

Isospin breaking and $f_0(980)$ - $a_0(980)$ mixing in the $\eta(1405) \rightarrow \pi^0 f_0(980)$ reactionF. Aceti,¹ W. H. Liang,² E. Oset,¹ J. J. Wu,³ and B. S. Zou^{4,5}¹*Departamento de Física Teórica and IFIC, Centro Mixto Universidad de Valencia-CSIC, Institutos de Investigación de Paterna, Apartado 22085, 46071 Valencia, Spain*²*Physics Department, Guangxi Normal University, Guilin 541004, China*³*Physics Division, Argonne National Laboratory, Argonne, Illinois 60439, USA*⁴*State Key Laboratory of Theoretical Physics, Institute of Theoretical Physics, Chinese Academy of Sciences, Beijing 100190, China*⁵*Theoretical Physics Center for Science Facilities, Institute of High Energy Physics, Chinese Academy of Sciences, Beijing 100049, China*

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We make a theoretical study of the $\eta(1405) \rightarrow \pi^0 f_0(980)$ and $\eta(1405) \rightarrow \pi^0 a_0(980)$ reactions with an aim to determine the isospin violation and the mixing of the $f_0(980)$ and $a_0(980)$ resonances. We make use of the chiral unitary approach where these two resonances appear as composite states of two mesons, dynamically generated by the meson-meson interaction provided by chiral Lagrangians. We obtain a very narrow shape for the $f_0(980)$ production in agreement with a BES experiment. As to the amount of isospin violation, or $f_0(980)$ and $a_0(980)$ mixing, assuming constant vertices for the primary $\eta(1405) \rightarrow \pi^0 K \bar{K}$ and $\eta(1405) \rightarrow \pi^0 \pi^0 \eta$ production, we find results which are much smaller than found in the recent experimental BES paper, but consistent with results found in two other related BES experiments. We have tried to understand this anomaly by assuming an $I = 1$ mixture in the $\eta(1405)$ wave function, but this leads to a much bigger width of the $f_0(980)$ mass distribution than observed experimentally. The problem is solved by using the primary production driven by $\eta' \rightarrow K^* \bar{K}$ followed by $K^* \rightarrow K \pi$, which induces an extra singularity in the loop functions needed to produce the $f_0(980)$ and $a_0(980)$ resonances. Improving upon earlier work along the same lines, and using the chiral unitary approach, we can now predict absolute values for the ratio $\Gamma(\pi^0, \pi^+ \pi^-)/\Gamma(\pi^0, \pi^0 \eta)$ which are in fair agreement with experiment. We also show that the same results hold if we had the $\eta(1475)$ resonance or a mixture of these two states, as seems to be the case in the BES experiment.

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I. INTRODUCTION

In a recent paper the BES team has reported an unusually large isospin violation in the decay of the $\eta(1405) \rightarrow \pi^0 f_0(980)$ compared to the $\eta(1405) \rightarrow \pi^0 a_0(980)$ reaction [1]. The $\eta(1405)$ being an isospin $I = 0$ object can decay naturally to $\pi^0 a_0(980)$, but the decay into $\pi^0 f_0(980)$ violates isospin. A mixture of the $f_0(980)$ and $a_0(980)$ is unavoidable because isospin is broken in meson rescattering due to the different masses of the charged and neutral kaons, as was early discussed in Ref. [2]. More recently the subject has been thoroughly discussed in Refs. [3,4] suggesting the study of the $J/\psi \rightarrow \phi \pi^0 \eta$ reaction as a test for it. This reaction has been done at BES [5], where one finds a narrow signal for the $J/\psi \rightarrow \phi \pi^0 \eta$ of the order of the difference of kaon masses, as predicted [2–4], with an intensity of about a half percent with respect to the one of the $J/\psi \rightarrow \phi \pi \pi$ in the $f_0(980)$ peak of the $\pi \pi$ mass distribution. Very recently, this reaction has been studied theoretically in Ref. [6] using the chiral unitary approach, as in Ref. [4], showing that one not only gets the shape of the experiment but also the absolute rate. Following the suggestion of Wu and Zou [7], the same experimental work of Ablikim *et al.* [5] also reports on the $\chi_{c1} \rightarrow \pi^0 \pi \pi$ in the region of the $f_0(980)$ peak of the $\pi \pi$ mass distribution, and

once again finds a narrow signal, with an intensity with respect to $\chi_{c1} \rightarrow \pi^0 \pi^0 \eta$ in the $a_0(980)$ region of the $\pi^0 \eta$ mass distribution of the order of also a half percent. These numbers are within expected values for isospin violation and the narrowness of the isospin forbidden signal is tied to the mass difference between charged and neutral kaons, reflecting that the isospin violation is tied to the difference of the loop functions of intermediate kaons in the rescattering of mesons that leads both to the $f_0(980)$ and $a_0(980)$ resonances. This provides support [4] to the chiral dynamical picture of these resonances [8–14], which appear as composite states of meson-meson, dynamically generated by the interaction of mesons provided by the chiral Lagrangians [15,16].

With this earlier experimental work, the recent work on the $\eta(1405) \rightarrow \pi^0 f_0(980)$ and $\eta(1405) \rightarrow \pi^0 a_0(980)$ reactions [1] has brought a surprise. The signal for the isospin violating channel $\eta(1405) \rightarrow \pi^0 f_0(980)$ is also very narrow, in agreement with previous findings in analogous reactions, but the reported ratio of the partial decay widths of the two channels is abnormally large: 18% for $\eta(1405) \rightarrow \pi^0 \pi^+ \pi^-$ to $\eta(1405) \rightarrow \pi^0 a_0(980)$, or summing the $\pi^0 \pi^0$ channel to the $\pi^+ \pi^-$, a ratio of 27% for the ratio of rates of $\eta(1405) \rightarrow \pi^0 f_0(980)$ to $\eta(1405) \rightarrow \pi^0 a_0(980)$. One anticipates difficulties in a theoretical

description of such a large rate, unless the same $\eta(1405)$ state already contains a large mixture of $I = 0$ and $I = 1$, in which case the rate of production of the $f_0(980)$ final state would be largely enhanced. However, in this case, the signal of the $f_0(980)$ would not be due to the difference of the kaon masses and the production of the $f_0(980)$ would proceed unhindered, showing the natural width of the $f_0(980)$ of about 50 MeV instead of the 9 MeV observed in the BES experiment [1].

In Ref. [17] a particular mechanism was proposed, consisting in the $\eta(1405)$ decay into $K^*\bar{K}$, the posterior K^* decay into $\pi^0 K$ and the rescattering of the $K\bar{K}$ to produce either the $f_0(980)$ and the $a_0(980)$ resonances. This leads technically to a triangular loop diagram that has two cuts (singularities in the integrand), which make it different from the standard G loop function from $K\bar{K}$, with only the $K\bar{K}$ on shell singularity. This latter G function would appear should the $\eta' \rightarrow \pi^0 K\bar{K}$ vertex be a contact term, or if it was coming from diagrams where an internal propagator is far off shell (contact like vertex).

In the present work we shall make first a thorough discussion of the issue assuming a contact (or contactlike) $\eta' \rightarrow \pi^0 K\bar{K}$ vertex. Under this assumption one can make a quite model independent study, and the conclusion is that the results obtained are in line with those of other reactions, like the $J/\psi \rightarrow \phi \pi^0 \eta(\pi\pi)$.

A second part is devoted to the explicit study of the triangular mechanism of Wu *et al.* [17] which is quite unique to the present reaction. Using the chiral unitary approach we shall see that we are able to evaluate the ratio for isospin violation rather reliably, beyond the reach of Wu *et al.* [17] where the ratio of widths for $\eta' \rightarrow \pi^0 \pi^+ \pi^-$ and $\eta' \rightarrow \pi^0 \pi^0 \eta$ was dependent on an unknown cutoff. We find that this ratio is sizeably increased with respect to the standard approach, in the line of the claims of Wu *et al.* [17]. We also show that the consideration of an extra mechanism driven by primary $\pi^0 \pi^0 \eta$ production and rescattering can moderately increase that ratio such that a good agreement with experiment is found at the end. We emphasize that the concept of $f_0 - a_0$ mixing is not very appropriate since the apparent mixing is so different in different reactions. We rather prefer to talk in terms of isospin violation, magnified due to the proximity of the f_0 and a_0 resonances, but which is very much tied to different reactions. The ability of the chiral unitary approach to provide a fair description of all these processes certainly gives support to this method and the underlying consequence in this case, that the f_0 and a_0 resonances are basically molecular states of meson-meson, mostly $K\bar{K}$ in both cases.

