any self-adjoint operator or, equivalently, the eigenfunctions of  $H$  cannot form the Riesz basis. We remark that the question whether eigenvectors of  $H$  form a basis remains open.

We established the existence of a bounded metric, which is in fact equivalent to the completeness of eigenfunctions that we proved and the reality and simplicity of eigenvalues. However, the singular nature of any metric is inevitable. The latter was established by semiclassical tools, namely the pseudomode construction due to Davies [35]. The method of proof implies that (1) possesses a very nontrivial pseudo-spectrum and regions of strong spectral instabilities, cf. (7) and (16). In the language of exceptional points, the imaginary cubic oscillator possesses an ‘‘intrinsic exceptional point’’ that is much stronger than any exceptional point associated with finite Jordan blocks, cf. Secs. II D and III C.

The method of this paper, namely the disproval of quasi-Hermiticity with bounded and boundedly invertible metric based on the localized semiclassical pseudomodes, does

not restrict to the particular Hamiltonian (1). It also applies to the already mentioned  $x^2 + ix^3$  potential and to many others. As a large class of admissible operators, let us mention, for instance, the Schrödinger operators considered by Davies [35]:

$$-\frac{d^2}{dx^2} + \sum_{m=1}^{2n} c_m x^m, \quad (17)$$

where the constant  $c_{2n}$  has positive real and imaginary parts; then the corresponding closed realization is an m-sectorial operator. Later, the results of Ref. [35] were substantially generalized to higher dimensions and pseudo-differential operators in Refs. [36,37].

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