In what follows we will assume that we have the $\eta(1405)$ decay, while in the BES experiment a mixture of the $\eta(1405)$ and $\eta(1475)$ is present. We also evaluate decay rates for the $\eta(1475)$ and find that the results are basically the same, independently of whether we have either resonance or a mixture of the two.

II. FORMALISM

The starting point in the following discussion is the acceptance that the $f_0(980)$ and $a_0(980)$ qualify as composite meson-meson states which are dynamically generated by the meson-meson interaction provided by the chiral Lagrangians. The Schrödinger equation is solved using the kernel (potential) from the chiral Lagrangians, which provide a scattering amplitude from where the $f_0(980)$ and $a_0(980)$ emerge as poles in the complex plane. In practice, the Bethe-Salpeter equation in coupled channels is used, accounting for dynamical and relativistic effects. The basic building blocks are $\pi\pi$ and $K\bar{K}$ for the $f_0(980)$ and $\pi\eta$ and $K\bar{K}$ for the $a_0(980)$ [8–14]. Once this is accepted, the next step is that, consistently with this picture, these resonances do not couple directly to external sources. It is the constituents, pairs of mesons, that couple directly to these sources and, upon unitarization (multiple scattering of these mesons), the resonances are formed. According to this picture, a series of reactions where these resonances are formed were studied and, with no extra parameters than those needed in the study of meson-meson scattering, predictions were made for cross sections or other observables in these reactions. Examples of it are the reactions $\phi \rightarrow \pi^0 \pi^0 \gamma$, $\pi^0 \eta \gamma$ [18], the $J/\psi \rightarrow \phi(\omega) f_0$ [19–22], the $J/\psi \rightarrow p\bar{p}\pi\pi$ reaction [23] or the photoproduction of $f_0(980)$ on nucleons [24].

The success in the study of these reactions gives strong support to the basic idea that we adopt here concerning these resonances.

A. Standard formalism assuming local primary $\eta(1405) \rightarrow \pi^0 PP$ vertices

After this introductory discussion, let us begin the first point where we shall assume that the first step consists of $\eta(1405) \rightarrow \pi^0 PP$ (P for pseudoscalar) well described by contact (or contactlike) vertices. We also accept that the $\eta(1405)$ is an isospin zero state. Then, the mechanism for production of either $\pi^+ \pi^-$ or $\pi^0 \eta$ in the final state, together with an extra π^0 , is given by Fig. 1.

Implicit in the picture of Fig. 1 is the fact that the π^0 of the upper line has an energy, when the other pair of mesons produce the $f_0(980)$ or $a_0(980)$, which does not match with the energy of the other mesons (in the case of $\pi^0 \eta$ production) to produce the $f_0(980)$ or $a_0(980)$ resonances. In the case of $\pi^0 \pi^+ \pi^-$, the π^0 would not produce either the $f_0(980)$, that has zero charge, nor the $a_0(980)$ which does not couple to two pions. But even if it had, it would not play a role in the reaction as we shall discuss below.

The pair of interacting mesons in Fig. 1 will have $I = 1$ if we invoke exact $I = 0$ for the $\eta(1405)$. Then the $K\bar{K}$ pair appears in the $I = 1$ combination

$$\frac{1}{\sqrt{2}}(K^+ K^- - K^0 \bar{K}^0), \quad (1)$$

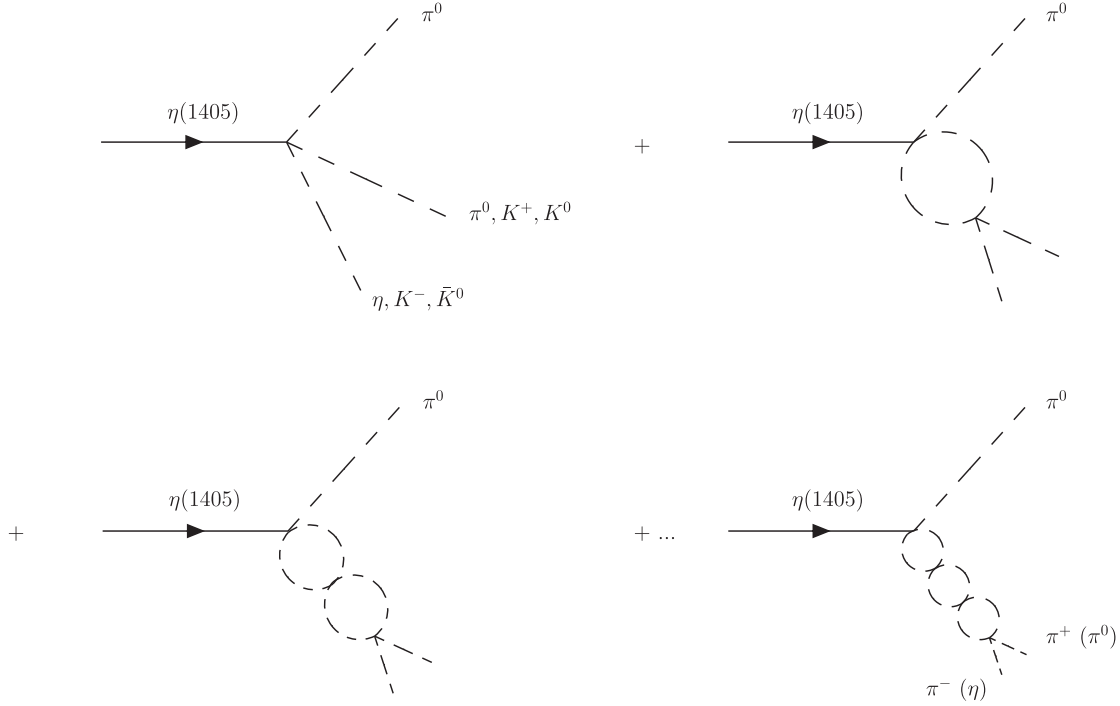


FIG. 1. Diagrammatic representation of the $\pi^0\pi^+\pi^-$, $\pi^0\pi^0\eta$ production in the $\eta(1405)$ decay.

where we take the convention that $|K^- \rangle \equiv -|1/2, -1/2 \rangle$ of isospin. Should the kaons have the same mass, the loop functions in the figure would be the same for charged and neutral kaons and the relative minus sign in Eq. (1) guarantees that $\pi^+\pi^-$ will not be produced, since there is an exact cancellation of the K^+K^- and $K^0\bar{K}^0$ contributions (the $\pi^0\eta \rightarrow \pi^+\pi^-$ would also not proceed). However, when the physical masses are considered, the exact cancellation turns into a partial cancellation, leading to an isospin breaking effect that we study in detail below.

So far we have only advocated isospin conservation in the $\eta \rightarrow \pi^0 MM$ vertex. Now we can go one step further to put some constraints on the $\pi^0\eta$ primary production using arguments of $SU(3)$.

By analogy to the η and η' , which are members of a nonet, with the η largely an octet and the η' basically a singlet, with a small mixing [25–27], we can also assume that in the next pair of η states, the $\eta(1295)$ is largely an octet and the $\eta(1405)$ is mostly a singlet (we shall release this constraint later on to quantify uncertainties).

In this case we have to place the interacting meson pair into an octet to produce a singlet with the octet of the spectator π^0 . Then, up to an undetermined reduced matrix element, the weight of K^+K^- , $K^0\bar{K}^0$ and $\pi^0\eta$ is determined by the $SU(3)$ Clebsch-Gordan coefficients of the $8 \otimes 8 \rightarrow 1$ decomposition, and we have up to a global factor

$$M_{K^+K^-} = \sqrt{\frac{3}{5}}, \quad M_{K^0\bar{K}^0} = -\sqrt{\frac{3}{5}}, \quad M_{\pi^0\eta} = \sqrt{\frac{4}{5}}. \quad (2)$$

Then, the scattering matrix for the production of the final state is given by

$$t_f = M_f + \sum_{i=1}^3 M_i G_i T_{if}, \quad (3)$$

where T_{if} is the 5×5 scattering matrix for the channels K^+K^- (1), $K^0\bar{K}^0$ (2), $\pi^0\eta$ (3), $\pi^+\pi^-$ (4), $\pi^0\pi^0$ (5) and M_i in the same basis is given by

$$M_i = A \left(\sqrt{\frac{3}{5}}, -\sqrt{\frac{3}{5}}, \sqrt{\frac{4}{5}}, 0, 0 \right), \quad (4)$$

with A a reduced matrix element.

The T matrix is obtained using the Bethe-Salpeter equation in the five coupled channels

$$T = [1 - VG]^{-1}V, \quad (5)$$

with V taken from Ref. [8] (care is taken to multiply by $1/\sqrt{2}$ the matrix elements in the case of $\pi^0\pi^0$ states, thus implementing the unitary normalization which is suited for the sum over intermediate states of identical particles).

The G function is the diagonal loop matrix of the propagators of the intermediate particles

$$G(P^2) = \int \frac{d^4q}{(2\pi)^4} \frac{1}{q^2 - m_1^2 + i\epsilon} \frac{1}{(P-q)^2 - m_2^2 + i\epsilon}, \quad (6)$$

with P the total four-momentum ($P^2 = s$) and m_1, m_2 the masses of the particles in the considered channel. Upon regularization with a cutoff one obtains [8]

$$G(P^2) = \int_{|\vec{q}| < q_{\max}} \frac{d^3q}{(2\pi)^3} \frac{\omega_1 + \omega_2}{2\omega_1\omega_2} \times \frac{1}{(P^{02} - (\omega_1 + \omega_2)^2 + i\epsilon)}, \quad (7)$$

where $\omega_i = \sqrt{\vec{q}^2 + m_i^2}$.

By using a cutoff of $q_{\max} = 900$ MeV we obtain a good description of the $f_0(980)$ and $a_0(980)$ resonances, as in Ref. [8].

In the literature one also uses G functions with dimensional regularization [28] and for a certain range of energies one can show that both prescriptions are equivalent [29]. In the present case we use deliberately the cutoff method because the knowledge of this cutoff will serve to regularize a different loop function found in Sec. II C.

Note that in Eq. (3) we have two sources of isospin violation: The one due to the G_i functions, which now are different for K^+K^- and $K^0\bar{K}^0$; and the T_{if} matrix elements, which are evaluated by means of Eq. (5) in the charge basis of the states and that also break isospin symmetry because the G_i functions are different for different members of the same isospin multiplets.

Now we would like to restrict the assumption of the $\eta(1405)$ being a $SU(3)$ singlet. Let us accept that it would also have a mixture with an octet. In the case of a pure octet for the $\eta(1405)$ then the interacting pair can belong to the 8, 10, and 27 representations.

Defining

$$R = \frac{M(\pi^0\eta)}{M(K^+K^-)}, \quad (8)$$

we have $R = \sqrt{4/3}$ for the octet, $R = 0$ for the decuplet and $R = -\sqrt{3}$ for the 27. It is quite unlikely that the $\eta(1405)$ would be a pure octet, and that in this case the interacting pair would couple only to the 27 representation, which leads us to values of R preferably positive. Note that with negative values of R (we have seen that this can happen for values around $R \simeq -1.5$) there is a destructive interference between $\pi^0\eta$ and $K\bar{K}$ induced $a_0(980)$ production such that $\pi^0 a_0(980)$ production would disappear in the $\eta(1405)$ decay, which is not the case experimentally [30]. The order of magnitude for R is determined with these simple arguments, but we can get help from experiment since we have the ratio [30,31]

$$R_\Gamma = \frac{\Gamma(\pi\pi\eta)}{\Gamma(\pi K\bar{K})} = 1.09 \pm 0.48. \quad (9)$$

Assuming the ratio to hold for the rates to $\pi^0\pi^0\eta$ and $\pi^0(K^+K^- + K^0\bar{K}^0)$ we obtain

$$R_\Gamma = \frac{1}{2} R^2 \frac{PS(\pi^0\pi^0\eta)}{PS(\pi^0 K\bar{K})}, \quad (10)$$

where PS stands for the phase space of each final state, which is obtained integrating $\frac{d\Gamma}{dm_f}$ of Eq. (12) over m_f (taking $\beta = |t_f| = 1$). By doing this we obtain

$$|R| = 0.75 \pm 0.17. \quad (11)$$

This result with positive sign would be in agreement with the prediction based on the assumption of the $\eta(1405)$ being a $SU(3)$ singlet, $R = \sqrt{4/3} = 1.15$. Yet, we shall explore the results within the range $R \in [-1, 1.2]$.

B. Results with the local vertices

We need to evaluate $\frac{d\Gamma}{dm_f}$ to compare with experiment, where m_f is the invariant mass of the final interacting pair ($\pi^+\pi^-$ and $\pi^0\eta$ in our case). Since the meson-meson interaction that leads to the $f_0(980)$ and $a_0(980)$ resonances is s -wave, there is no angular dependence in the t_f matrix and, since we are concerned only around the $m_f = 980$ MeV region, the magnitude A in Eq. (4) can be considered constant. In this case we have [32]

$$\frac{d\Gamma}{dm_f} = \beta p_1 \tilde{p}_2 |t_f|^2, \quad (12)$$

with β a constant factor, where p_1, \tilde{p}_2 are the momentum of the spectator π^0 in the $\eta(1405)$ rest frame and the momentum of the interacting pair in the rest frame of the pair, respectively

$$p_1 = \frac{\lambda^{1/2}(m_\eta^2, m_{\pi^0}^2, m_f^2)}{2m_\eta}, \quad \tilde{p}_2 = \frac{\lambda^{1/2}(m_f^2, m_2^2, m_3^2)}{2m_f}. \quad (13)$$

In Eq. (13), λ is the Källén function and m_2, m_3 the masses of the mesons of the interacting pair.

In Figs. 2 and 3, we plot $\frac{d\Gamma}{dm_f}$ for f equal to $\pi^+\pi^-$ and $\pi^0\eta$, taking A from Eq. (4) equal to 1. We can rightly say that the unitarization from the meson-meson pairs should be implemented in other pairs too. Think for instance of primary production of $\pi^0 K\bar{K}$ and then $\pi^0 K$ interaction producing an effective $\eta(1405)K\bar{K}\pi^0$ vertex that will depend on $m(\pi^0 K)$. After this, the $K\bar{K}$ will interact again to finally produce the f_0 or a_0 . The isospin, or $SU(3)$

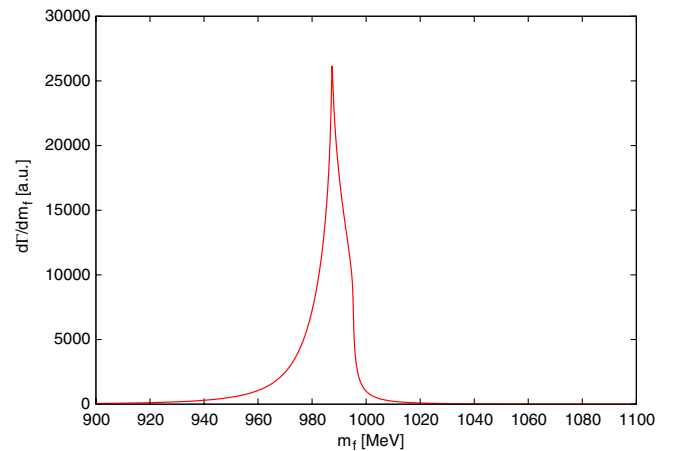


FIG. 2 (color online). $\frac{d\Gamma}{dm_f}$ for $\eta' \rightarrow \pi^0\pi^+\pi^-$ decay in the $f_0(980)$ region.

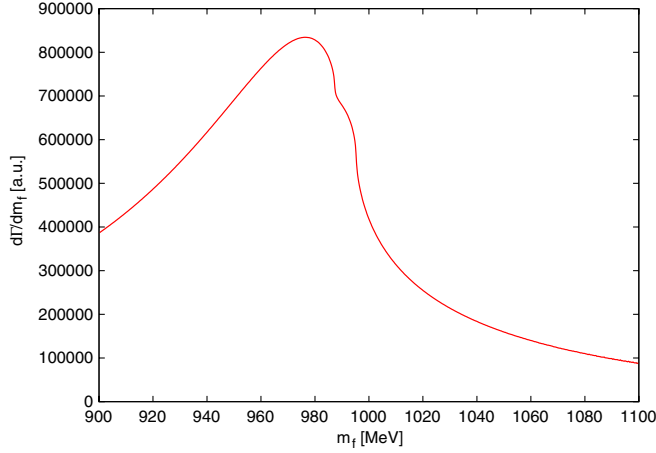


FIG. 3 (color online). $\frac{d\Gamma}{dm_f}$ for $\eta' \rightarrow \pi^0\pi^0\eta$ decay in the $a_0(980)$ region.

argument used before, should also hold, but the coefficient A would now be dependent on $m(\pi^0 K)$ which also introduces an angular dependence on this coefficient. However, upon projection over s -wave, needed to generate the f_0 or a_0 resonances, and the selection of a narrow window for $m(K\bar{K})$ around 980 MeV, the coefficient A turns again into a constant. Similar arguments can be made with respect to the symmetrization of the two pions in the $\pi^0\pi^0\eta$ channel.

What we can see in Figs. 2 and 3 is that in the case of the $\pi^+\pi^-$ production we obtain a very narrow peak around 980 MeV like in the experiment [1]. The width of this peak is about 10 MeV, in agreement with experimental observations. As we discussed above, the peak appears in the $f_0(980)$ region, in between the thresholds of K^+K^- and $K^0\bar{K}^0$, because now $G_{K^+K^-} - G_{K^0\bar{K}^0}$ is different from zero. However, the difference, which is due to the different kaon masses, is only significant in a region of energies around the $K\bar{K}$ thresholds, where $\Delta(\sqrt{s})$ is of the order of $m_{K^+} - m_{K^0}$; see Fig. 4. Away from the thresholds the difference of the two G functions due to the mass difference becomes gradually smaller and this leads to the peculiar narrow

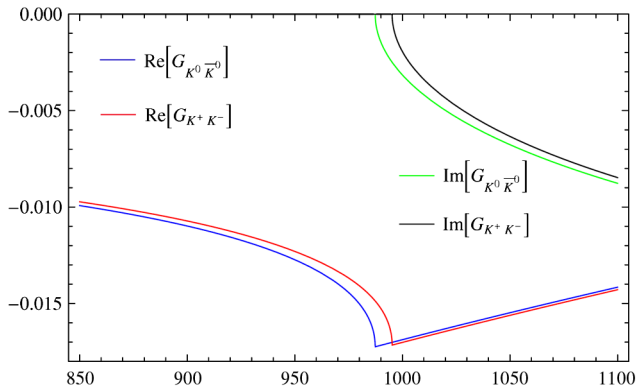


FIG. 4 (color online). Lines upper side: higher $\text{Im}[G_{K^+K^-}]$, lower $\text{Im}[G_{K^0\bar{K}^0}]$; lines lower side: higher $\text{Re}[G_{K^+K^-}]$, lower $\text{Re}[G_{K^0\bar{K}^0}]$.

shape of the $f_0(980)$ excitation in the $\pi^+\pi^-$ channel, already anticipated in Refs. [2–4].

One should stress here that the shape of Fig. 2 is not the standard one of the $f_0(980)$ seen in isospin allowed reactions and the width is tied to the mass difference $m_{K^+} - m_{K^0}$. This comment is pertinent in view of the comment in Ref. [1] quoting that ‘‘The measured width of the $f_0(980)$ is much narrower than the world average.’’ It is clear that the shape of $\pi^+\pi^-$ production here is not the shape of the $f_0(980)$.

In Fig. 3 we see the signal for the $a_0(980)$ excitation, which is isospin allowed. The width is much larger and the strength at the peak is also much larger. If we compare the strength of the peak for $\pi^+\pi^-$ of f_0 and $\pi^0\eta$ of a_0 production, we find that the ratio is of the order of 3%. However if we integrate the strength over m_f in the region of the peaks for the two cases, we find a smaller ratio

$$\frac{\Gamma(\pi^0, \pi^+\pi^-)}{\Gamma(\pi^0, \pi^0\eta)} = 0.015, \quad (14)$$

of the order of 1.5%, which is along the lines of the 0.6% observed in the two reactions $J/\psi \rightarrow \phi\pi^0\eta(\pi^+\pi^-)$ or $\chi_{c1} \rightarrow \pi^0(\pi^+\pi^-)(\pi^0\eta)$ [5]. In Fig. 5, we show the ratio of $d\Gamma(\pi^+\pi^-)/d\Gamma(\pi^0\eta)$ as a function of the energy. We observe a peculiar structure, where the K^+K^- , $K^0\bar{K}^0$ thresholds show up as cusps, as predicted in Refs. [3,4] and also shown in Ref. [6].

We come now to see the uncertainties due to the diversion from the $SU(3)$ hypothesis assumed. We allow R of Eq. (8) to vary between -1 and 1.2 , as discussed in the previous section. In Fig. 6 we can see that the ratio of strengths at the peak of each resonance changes within a factor of 2 in such a large range. In terms of the m_f integrated over the peak, removing background, the range is

$$\frac{\Gamma(\pi^0, \pi^+\pi^-)}{\Gamma(\pi^0, \pi^0\eta)} \in [0.01-0.04]. \quad (15)$$

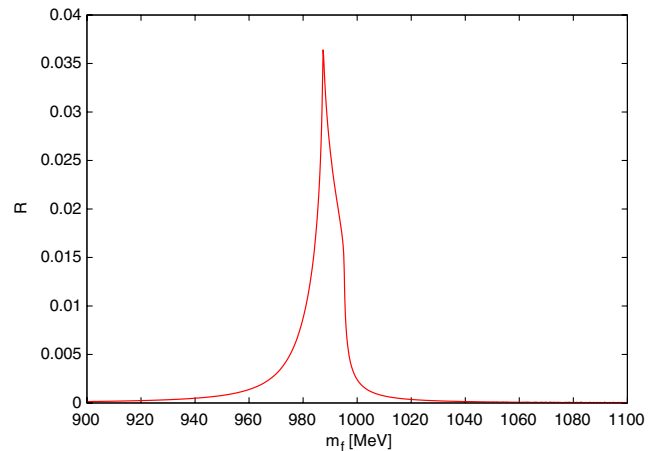


FIG. 5 (color online). Ratio $(\frac{d\Gamma}{dm_f})_{\pi^+\pi^-} / (\frac{d\Gamma}{dm_f})_{\pi^0\eta}$ as a function of m_f .

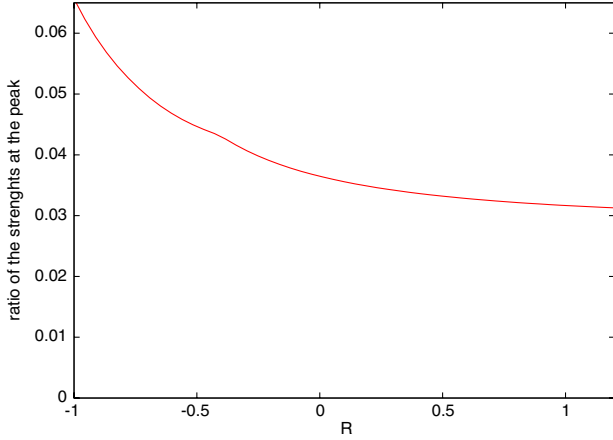


FIG. 6 (color online). Ratio of strengths at the peak as a function of R .

The results are shown in Fig. 7. At the extreme negative value of R , not preferred by the theory, the ratio reaches the value of 0.042. In the range from $R = 0$ (the value implicitly taken in Ref. [17]) to $R = 1.2$ [$R = 1.15$ corresponds to the $SU(3)$ singlet for the $\eta(1405)$] the value of the ratio of Γ 's ranges from 1 to 1.5%. Even with this theoretical uncertainty, it is thus clear that we cannot obtain a ratio as big as the 18% reported in the experiment of Ablikim *et al.* [1].

There could be a scope, since so far we have always assumed the $\eta(1405)$ to be a pure $I = 0$ state. Let us assume that we have a mixture of $I = 0$ and $I = 1$ in that state (the same conclusions would hold if we say instead that there is isospin violation in the production of mesons of the first step—something that is very unusual in chiral theories [4]). In the case of $I = 1$ for the $\eta(1405)$ the interacting meson pair can have $I = 0$, which we assume in the $SU(3)$ octet, to magnify the $f_0(980)$ production. Then the channels are $\pi\pi$ and $K\bar{K}$, but the $\pi\pi$ channel is weak in this process. For the exercise that we do, the $\pi\pi$ channel can be safely ignored in the production

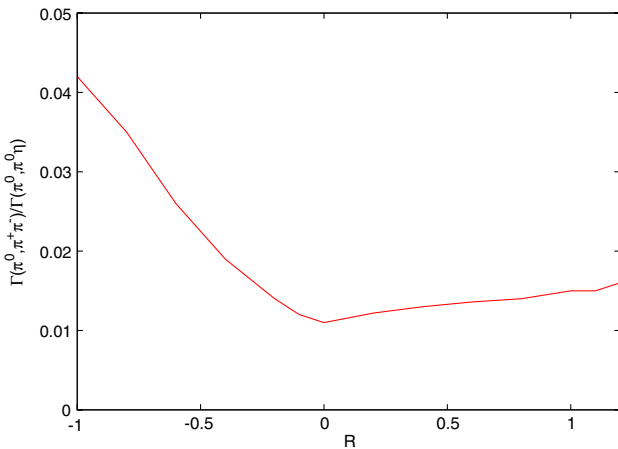


FIG. 7 (color online). Ratio $\frac{\Gamma(\pi^0, \pi^+ \pi^-)}{\Gamma(\pi^0, \pi^0 \eta)}$ as a function of R .

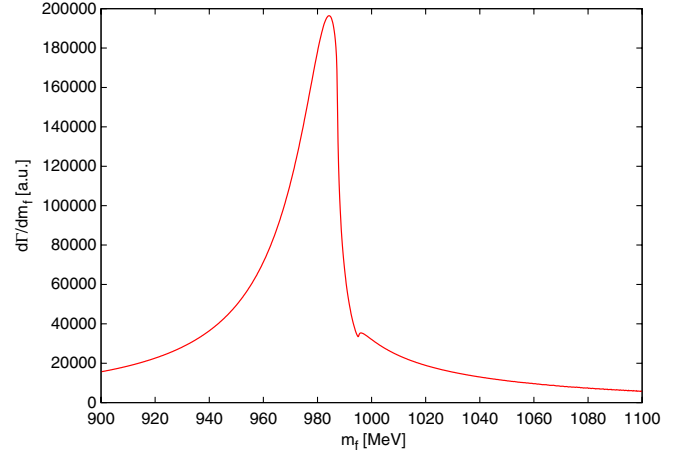


FIG. 8 (color online). $\frac{d\Gamma}{dm_f}$ for $\eta \rightarrow \pi^0 \pi^+ \pi^-$ decay in the $f_0(980)$ region, for $\alpha = 0.54$.

vertices, but not in the T_{if} matrix of Eq. (3). Then the $K\bar{K}$, $I = 0$ combination is

$$\frac{1}{\sqrt{2}}(K^+ K^- + K^0 \bar{K}^0). \quad (16)$$

Taking into account the isospin mixture and a different reduced matrix element for $I = 0$ pair production and putting the product in a coefficient α , we have now $M_i \rightarrow \tilde{M}_i$, with \tilde{M}_i given by

$$\tilde{M}_i = A \left((1 + \alpha) \sqrt{\frac{3}{5}}, (\alpha - 1) \sqrt{\frac{3}{5}}, \sqrt{\frac{4}{5}}, 0, 0 \right). \quad (17)$$

We vary the parameter α until we find a ratio $\Gamma(\pi^0, \pi^+ \pi^-) / \Gamma(\pi^0, \pi^0 \eta) = 0.18$. The parameter α has the value 0.54 which implies a massive isospin violation in a physical state. This would be difficult to accept in physical terms, but there is one stronger reason to reject this solution. Indeed, as seen in Fig. 8, since the $f_0(980)$ production proceeds unhindered because we have an $I = 0$ pair to begin with, the $f_0(980)$ is produced with its natural width and the combination of Eq. (17) leads to an effective width of about 20 MeV, much bigger than the experimentally observed 9 MeV of Ref. [1]. In Fig. 9 we can see that the $a_0(980)$ resonance is also produced in this case with a shape like the ordinary one.

C. The primary production vertex with the $K^* \bar{K}$ singularity

In the former section we showed that it is not possible to get such a large isospin violation as found in Ref. [1] assuming a local vertex production. In Ref. [17] it was shown that using the $\eta(1405)$ decay mode to $K^* \bar{K}$ and the successive decay of K^* into $K\pi$ one obtains a mechanism for $K\bar{K}\pi$ production at tree level by means of which one could obtain good agreement with experimental data on this channel. This production mechanism is depicted in Fig. 10. After rescattering of the $K\bar{K}$ pair, as shown in

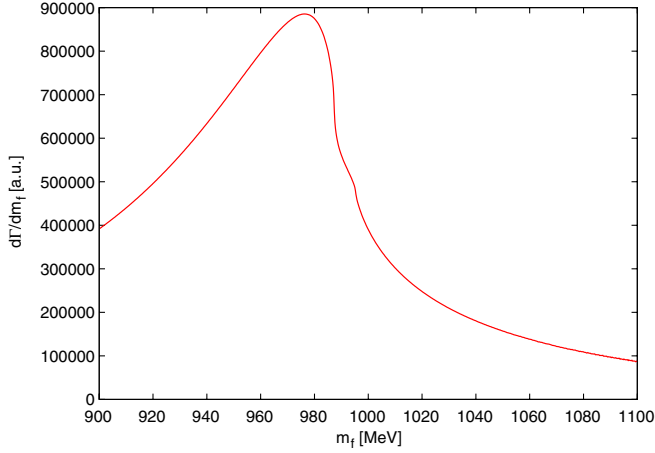


FIG. 9 (color online). $\frac{d\Gamma}{dm_\eta}$ for $\eta \rightarrow \pi^0\pi^0\eta$ decay in the $a_0(980)$ region, for $\alpha = 0.54$.

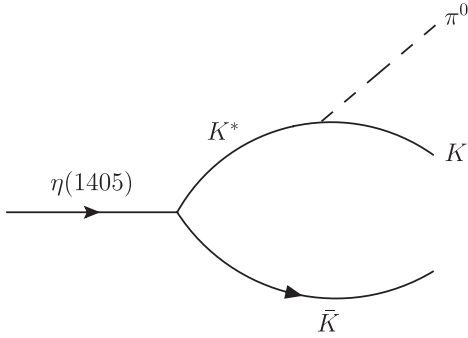


FIG. 10. Singular mechanism for $\pi^0 K\bar{K}$ production.

Fig. 11, the f_0 and a_0 resonances will be produced in our approach. The novelty now is that the first loop depicted in Fig. 11 is rather different than the one of the ordinary G function for $K\bar{K}$ propagation shown in the second diagram of Fig. 1. The difference is substantial because the structure of the loop function (through dispersion relations) is determined by the singularities (pairs of intermediate particles that can be simultaneously placed on shell in the loop integration). The loop in Fig. 11 has two singularity cuts, indicated by the dashed lines: one for the $K^*\bar{K}$ on shell and

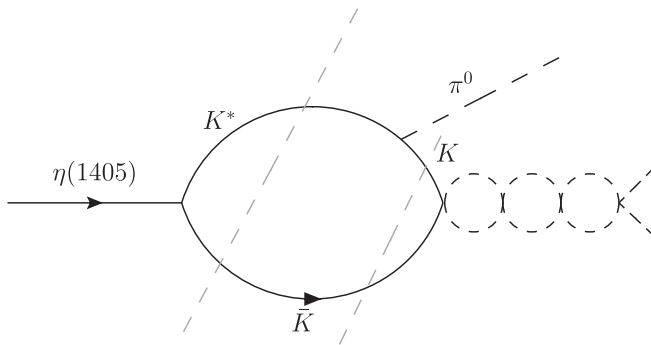


FIG. 11. Rescattering mechanism for the production of the f_0 and a_0 .

the other one for the $K\bar{K}$ on shell. The kinematics of the two cuts are not too far away, which magnifies the difference in the loop functions in the charged and neutral cases due to the different masses amongst the kaons and the K^* .

Note that the situation for $J/\psi \rightarrow \phi f_0$ is very different, because even if the highly suppressed $J/\psi \rightarrow K^*\bar{K}$ decay would be followed by the $K^* \rightarrow \phi K$ vertex, this latter process is kinematically forbidden and then the K^* is highly off shell. So, this mechanism for $J/\psi \rightarrow \phi K\bar{K}$ qualifies as a contact term for $\phi K\bar{K}$ production. Then the approach followed in the former section is most appropriate for this case and it is in essence the one followed in Refs. [4,6]. The experimental ratio for the J/ψ decay widths in this reaction are in line with the results obtained in the former sections.

On the other hand, the mechanism depicted in Fig. 10 reminds one of the $\phi \rightarrow \pi^0\pi^0\gamma$ decay which has the same structure with $\phi \rightarrow K\bar{K}$, the K (or \bar{K}) radiating a photon and the resulting $K\bar{K}$ pair interacting to give $\pi^0\pi^0$ or $\pi^0\eta$ (same diagram as Fig. 11 substituting the π^0 by γ and the K^* by K). One has there two cuts for $K\bar{K}$ before and after the radiation of the photon. One should then recall that the mechanism outlined above was very successful [33–36] reproducing the experimental data for $\phi \rightarrow \pi^0\pi^0\gamma$, $\pi^0\eta\gamma$.

Let us proceed to the explicit evaluation of the amplitude for the mechanism of Fig. 11. The loop function is evaluated using the momenta described in Fig. 12. For convenience we make the evaluation in the frame where $\vec{P} = 0$ and thus $\vec{p}_{\eta'} = \vec{p}_\pi$.

Given the structure of the $V \rightarrow PP$ vertices, $\epsilon^\mu(p_1 - p_2)_\mu$, we obtain

$$\begin{aligned} \tilde{G}(P, p_\pi, m_K, m_{K^*}) &= i \int \frac{d^4q}{(2\pi)^4} \epsilon^\mu(P + p_\pi + P - q)_\mu \epsilon^\nu(p_\pi - q)_\nu \\ &\times \frac{1}{(p_\pi + q)^2 - m_{K^*}^2 + i\epsilon} \frac{1}{q^2 - m_K^2 + i\epsilon} \\ &\times \frac{1}{(P - q)^2 - m_K^2 + i\epsilon}. \end{aligned} \quad (18)$$

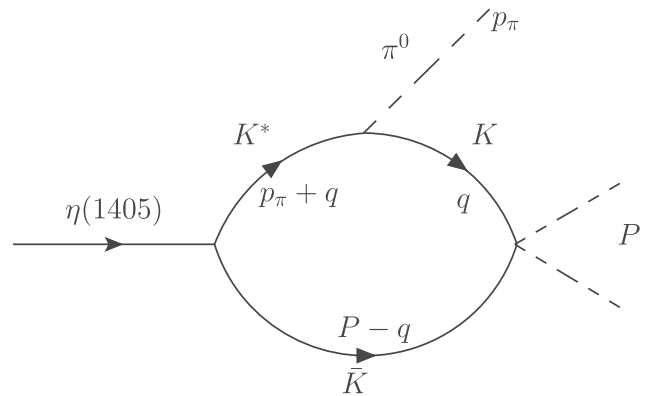


FIG. 12. Loop for the function \tilde{G} .

By summing over the polarizations

$$\sum \epsilon_\mu \epsilon_\nu \rightarrow -g_{\mu\nu} + \frac{(p_\pi + q)_\mu (p_\pi + q)_\nu}{m_{K^*}^2}, \quad (19)$$

we get

$$\begin{aligned} \tilde{G}(P, p_\pi, m_K, m_{K^*}) \\ = i \int \frac{d^4 q}{(2\pi)^4} \frac{F_{num}}{(p_\pi + q)^2 - m_{K^*}^2 + i\epsilon} \frac{1}{q^2 - m_K^2 + i\epsilon} \\ \times \frac{1}{(P - q)^2 - m_K^2 + i\epsilon}, \end{aligned} \quad (20)$$

where

$$\begin{aligned} F_{num} &= -(2P(p_\pi - q) + m_\pi^2 + q^2 - 2p_\pi q) \\ &\quad + \frac{(m_\pi^2 - q^2)}{m_{K^*}^2} [2P(p_\pi + q) + m_\pi^2 - q^2] \\ &= 2p_{\eta'}(p_\pi - q) + \frac{(m_\pi^2 - q^2)}{m_{K^*}^2} [2P(p_\pi + q) \\ &\quad + m_\pi^2 + m_{K^*}^2 - q^2]. \end{aligned} \quad (21)$$

One technical problem faced in Ref. [17] is that the integral of Eq. (20) is highly superficially divergent ($d^4 q/q^2$) and some form factor or cutoffs were used to implement convergence. However, we shall see below that the integral is only logarithmically divergent. When performing the evaluation of the $\eta' \rightarrow \pi^0 \pi^+ \pi^-$ amplitude one has the difference of \tilde{G} for the charged $K^- K^+$ and the neutral one and the results are convergent, but then the ratio to the $\eta' \rightarrow \pi^0 \pi^0 \eta$ is tied to an unknown form factor.

Our approach solves naturally the former problem. To see this, recall that in Eq. (5), for the scattering, the G function is also formally divergent and is regularized by a cutoff which is fitted to the meson-meson scattering data. The natural choice is to use this cutoff in the new loop, but this becomes a necessity when one recalls that the results of the chiral unitary approach with the G function implementing a cutoff $\theta(q_{\max} - |\vec{q}|)$ in the integration are obtained formally in a quantum mechanical formulation starting with a potential (for s -waves that we study here)

$$V(\vec{q}, \vec{q}') = v\theta(q_{\max} - |\vec{q}|)\theta(q_{\max} - |\vec{q}'|). \quad (22)$$

Then in Fig. 11 the cutoff $\theta(q_{\max} - |\vec{q}|)$ appears automatically in the loop function from the first $K\bar{K} \rightarrow PP$ potential in the sum of the diagrams implicit in the figure. Observe that the cutoff is in three-momentum. The q^0 integration must be done analytically and it is convergent.

The expressions are simplified and equally accurate if we just take the positive energy part of the relativistic K^* propagator

$$\frac{1}{2\omega_{K^*}(\vec{p}_\pi + \vec{q})} \frac{1}{p_\pi^0 + q^0 - \omega_{K^*}(\vec{p}_\pi + \vec{q}) + i\epsilon}, \quad (23)$$

where $\omega_{K^*}(\vec{p}) = \sqrt{\vec{p}^2 + m_{K^*}^2}$. Using Cauchy's theorem for the q^0 integration, we obtain then

$$\begin{aligned} \tilde{G}(P, p_\pi, m_K, m_{K^*}) \\ = \int_{|\vec{q}| < q_{\max}} \frac{d^3 q}{(2\pi)^3} \frac{1}{2\omega} \frac{1}{P^0} \frac{1}{2\omega_{K^*}} \left[\frac{F_{num}(q^0 = -\omega)}{P^0 + 2\omega} \right. \\ \times \frac{1}{p_\pi^0 - \omega - \omega_{K^*}} + \frac{F_{num}(q^0 = P^0 - \omega)}{P^0 - 2\omega + i\epsilon} \\ \left. \times \frac{1}{P^0 + p_\pi^0 - \omega - \omega_{K^*} + i\epsilon} \right], \end{aligned} \quad (24)$$

where $\omega = \sqrt{\vec{q}^2 + m_K^2}$ and $\omega_{K^*} = \sqrt{\vec{q}^2 + m_{K^*}^2}$. Equation (24) shows explicitly in the second term the two singularities corresponding to the cuts depicted in Fig. 11. One can show from Eq. (24) that \tilde{G} is only logarithmically divergent. The apparent two extra powers of q introduced by the K^* polarization sum of Eq. (19) do not result into extra degrees of divergence once the value of q^0 at the poles is substituted in Eq. (21) in the Wick rotation leading to Eq. (24).

Taking into account that the $\eta(1405)$ is an $I = 0$ object and that the $K^+ K^-$ and $K^0 \bar{K}^0$ vertices appear with different sign, the amplitude of Eq. (3) is substituted now by

$$\begin{aligned} t_f &= \tilde{G}(P, p_\pi, m_{K^+}, m_{K^{*+}}) t_{K^+ K^-, f} \\ &\quad - \tilde{G}(P, p_\pi, m_{K^0}, m_{K^{*0}}) t_{K^0 \bar{K}^0, f}, \end{aligned} \quad (25)$$

where f now stands for $\pi^+ \pi^-$ or $\pi^0 \eta$, as before.

D. Results with the triangular diagram

In Fig. 13 we show the result for $d\Gamma/dm_f$ for $\eta(1405) \rightarrow \pi^0 \pi^+ \pi^-$ and in Fig. 14 for $\eta(1405) \rightarrow \pi^0 \pi^0 \eta$. The shapes are similar to those in Figs. 2 and 3, and however we can already observe that the ratio, depicted in Fig. 15 is much bigger than that of Fig. 5—about a factor 9 bigger.

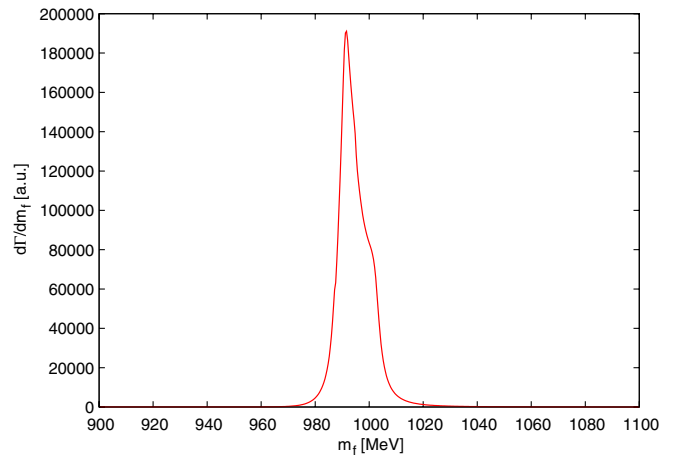


FIG. 13 (color online). $\frac{d\Gamma}{dm_f}$ for $\eta' \rightarrow \pi^0 \pi^+ \pi^-$ decay in the $f_0(980)$ region.

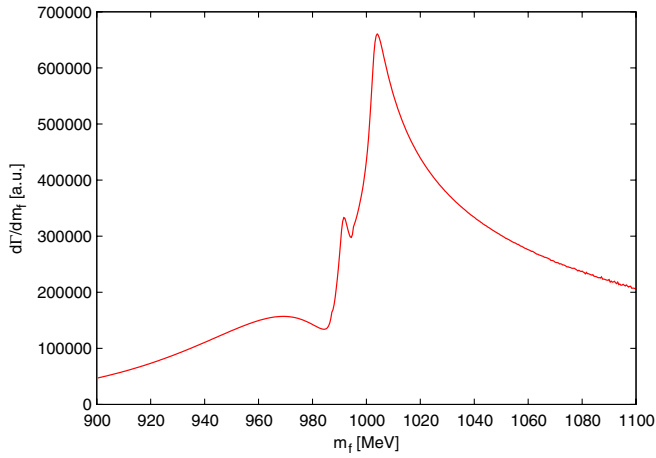


FIG. 14 (color online). $\frac{d\Gamma}{dm_f}$ for $\eta' \rightarrow \pi^0 \pi^0 \eta$ decay in the $a_0(980)$ region.

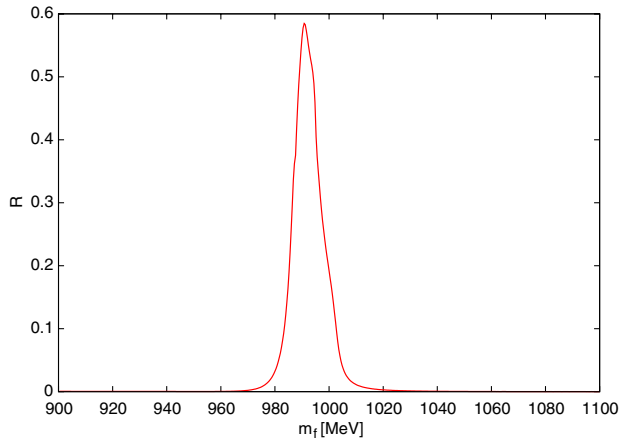


FIG. 15 (color online). Ratio $(\frac{d\Gamma}{dm_f})_{\pi^+ \pi^-} / (\frac{d\Gamma}{dm_f})_{\pi^0 \eta}$ as a function of m_f .

From these spectra we find that the ratio of integrated decay widths is now

$$\frac{\Gamma(\pi^0, \pi^+ \pi^-)}{\Gamma(\pi^0, \pi^0 \eta)} \approx 0.13. \quad (26)$$

This 13% is much closer to the experimental value of $(17.9 \pm 4.2)\%$, which has a lower limit of 13.7%. Assuming similar theoretical uncertainties the results are compatible. We have made some estimates of the errors by changing the cutoff q_{\max} by ± 20 MeV, which moves the $f_0(980)$ and $a_0(980)$ peak in $\pi\pi$ and $\pi\eta$ scattering by about 8 MeV. We find that this change induces changes in the ratio of Eq. (26) by 0.01. However, an uncertainty of 0.02 is more appropriate to account also for the uncertainties in the background subtraction. So we would be obtaining (0.13 ± 0.02) for the fraction of decay rates.

This increase by about one order of magnitude with respect to the standard calculation is a consequence of the two neighboring singularities in the triangle diagram which is peculiar to the $\eta(1405)$ case.

We can now estimate the effect of having also $\pi^0 \pi^0 \eta$ in the primary production process. A triangular diagram of the type used for $\pi^0 K \bar{K}$ production is not possible now. Indeed, one would have to substitute the K^* by a ρ , but this is dynamically forbidden (no $\rho \pi^0 \eta$ coupling). Then we must rely upon a contact term like the one assumed in the previous section.

By recalling the exercise done in the previous section [Eqs. (9)–(11)] and the conclusion that positive values of R (with respect to an equivalent local $\pi^0 K \bar{K}$ production mechanism) were preferred, the inspection of Fig. 7 can give us a qualitative estimate of what adding this new primary $\pi^0 \pi^0 \eta$ production vertex can do to the widths, which is a moderate increase of the ratio $\Gamma(\pi^0, \pi^+ \pi^-) / \Gamma(\pi^0, \pi^0 \eta)$ by about 26%. This would provide a ratio around 16.4% with an uncertainty of 2.5% or, with rounding errors, a ratio of (0.16 ± 0.03) , in good agreement with the experimental values.

Now we come back to the BES experiment [1]. In this experiment the authors cannot distinguish whether they have the $\eta(1405)$ or the $\eta(1475)$ resonance, so we must assume that they have a mixture of both. In order to account for this possibility, we have evaluated the same ratio of rates as before assuming that we have now the $\eta(1475)$ resonance. The result that we obtain is

$$\frac{\Gamma(\pi^0, \pi^+ \pi^-)}{\Gamma(\pi^0, \pi^0 \eta)} \Big|_{\eta(1475)} \approx 0.16. \quad (27)$$

This coincides with the centroid of our result of (0.16 ± 0.03) . We might also think about the possibility of a contribution from the original $\pi\pi\eta$ channel. However, the same collaboration team reports for the mixture of the resonances in the $J/\psi \rightarrow \gamma \pi^+ \pi^- \eta$, $J/\psi \rightarrow \gamma K \bar{K} \eta$ a large dominance of the second process by nearly one order of magnitude [37,38], which means we can neglect the primary $\pi\pi\eta$ channel in this case. Hence, assuming the same uncertainties as before, our final results for the $\eta(1405)$, the $\eta(1475)$, or a mixture of both are given by

$$\frac{\Gamma(\pi^0, \pi^+ \pi^-)}{\Gamma(\pi^0, \pi^0 \eta)} = 0.16 \pm 0.03. \quad (28)$$

We discuss here also the case of the $\eta(1295)$. Little is known about the couplings of this resonance to different channels. One might intuitively think that, by complementarity and orthogonality, if the $\eta(1405)$ couples strongly to $K^* \bar{K}$ it indicates that it has a large $s\bar{s}$ component, in which case the $\eta(1295)$ would mostly account for $u\bar{u}$ ($d\bar{d}$). In this case the coupling of the $\eta(1295)$ to $K^* \bar{K}$ would be highly suppressed.

We have evaluated the ratio $\Gamma(\pi^0, \pi^+ \pi^-) / \Gamma(\pi^0, \pi^0 \eta)$ at the peak of the f_0 , a_0 for the two situations as before: (a) contact primary vertices, (b) triangular mechanism via $K^* \bar{K}$ production.

In case (a) we find

$$\frac{\Gamma(\pi^0, \pi^+ \pi^-)}{\Gamma(\pi^0, \pi^0 \eta)} = 0.017, \quad (29)$$

while in case (b) we find

$$\frac{\Gamma(\pi^0, \pi^+ \pi^-)}{\Gamma(\pi^0, \pi^0 \eta)} = 0.12. \quad (30)$$

In this latter case the channel $\eta(1295) \rightarrow K^* \bar{K}$ is not open, but close by, such that its near singularity still has an effect on the ratio, similar to the one of Eq. (26). In the former case, Eq. (29), the results are also similar to all other cases where we have assumed dominance by primary contact production vertices.

Given the argumentation above, where we expect the $\eta(1295)$ to have small $s\bar{s}$ component, and hence small couplings to $K^* \bar{K}$, we would expect rates for $\Gamma(\pi^0, \pi^+ \pi^-)/\Gamma(\pi^0, \pi^0 \eta)$ of the order of 0.017. Should the experiment find a large value of this ratio, comparable to the one of the $\eta(1405)$, we would face an unexpected situation that could bring new light into the quest for the nature of the $\eta(1295)$ and $\eta(1405)$ resonances which has stimulated much work [39–42].

III. CONCLUSIONS

In this paper we have carried out a calculation of the decay rates of the $\eta(1405) \rightarrow \pi^0 f_0(980)(\pi^+ \pi^-)$ and $\eta(1405) \rightarrow \pi^0 a_0(980)(\pi^0 \eta)$ reactions with the aim of investigating the isospin violation in the first reaction which is tied to the $f_0(980)$ - $a_0(980)$ mixing in the terminology of other works. We have abstained from talking about a measure of the mixing since in our formalism there is no transition of one resonance to the other but a simultaneous production of both once the problem is tackled with meson states in charge basis with different masses, where a small violation of isospin is immediately obtained. Since the two resonances are produced from the interaction of meson pairs, the process proceeds via a first step in which a π^0 and a pair of mesons are produced, and a second step in which the pair of mesons interacts. Isospin violation has then two sources: the first loop after the production, and the scattering matrices of meson-meson interaction. But in both cases the violation is tied to the difference of masses between the charged and neutral kaons. As a consequence the shape of the peak obtained for the $\pi^+ \pi^-$ production in the first reaction has a very narrow width of the size of this mass difference, of the order of 9 MeV. This comes naturally in the approach and is in perfect agreement with the observation in the experiment.

In the first part we avoided making an explicit model for the reaction, but we assumed the primary production of $\pi^0 PP$ to be given by a contact term and we could see that, invoking general principles and admitting large uncertainties in the input, we obtained a rate of $\pi^+ \pi^-$ production versus $\pi^0 \eta$ production which was rather small, of the order

of one percent, which is in good agreement with the $f_0(980)$ and $a_0(980)$ mixing of the two BES experiments on $J/\psi \rightarrow \phi \pi \eta$ and $\chi_{c1} \rightarrow \pi^0 \pi \pi$, with respect to the isospin allowed counterparts of $J/\psi \rightarrow \phi \pi \pi$ and $\chi_{c1} \rightarrow \pi^0 \pi \eta$ [5]. The rates obtained are also in agreement with those obtained in theoretical papers of the $J/\psi \rightarrow \phi \pi \eta$ versus $J/\psi \rightarrow \phi \pi \pi$ [4,6]. However, the rates obtained for the $\eta \rightarrow \pi^0 \pi^+ \pi^-$ versus $\eta \rightarrow \pi^0 \pi^0 \eta$ are very small compared to those claimed in the experiment [1]—about one order of magnitude smaller. We tried to understand the situation by admitting a large admixture of $I = 1$ in the $\eta(1405)$ wave function, but it required a very large $I = 1$ component, not easily acceptable, and worse, it gave a signal for $f_0(980)$ production which had a width of the order of 20 MeV, which was much larger than the experimental one of the order of 9 MeV.

In the second part we followed the approach of Wu *et al.* [17] using the dominant primary production mechanism given by $\eta' \rightarrow K^* \bar{K}$ followed by $K^* \rightarrow K \pi$. The first loop now was quite different than for the contact interactions since the new singularity associated to $\eta' \rightarrow K^* \bar{K}$ played a very important role in the reaction. We found that using this new mechanism of production, the ratio of $\Gamma(\pi^0, \pi^+ \pi^-)/\Gamma(\pi^0, \pi^0 \eta)$ was increased by about one order of magnitude with respect to the results using the contact production vertices, providing results very close to those in the experiment. These results confirm the claims of Wu *et al.* [17], where, however, a precise determination of that ratio could not be given since it was tied to unknown form factors needed to regularize the divergent loops. The use of the chiral unitary approach in the present work solved this problem since one could associate the regularizing cutoff in the new loops to the one used in meson-meson scattering to generate the $f_0(980)$ and $a_0(980)$ resonances dynamically. This allowed us to make quantitative predictions for the $\Gamma(\pi^0, \pi^+ \pi^-)/\Gamma(\pi^0, \pi^0 \eta)$ ratio, with a value of (0.16 ± 0.03) , in basic agreement with experiment, of (0.179 ± 0.04) .

We also showed that the results obtained for that ratio were the same if we had the $\eta(1475)$ resonance instead of the $\eta(1405)$, or a mixture of the two, as seems to be the case in the BES experiment.

A final conclusion to be drawn is that the concept of $f_0(980)$ - $a_0(980)$ mixing is not very appropriate and different apparent ratios are obtained in different reactions. Then, the chiral unitary approach appears as an appropriate and accurate tool to use in order to analyze these reactions, and the present results, together with other results in different reactions on the mixing of these resonances, come to strengthen the support for the $f_0(980)$ and $a_0(980)$ resonances as dynamically generated from the meson-meson interaction.

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